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TORSIONS OF CONNECTIONS ON HIGHER ORDER
COTANGENT BUNDLES

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Abstract. By a torsion of a general connection Γ on a fibered manifold $Y \rightarrow M$ we understand the Frölicher-Nijenhuis bracket of Γ and some canonical tangent valued one-form (affinor) on Y . Using all natural affinors on higher order cotangent bundles, we determine all torsions of general connections on such bundles. We present the geometrical interpretation and study some properties of the torsions.

Keywords: affinor, general connection, torsion

MSC 2000: 53C05, 58A20

1. INTRODUCTION

Given a linear connection Γ on a manifold M , there are two classical approaches to the concept of the torsion of Γ . First, if we interpret Γ as a linear connection on the tangent bundle TM , we can define the torsion of Γ as the covariant exterior differential in the sense of Koszul of the identity tensor on M . This approach leads to the well known formula

$$(1) \quad \tau(X, Y) = \frac{1}{2}(\nabla_X Y - \nabla_Y X - [X, Y])$$

for every two vector fields X, Y on M . On the other hand, Γ can be also interpreted as a principal connection on the frame bundle PM of M . In this case we can introduce the torsion of Γ as the standard covariant differential of the canonical \mathbb{R}^m -valued

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form $\Theta: TPM \rightarrow \mathbb{R}^m$ of PM , $m = \dim M$. Clearly, both above definitions of the torsion of Γ give the same result. Further, if Γ^* is the dual linear connection on the cotangent bundle T^*M , the torsion of Γ^* can be introduced as the composition

$$(2) \quad \sigma \circ \Gamma^*$$

of the formal exterior differential $\sigma: J^1T^*M \rightarrow \wedge^2T^*M$ with the connection Γ^* itself, [8]. But the composition (2) can be defined for arbitrary (not only linear) connections on the cotangent bundle.

If $Y \rightarrow M$ is a fibered manifold, the (general) connection on Y is a smooth section $\Gamma: Y \rightarrow J^1Y$ of the first jet prolongation of Y . Obviously, Γ can be identified with its horizontal projection $TY \rightarrow TY$, which is a tangent valued one-form on Y . If $Q: TY \rightarrow TY$ is an arbitrary canonical tangent valued one-form on Y (in other words an affiner), the Frölicher-Nijenhuis bracket $[\Gamma, Q]$ of Γ and Q is called a (general) torsion of Γ . Let F be a natural bundle on the category $\mathcal{M}f_m$ of all m -dimensional manifolds and their local diffeomorphisms and $\Gamma: FM \rightarrow J^1FM$ be a connection on FM . If we determine all natural affiners on FM , then we can completely describe all general torsions of Γ . Such an approach has been used e.g. in [2], [6] and [9] for many particular cases of F .

In this paper we determine all general torsions of connections on r -th order cotangent bundles $T^{r*}M$. We also study the geometrical interpretation and properties of the torsions. In the geometrical construction of the torsions we essentially use the vertical prolongation of connections on vector bundles. We characterize linear connections and projectable connections on $T^{r*}M$ by means of certain properties of their torsions. We also show the interpretation of (2) for arbitrary (not only linear) connections. We remark that higher order cotangent bundles and their dual bundles (which are called higher order tangent bundles) are used e.g. in higher order mechanics. For example, in the study, within a continuum mechanical context, of higher order gross bodies, it is useful to study fields with values in tensor products of tangent and cotangent spaces of “higher order contact”, cf. [10]. All manifolds and maps are assumed to be infinitely differentiable.

2. AFFINORS AND GENERAL TORSIONS

Denoting by $C^\infty(TM \otimes \wedge^p T^*M)$ the space of all tangent valued p -forms on M , the Frölicher-Nijenhuis bracket is a map

$$[\cdot, \cdot]: C^\infty(TM \otimes \wedge^p T^*M) \times C^\infty(TM \otimes \wedge^q T^*M) \rightarrow C^\infty(TM \otimes \wedge^{p+q} T^*M),$$

(cf. [3]). In particular, for $p = 0 = q$ the Frölicher-Nijenhuis bracket coincides with the Lie bracket of vector fields on M . By the Frölicher-Nijenhuis theory, tangent valued 1-forms on M are exactly *affinors* on M , which are defined as linear morphisms $TM \rightarrow TM$ over the identity of M . Then the Frölicher-Nijenhuis bracket of two such affinors K, L is a tangent valued 2-form $[K, L]$ given by

$$(3) \quad [K, L](X, Y) = [KX, LY] + [LX, KY] + KL[X, Y] + LK[X, Y] \\ - K[X, LY] - K[LX, Y] - L[X, KY] - L[KX, Y],$$

X, Y being arbitrary vector fields on M .

Let $p: Y \rightarrow M$ be a fibered manifold and denote by VY the vertical tangent bundle. An affinor $\varphi \in C^\infty(TY \otimes T^*Y)$ on Y is called *vertical* if $\varphi \in C^\infty(VY \otimes T^*Y)$. Consider now the inclusion $T^*M \subset T^*Y$ of cotangent bundles.

Definition. A vertical affinor $\varphi \in C^\infty(VY \otimes T^*M)$ is called a *soldering form*.

Clearly, $J^1Y \rightarrow Y$ is an affine bundle, the associated vector bundle of which is $VY \otimes T^*M$. In this way soldering forms on Y can be considered as “deformations” of connections on Y in the following sense: If $\Gamma: Y \rightarrow J^1Y$ is a connection and $\varphi: Y \rightarrow VY \otimes T^*M$ a soldering form on Y , then their sum $(\Gamma + \varphi)$ is a connection on Y . On the other hand, the difference of two connections on Y is a soldering form.

If we identify a connection $\Gamma: Y \rightarrow J^1Y$ with its horizontal projection $TY \rightarrow TY$ (denoted by the same symbol Γ), we obtain an affinor on Y . In this situation the Frölicher-Nijenhuis bracket turns out to be a very powerful tool in the theory of connections. For example, by [3], the curvature of Γ is given by

$$(4) \quad C\Gamma = \frac{1}{2}[\Gamma, \Gamma]$$

and the Bianchi identity can be simply expressed in the form $[\Gamma, C\Gamma] = 0$. Consider now a soldering form $\varphi: Y \rightarrow VY \otimes T^*M$ (i.e. a deformation of Γ) and modify the curvature formula (4) by

$$(5) \quad \tau_\varphi := [\Gamma, \varphi].$$

By Mangiarotti and Modugno, [8], $\tau_\varphi: Y \rightarrow VY \otimes \wedge^2 T^*M$ is called a *torsion of Γ with respect to the soldering form φ* . By [8], τ_φ generalizes the classical concept of the torsion of a connection. Finally, the most general definition of a torsion was given by Kolář and Modugno, [2]. The main idea is to replace the soldering form φ in (5) by an arbitrary affinor on Y , which is canonical in the following sense:

Definition. A *natural affinor* on a natural bundle F over manifolds is a system of affinors $Q_M: TFM \rightarrow TFM$ for every m -manifold M satisfying $TFf \circ Q_M = Q_N \circ Tff$ for every local diffeomorphism $f: M \rightarrow N$.

Let F be a natural bundle on the category $\mathcal{M}f_m$ and $\Gamma: FM \rightarrow J^1(FM \rightarrow M)$ be a connection on FM .

Definition. Let Q be a non identical natural affinor on F . The Frölicher-Nijenhuis bracket $[\Gamma, Q]$ is called the (general) *torsion* of a connection Γ .

Clearly, $[\Gamma, \varphi] \in C^\infty(TY \otimes \wedge^2 T^*Y)$ and if $\text{Id}: FM \rightarrow FM$ is a trivial identical affinor, then we have $[\Gamma, \text{Id}] = 0$. For many particular cases of a natural bundle F we are able to determine all natural affinors on F and in this way all general torsions of a given connection Γ on FM . For example, any product preserving functor F on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps is a Weil functor of the form $F = T^A$, where $A = F\mathbb{R}$ is the corresponding Weil algebra, [3]. By Kolář and Modugno, [2], all natural affinors on F are parametrized by $F\mathbb{R}$. An important example of a product preserving functor is the functor T_k^r of k -dimensional velocities of order r , which is defined by

$$T_k^r M = J_0^r(\mathbb{R}^k, M), \quad T_k^r f(j_0^r g) = j_0^r(f \circ g)$$

for an arbitrary smooth manifold M and a smooth map $f: M \rightarrow N$. On the basis of the complete list of all natural affinors on product preserving bundles, Kolář and Modugno have described all general torsions of connections on the bundles TM , $T_k^1 M$, $T_1^2 M$ and on the frame bundle PM , [2]. The same authors have also computed general torsions on the cotangent bundle.

From (3) it follows that the Frölicher-Nijenhuis bracket of a connection $\Gamma: Y \rightarrow J^1 Y$ and a soldering form $\varphi: Y \rightarrow VY \otimes T^*M$ is of the form

$$(6) \quad [\Gamma, \varphi](U, V) = \frac{1}{2}([\Gamma U, V] - [\Gamma V, \varphi U] - \varphi[U, V])$$

for every two vector fields U, V on M . Denoting by (x^i, y^p) the local fibered coordinates on Y , the equations of a connection $\Gamma: Y \rightarrow J^1 Y$ are

$$dy^p = F_i^p(x, y) dx^i$$

so that the corresponding horizontal projection $\Gamma: TY \rightarrow TY$ is of the form $(dx^i, dy^p) \mapsto (dx^i, F_i^p dx^i)$, i.e.

$$\delta_j^i \frac{\partial}{\partial x^i} \otimes dx^j + F_i^p \frac{\partial}{\partial y^p} \otimes dx^i.$$

Further, if

$$\varphi_i^p(x, y) \frac{\partial}{\partial y^p} \otimes dx^i$$

is the coordinate expression of a soldering form $\varphi \in C^\infty(VY \otimes T^*M)$, then (6) is of the form

$$(7) \quad \left(\frac{\partial \varphi_j^p}{\partial x^i} + F_i^q \frac{\partial \varphi_j^p}{\partial y^q} - \varphi_j^q \frac{\partial F_i^p}{\partial y^q} \right) \frac{\partial}{\partial y^p} \otimes (dx^i \wedge dx^j).$$

3. NATURAL AFFINORS ON HIGHER ORDER COTANGENT BUNDLES

Let M be a smooth manifold. The space

$$T^{r*}M = J^r(M, \mathbb{R})_0$$

is called the r -th order cotangent bundle. Every local diffeomorphism $f: M \rightarrow N$ extends to a vector bundle morphism $T^{r*}f: T^{r*}M \rightarrow T^{r*}N$, $j_x^r \varphi \mapsto j_{f(x)}^r(\varphi \circ f^{-1})$, where f^{-1} is constructed locally. Then T^{r*} is a vector bundle functor defined on the category $\mathcal{M}f_m$. Clearly, the functor T^{r*} does not preserve products and for $r = 1$ we obtain the classical cotangent functor T^* . Denoting by (x^i) some local coordinates on M , the induced coordinates $(u_i, u_{ij}, \dots, u_{i_1 \dots i_r})$ on $T^{r*}M$ (symmetric in all indices) are given by

$$u_i(j_x^r f) = \left. \frac{\partial f}{\partial x^i} \right|_x, \quad u_{ij}(j_x^r f) = \left. \frac{\partial^2 f}{\partial x^i \partial x^j} \right|_x, \quad u_{i_1 \dots i_r}(j_x^r f) = \left. \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_r}} \right|_x.$$

Denote by $\pi_M: T^{r*}M \rightarrow M$ and $p_M: TM \rightarrow M$ the bundle projections and $q_M: T^{r*}M \rightarrow T^*M$ the projection given by $q_M(j_x^r f) = j_x^1 f$. Let $\lambda_M: TT^{r*}M \rightarrow \mathbb{R}$ be the generalized Liouville form on $T^{r*}M$ defined by

$$\lambda_M(X) = \langle q_M(p_{T^{r*}M}(X)), T\pi_M(X) \rangle$$

and $A_s^r: T^{r*}M \rightarrow T^{r*}M$ be the s -th power natural transformation of $T^{r*}M$ into itself defined by $A_s^r(j_x^r f) = j_x^r(f)^s$, where $(f)^s$ denote the s -th power of f . Since $\pi_M: T^{r*}M \rightarrow M$ is a vector bundle, the vertical tangent bundle $VT^{r*}M$ can be identified with the Whitney sum $T^{r*}M \oplus T^{r*}M$. Taking into account this identification, we can define natural affinors $Q_M^s: TT^{r*}M \rightarrow VT^{r*}M$ by

$$(8) \quad Q_M^s(X) = (p_{T^{r*}M}(X), \lambda_M(X)A_s^r(p_{T^{r*}M}(X))).$$

In coordinates,

$$\begin{aligned}
 (9) \quad Q_M^s &= u_{i_1 \dots i_s} u_j \frac{\partial}{\partial u_{i_1 \dots i_s}} \otimes dx^j \\
 &+ \frac{(s+1)!}{(s-1)! 2!} u_{(i_1 \dots i_{s-1} i_{s+1})} u_j \frac{\partial}{\partial u_{i_1 \dots i_{s+1}}} \otimes dx^j \\
 &+ \dots + \frac{r!}{(s-1)! (r-s+1)!} u_{(i_1 \dots i_{s-1} i_s \dots i_r)} u_j \frac{\partial}{\partial u_{i_1 \dots i_r}} \otimes dx^j
 \end{aligned}$$

where $(i_1 \dots i_r)$ denote the symmetrization. The second author has proved, [4]:

Lemma 1. *All natural affiners on the r -th order cotangent bundle $T^{r*}M$ are of the form*

$$(10) \quad k_0 \text{Id} + k_1 Q_M^1 + \dots + k_r Q_M^r$$

with any real parameters k_0, k_1, \dots, k_r .

Obviously, all nontrivial natural affiners on $T^{r*}M$ are vertical and we have $Q_M^i \in C^\infty(VT^{r*}M \otimes T^*M)$, i.e. all Q_M^i are even soldering forms, $i = 1, \dots, r$. The second author has also described all natural transformations of T^{r*} into itself, [5].

Lemma 2. *All natural transformations $T^{r*} \rightarrow T^{r*}$ are of the form*

$$(11) \quad k_1 A_1^r + \dots + k_r A_r^r$$

with any real parameters k_1, \dots, k_r .

4. VERTICAL PROLONGATION OF CONNECTIONS ON VECTOR BUNDLES

We first recall the concept of the vertical prolongation of a connection, see [3]. Let $\Gamma: Y \rightarrow J^1Y$ be a connection on a fibered manifold $p: Y \rightarrow M$. Applying the vertical tangent functor V , we obtain a map $V\Gamma: VY \rightarrow VJ^1Y$. Let

$$i_Y: VJ^1Y \rightarrow J^1VY$$

be the canonical exchange isomorphism, [3]. Then the composition $\mathcal{V}\Gamma := i_Y \circ V\Gamma: VY \rightarrow J^1VY$ is a connection on $VY \rightarrow M$. Denote by $(x^i, y^p, \eta^p = dy^p)$ the

canonical coordinates on VY . If $dy^p = F_i^p(x, y)dx^i$ is the coordinate form of Γ , then the vertical prolongation $\mathcal{V}\Gamma$ has equations

$$\begin{aligned} dy^p &= F_i^p(x, y)dx^i, \\ d\eta^p &= \frac{\partial F_i^p(x, y)}{\partial y^q} \eta^q dx^i. \end{aligned}$$

Now let $p: E \rightarrow M$ be a vector bundle. Then $J^1E \rightarrow M$ is a vector bundle too and we have the canonical identifications $VE \cong E \oplus_M E$ and $VJ^1E \cong J^1E \oplus_M J^1E$. Using the exchange isomorphism $i_E: VJ^1E \rightarrow J^1VE$ we obtain an identification

$$\alpha_E: J^1VE \cong J^1E \oplus_M J^1E.$$

Consider the classical Liouville vector field

$$L_E: E \rightarrow VE \subset TE$$

which is generated by the one-parameter group of homotheties of the vector bundle E , in coordinates $L_E = y^p \frac{\partial}{\partial y^p}$. If $\Gamma: E \rightarrow J^1E$ is a connection on E (not necessarily linear), then the composition

$$(12) \quad \bar{\Gamma} := \text{pr}_2 \circ \alpha_E \circ \mathcal{V}\Gamma \circ L_E: E \rightarrow J^1E$$

is a section of J^1E . Clearly, $\bar{\Gamma}$ is a connection on $E \rightarrow M$ with the coordinate expression

$$dy^p = \frac{\partial F_i^p(x, y)}{\partial y^q} y^q dx^i.$$

We have

Proposition 1. $\Gamma = \bar{\Gamma}$ if and only if the connection Γ is linear.

Proof. Linearity of Γ means $F_i^p(x, y) = F_{ir}^p(x) y^r$. On the other hand, $\Gamma = \bar{\Gamma}$ iff $F_i^p(x, y) = \frac{\partial F_i^p(x, y)}{\partial y^q} y^q$. □

One verifies directly

Proposition 2. Let Γ be a connection on the vector bundle $E \rightarrow M$. Then the vertical prolongation $\mathcal{V}\Gamma: VE \rightarrow J^1VE$ is of the form

$$\mathcal{V}\Gamma = (\Gamma \times \bar{\Gamma}): E \oplus_M E \cong VE \rightarrow J^1VE \cong J^1E \oplus_M J^1E.$$

Suppose now that $E = FM$, where F is a natural vector bundle functor. To construct the connection $\bar{\Gamma}$, we have used the Liouville vector field $L_{FM}: FM \rightarrow VFM$. This vector field is canonical (more precisely natural) in the following sense:

Definition. A natural vector field ξ on F is a system of vector fields $\xi_M: FM \rightarrow TFM$ for every m -manifold M satisfying $Tff \circ \xi_M = \xi_N \circ Ff$ for all local diffeomorphisms $f: M \rightarrow N$.

Roughly speaking, natural vector fields on F can be interpreted as absolute (or constant) natural operators transforming vector fields on M into vector fields on FM , [3]. By [3], every natural vector field on FM is vertical.

Example. Let $\Phi(t)$ be a smooth one-parameter family of natural transformations $F \rightarrow F$, where smoothness means that the map $(\Phi(t))_M: \mathbb{R} \times FM \rightarrow FM$ is smooth for every manifold M . Then the formula $\xi_M = \frac{\partial}{\partial t} \Big|_0 (\Phi(t))_M$ defines a natural vector field $\xi_M: FM \rightarrow VFM$.

If we replace the Liouville vector field L_{FM} in (12) by an arbitrary natural vector field $L: FM \rightarrow VFM$, we obtain

Proposition 3. Let $\Gamma: FM \rightarrow J^1FM$ be a connection on a natural vector bundle F . Then every natural vector field $L: FM \rightarrow VFM$ induces a connection $\Gamma_L: FM \rightarrow J^1FM$ by

$$(13) \quad \Gamma_L = \text{pr}_2 \circ \alpha_{FM} \circ \mathcal{V}\Gamma \circ L.$$

If L is the Liouville vector field L_{FM} , then $\Gamma_L = \bar{\Gamma}$.

Let $\Gamma: E \rightarrow J^1E$ be a connection on the vector bundle $E \rightarrow M$ and $f: E \rightarrow E$ be a vector bundle morphism over the identity of M . In the rest of this section we describe the construction of a connection on E by means of the vector bundle morphism f . Denoting by $\text{Id}_E: E \rightarrow E$ the identity morphism, the difference $(\text{Id}_E - f): E \rightarrow E$ is also a vector bundle morphism over id_M . Using the inclusion $E \subset J^1E$ we obtain a vector bundle morphism

$$\tilde{f} := (\text{Id}_E - f): E \rightarrow J^1E$$

over the identity of M . On the other hand, the composition

$$g := J^1 f \circ \Gamma: E \rightarrow J^1 E$$

is another vector bundle morphism over the identity of M . Then the sum of \tilde{f} and g on the vector bundle structure $J^1 E$ is a section of $J^1 E$,

$$(14) \quad \Gamma_f := (\tilde{f} + g): E \rightarrow J^1 E.$$

Clearly, Γ_f is a connection on E and for $f = \text{Id}_E$ we have $\Gamma_{\text{Id}_E} = \Gamma$.

5. THE TORSION ON THE COTANGENT BUNDLE

By Lemma 1, we have one nontrivial affiner $Q_M^1: TT^*M \rightarrow VT^*M$ on the cotangent bundle, $Q_M^1(dx^i, du_i) = (0, u_i u_k dx^k)$. If

$$du_i = \Gamma_{i,l}(x, u) dx^l$$

are equations of a connection Γ on T^*M , then by (7) the torsion $\tau := [\Gamma, Q_M^1]: T^*M \rightarrow VT^*M \otimes \wedge^2 T^*M$ is of the form

$$\tau = \left(\Gamma_{k,i} u_j + \Gamma_{j,i} u_k - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m u_j \right) \frac{\partial}{\partial u_k} \otimes (dx^i \wedge dx^j).$$

Write

$$\begin{aligned} \tau^L &= \Gamma_{j,i}(dx^i \wedge dx^j) \otimes \left(u_k \frac{\partial}{\partial u_k} \right), \\ \tau^* &= \left(\Gamma_{k,i} - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m \right) u_j \frac{\partial}{\partial u_k} \otimes (dx^i \wedge dx^j). \end{aligned}$$

Then we have

Proposition 4. *The torsion on the cotangent bundle T^*M is of the form*

$$\tau = \tau^L + \tau^*.$$

Further, $\tau^* = 0$ if and only if the connection Γ is linear.

Proof. If Γ is a linear connection, then $\Gamma_{k,i}(x, u) = \Gamma_{k,i}^r(x) u_r$ which yields $\tau^* = 0$. On the other hand, if $\tau^* = 0$, then we obtain the differential equation $\Gamma_{k,i} = \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m$. Clearly, the differential equation $y = y' \cdot x$ has the general solution $y = cx$, i.e. $\Gamma_{k,i}(x, u) = \Gamma_{k,i}^r(x) u_r$. \square

In other words, a connection Γ on T^*M is linear if and only if $\tau = \tau^L$. In the rest of this section we describe the geometrical construction of τ^L and τ^* . We remark that the construction of the “linear” term τ^L was given also by Kolář and Modugno in [2].

I. The construction of τ^L . By the theory of sesquiholonomic 2-jets, there is a natural map

$$\sigma: J^1T^*M \rightarrow \wedge^2T^*M$$

which is called the formal exterior differential, [7]. Denoting by $(x^i, u_i, u_{i,j})$ the local coordinates on J^1T^*M , the coordinate form of σ is $u_{i,j}(dx^i \wedge dx^j)$. Then the composition $\sigma \circ \Gamma: T^*M \rightarrow \wedge^2T^*M$ has the coordinate form

$$\Gamma_{i,j}(dx^i \wedge dx^j)$$

and $\tau^L: T^*M \rightarrow VT^*M \otimes \wedge^2T^*M$ is of the form

$$\tau^L(u) = (\sigma \circ \Gamma)(u) \otimes L(u)$$

where $L: T^*M \rightarrow VT^*M$ is the classical Liouville vector field on T^*M .

Remark. By Proposition 4, the torsion of every connection on the cotangent bundle can be written in the form $\tau = \tau^L + \tau^*$. Using such a point of view, the “linear” part τ^L can be considered as the interpretation of the classical formula (2) for arbitrary connections.

II. The construction of τ^* . Let L be the classical Liouville vector field on T^*M and denote by Γ_L the induced connection (13) from Proposition 3, where we put $F = T^*$. Then the equations of $\Gamma_L: T^*M \rightarrow J^1T^*M$ are

$$du_i = \frac{\partial \Gamma_{i,k}}{\partial u_m} u_m dx^k.$$

Since $J^1T^*M \rightarrow T^*M$ is an affine bundle with the associated vector bundle $VT^*M \otimes T^*M$, the difference of two connections Γ and Γ_L on T^*M is a section of $VT^*M \otimes T^*M$, in coordinates

$$\left(\Gamma_{k,i} - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m \right) \frac{\partial}{\partial u_k} \otimes dx^i.$$

Multiplying by the Liouville one-form $(u_j dx^j)$ and then using the antisymmetrization $T^*M \otimes T^*M \rightarrow \wedge^2T^*M$, we obtain τ^* .

6. THE FIRST TORSION ON THE r -TH ORDER COTANGENT BUNDLE

A connection Γ on $T^{r*}M$ has equations

$$(15) \quad du_i = \Gamma_{i,l}(x, u)dx^l, \quad du_{ij} = \Gamma_{ij,l}(x, u)dx^l, \dots, \quad du_{i_1 \dots i_r} = \Gamma_{i_1 \dots i_r, l}(x, u)dx^l$$

and the first natural affinor $Q_M^1: TT^{r*}M \rightarrow VT^{r*}M$ is of the form

$$Q_M^1(dx^i, du_i, du_{ij}, \dots, du_{i_1 \dots i_r}) = (0, u_i u_p dx^p, u_{ij} u_p dx^p, \dots, u_{i_1 \dots i_r} u_p dx^p),$$

see (8) and (9). In this section we show that the first torsion

$$\tau_1 := [\Gamma, Q_M^1]: T^{r*}M \rightarrow VT^{r*}M \otimes \wedge^2 T^*M$$

has quite similar properties and interpretation as the torsion τ on the cotangent bundle. Write

$$(16) \quad \begin{aligned} \tau_1^L &= \Gamma_{j,i}(dx^i \wedge dx^j) \otimes \left(u_k \frac{\partial}{\partial u_k} + u_{kl} \frac{\partial}{\partial u_{kl}} + \dots + u_{k_1 \dots k_r} \frac{\partial}{\partial u_{k_1 \dots k_r}} \right), \\ \tau_1^* &= \left(\Gamma_{k,i} - \frac{\partial \Gamma_{k,i}}{\partial u_m} u_m - \dots - \frac{\partial \Gamma_{k,i}}{\partial u_{m_1 \dots m_r}} u_{m_1 \dots m_r} \right) u_j \\ &\quad \times \frac{\partial}{\partial u_k} \otimes (dx^i \wedge dx^j) \\ &\quad + \left(\Gamma_{kl,i} - \frac{\partial \Gamma_{kl,i}}{\partial u_m} u_m - \dots - \frac{\partial \Gamma_{kl,i}}{\partial u_{m_1 \dots m_r}} u_{m_1 \dots m_r} \right) u_j \frac{\partial}{\partial u_{kl}} \otimes (dx^i \wedge dx^j) \\ &\quad \vdots \\ &\quad + \left(\Gamma_{k_1 \dots k_r, i} - \frac{\partial \Gamma_{k_1 \dots k_r, i}}{\partial u_m} u_m - \dots - \frac{\partial \Gamma_{k_1 \dots k_r, i}}{\partial u_{m_1 \dots m_r}} u_{m_1 \dots m_r} \right) u_j \\ &\quad \times \frac{\partial}{\partial u_{k_1 \dots k_r}} \otimes (dx^i \wedge dx^j). \end{aligned}$$

We prove

Proposition 5. *The first torsion τ_1 on $T^{r*}M$ is of the form*

$$(17) \quad \tau_1 = \tau_1^L + \tau_1^*.$$

Further, $\tau_1^* = 0$ if and only if the connection Γ is linear.

Proof. Formula (17) follows from (7) by a direct computation. If Γ is linear, then

$$\Gamma_{k,i} = \Gamma_{k,i}^m u_m + \Gamma_{k,i}^{mn} u_{mn} + \dots + \Gamma_{k,i}^{m_1 \dots m_r} u_{m_1 \dots m_r}$$

so that the first term of τ_1^* vanishes. Quite analogously we show that all the remaining terms of τ_1^* are zero. On the other hand, if $\tau_1^* = 0$, then all the terms of τ_1^* vanish. In this case we obtain a partial differential equation $f(x, u_1, \dots, u_r) = \frac{\partial f}{\partial u_1} u_1 + \dots + \frac{\partial f}{\partial u_r} u_r$, whose solution is of the form $f = \tilde{f}_1(x)u_1 + \dots + \tilde{f}_r(x)u_r$. \square

I. The geometrical construction of τ_1^L . Denote by $L: T^{r*}M \rightarrow VT^{r*}M$ the Liouville vector field. Using the formal exterior differential $\sigma: J^1T^*M \rightarrow \wedge^2T^*M$ and the projection $q_M: T^{r*}M \rightarrow T^*M$ we obtain a map $\tilde{\sigma} := \sigma \circ J^1q_M: J^1T^{r*}M \rightarrow \wedge^2T^*M$ with the coordinate form $u_{i,j}(dx^i \wedge dx^j)$. Then τ_1^L is of the form

$$(18) \quad \tau_1^L(u) = (\tilde{\sigma} \circ \Gamma)(u) \otimes L(u).$$

II. The construction of τ_1^* . Let L be the Liouville vector field on $T^{r*}M$, $F = T^{r*}$ and Γ_L be the connection (13) from Proposition 3. Taking into account that $J^1T^{r*}M \rightarrow T^{r*}M$ is an affine bundle with the corresponding associated vector bundle $VT^{r*}M \otimes T^*M$, the difference of connections Γ and Γ_L on $T^{r*}M$ is the section of the associated vector bundle. Using multiplication by the Liouville form $(u_j dx^j)$ and then the antisymmetrization $\otimes^2 T^*M \rightarrow \wedge^2 T^*M$, we obtain τ_1^* .

7. TORSIONS ON THE r -TH ORDER COTANGENT BUNDLE

Let Γ be a connection (15) on $T^{r*}M$. By Lemma 1, all natural affinors on $T^{r*}M$ are linearly generated by the identity affinor and by nontrivial affinors Q_M^1, \dots, Q_M^r . In this way we have the list of r torsions on $T^{r*}M$,

$$\tau_s := [\Gamma, Q_M^s] \in C^\infty(VT^{r*}M \otimes \wedge^2 T^*M), \quad s = 1, \dots, r.$$

We show that analogously to Proposition 5, each such torsion can be expressed in the form $\tau_s = \tau_s^{L_s} + \tau_s^*$.

We first construct the terms $\tau_s^{L_s}$. Let $A_s^r: T^{r*}M \rightarrow T^{r*}M$ be the s -th power natural transformation from Lemma 2. Multiplying by a real number t , we obtain a smooth one-parameter family of natural transformations. Denote by

$$L_s: T^{r*}M \rightarrow VT^{r*}M$$

the natural vector field generated by this smooth one-parameter family, see Example. Then L_1 is the classical Liouville vector field. The vector field L_s can be also defined as a map $T^{r*}M \rightarrow T^{r*}M \oplus T^{r*}M \cong VT^{r*}M$ of the form $u \mapsto (u, A_s^r(u))$. In coordinates,

$$L_s = u_{i_1} \dots u_{i_s} \frac{\partial}{\partial u_{i_1 \dots i_s}} + \dots + \frac{r!}{(s-1)!(r-s+1)!} u_{(i_1 \dots u_{i_{s-1}} u_{i_s \dots i_r})} \frac{\partial}{\partial u_{i_1 \dots i_r}}.$$

Now it suffices to modify (18) by

$$(19) \quad \tau_s^{L_s}(u) = (\tilde{\sigma} \circ \Gamma)(u) \otimes L_s(u), \quad s = 1, \dots, r.$$

In coordinates, $\tau_s^{L_s} = \Gamma_{i,j}(dx^i \wedge dx^j) \otimes L_s$.

It remains to construct τ_s^* . Write $f_s := A_s^r: T^{r*}M \rightarrow T^{r*}M$ for the s -th power natural transformation. Let Γ_{f_s} be the connection on $T^{r*}M$ defined by (14) by means of the vector bundle morphism $f = f_s$. Since $f_1 = \text{id}_{T^{r*}M}$, we have $\Gamma_{f_1} = \Gamma$. Further, let Γ_{L_s} be the connection (13) from Proposition 3 corresponding to the natural vector field L_s on $T^{r*}M$. Then the difference of connections Γ_{f_s} and Γ_{L_s} is a section $t: T^{r*}M \rightarrow VT^{r*}M \otimes T^*M$ of the associated vector bundle. Multiplication by the Liouville one-form $(u_j dx^j)$ yields an element of $(VT^{r*}M \otimes T^*M) \otimes T^*M$. Finally, using antisymmetrization $\otimes^2 T^*M \rightarrow \wedge^2 T^*M$ we obtain $\tau_s: T^{r*}M \rightarrow VT^{r*}M \otimes \wedge^2 T^*M$. In what follows we shall need only the coordinate form of τ_r^* ,

$$(20) \quad \begin{aligned} \tau_r^* = & \left(-\frac{\partial \Gamma_{k,i}}{\partial u_{m_1 \dots m_r}} u_{m_1} \dots u_{m_r} \right) u_j \frac{\partial}{\partial u_k} \otimes (dx^i \wedge dx^j) + \dots \\ & + \left(-\frac{\partial \Gamma_{k_1 \dots k_{r-1}, i}}{\partial u_{m_1 \dots m_r}} u_{m_1} \dots u_{m_r} \right) u_j \frac{\partial}{\partial u_{k_1 \dots k_{r-1}}} \otimes (dx^i \wedge dx^j) \\ & + \left(\Gamma_{l,i} \delta_{(k_1}^{l} u_{k_2} \dots u_{k_r)} - \frac{\partial \Gamma_{k_1 \dots k_r, i}}{\partial u_{m_1 \dots m_r}} u_{m_1} \dots u_{m_r} \right) u_j \\ & \times \frac{\partial}{\partial u_{k_1 \dots k_r}} \otimes (dx^i \wedge dx^j). \end{aligned}$$

Using (7), (9) and (15) we prove by a direct evaluation

Proposition 6. *All torsions of a connection on $T^{r*}M$ are of the form*

$$\tau_s = \tau_s^{L_s} + \tau_s^*, \quad s = 1, \dots, r.$$

By Proposition 5, $\tau_1^* = 0$ if and only if the connection Γ is linear. We show that the vanishing of τ_r^* expresses the projectability of Γ in the following sense:

Definition. We say that a connection Γ on $T^{r*}M$ is *projectable* if there is a connection Δ on $T^{(r-1)*}M$ such that

$$J^1 \pi_M^{r,r-1} \circ \Gamma = \Delta \circ \pi_M^{r,r-1},$$

where $\pi_M^{r,r-1}: T^{r*}M \rightarrow T^{(r-1)*}M$ is the bundle projection.

Proposition 7. *If $\tau_r^* = 0$, then the connection Γ is projectable.*

Proof. By (20), if $\tau_r^* = 0$, then the first $(r - 1)$ components of Γ in the coordinate expression (15) are independent of $u_{m_1 \dots m_r}$. \square

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