Ladislav Nebeský

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THE DIRECTED GEODETIC STRUCTURE OF A STRONG DIGRAPH

Ladislav Nebeský, Praha

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Abstract. By a ternary structure we mean an ordered pair (U_0, T_0), where U_0 is a *finite* nonempty set and T_0 is a ternary relation on U_0. A ternary structure (U_0, T_0) is called here a directed geodetic structure if there exists a strong digraph D with the properties that $V(D) = U_0$ and

$$T_0(u,v,w) \quad \text{if and only if} \quad d_D(u,v) + d_D(v,w) = d_D(u,w)$$

for all $u, v, w \in U_0$, where d_D denotes the (directed) distance function in D. It is proved in this paper that there exists no sentence **s** of the language of the first-order logic such that a ternary structure is a directed geodetic structure if and only if it satisfies **s**.

Keywords: strong digraph, directed distance, ternary relation, finite structure

MSC 2000: 05C12, 05C20, 03C13

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The letters $e, f, g, \ldots, n$ (possibly with indices) will be reserved for denoting integers. All graphs and digraphs considered here are finite. For the graph theory terminology, the reader in referred to [1].

Let G be a connected graph, and let $V(G)$, $E(G)$ and d_G denote the vertex set of G, the edge set of G and the distance function of G, respectively. By the geodetic relation of G we will mean the ternary relation Γ_G on $V(G)$ defined as follows:

$$\Gamma_G(u,v,w) \quad \text{if and only if} \quad d_G(u,v) + d_G(v,w) = d_G(u,w)$$

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for all $u, v, w \in V(G)$. By the interval function of G we mean the mapping $I_G\colon V(G) \times V(G) \to 2^{V(G)}$ defined as follows:

$$I_G(x, y) = \{z \in V(G);\ \Gamma_G(x, z, y)\}$$

for all $x, y \in V(G)$. Note that the interval function of a connected graph was extensively studied in Mulder [3].

Let D be a strong digraph, and let $V(D)$, $E(D)$ and d_D denote the vertex set of D, the edge set of D and the (directed) distance function of D, respectively. By the directed geodetic relation of D we shall mean the ternary relation Γ_D^{dir} on $V(D)$ defined as follows:

$$\Gamma_D^{\mathrm{dir}}(u, v, w) \quad \text{if and only if} \quad d_D(u, v) + d_D(v, w) = d_D(u, w)$$

for all $u, v, w \in V(D)$.

By a *ternary structure* we shall mean an ordered pair (U_0, T_0), where U_0 is a *finite* nonempty set and T_0 is a ternary relation on U_0. We say that a ternary structure (U_0, T_0) is a *geodetic structure* if there exists a connected graph G such that $U_0 = V(G)$ and $T_0 = \Gamma_G$. We say that a ternary structure (U_0, T_0) is a *directed geodetic structure* if there exists a strong digraph D such that $U_0 = V(D)$ and $T_0 = \Gamma_D^{\mathrm{dir}}$.

In [4], [5] and [6] the present author gave an axiomatic characterization of the interval function of a connected graph. This characterization can be easily reformulated to an axiomatic characterization of the geodetic structure: a ternary structure is a geodetic structure if and only if it satisfies a certain finite set of axioms, or said more strictly, if and only if it satisfies a certain axiom in the language of first-order logic. In the present paper we will prove that a similar result for a directed geodetic structure does not hold.

To prove this, we need to introduce some logical notions and to use a result of model theory. For further details and more explicit formulations, the reader is referred to [2], Chapter 0.

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By an atomic formula of the first-order logic of vocabulary $\{T\}$, where T is the ternary relation symbol, we mean an expression $x = y$, where x and y are variables, or an expression $T(x, y, z)$, where x, y, z are variables. By a formula of the first order logic of vocabulary $\{T\}$ (shortly: a formula) we mean an atomic formula of the first-order logic of vocabulary $\{T\}$, or an expression $\neg\mathbf{a}$, where $\mathbf{a}$ is a formula, or an expression $\mathbf{a}_1 \vee \mathbf{a}_2$, where $\mathbf{a}_1$ and $\mathbf{a}_2$ are formulae, or an expression $\exists x\mathbf{a}$, where x is a variable and $\mathbf{a}$ is a formula.

Following [2], we define the quantifier rank qr of a formula:

if $\mathbf{a}$ is an atomic formula, then $\mathrm{qr}(\mathbf{a}) = 0$;

if $\mathbf{a}$ is a formula, then $\mathrm{qr}(\neg\mathbf{a}) = \mathrm{qr}(\mathbf{a})$;

if $\mathbf{a}_1$ and $\mathbf{a}_2$ are formulae, then $\mathrm{qr}(\mathbf{a}_1 \vee \mathbf{a}_2) = \max(\mathrm{qr}(\mathbf{a}_1), \mathrm{qr}(\mathbf{a}_2))$;

if $\mathbf{a}$ is a formula and x is a variable, then $\mathrm{qr}(\exists x\mathbf{a}) = \mathrm{qr}(\mathbf{a}) + 1$.

By a sentence of the first-order logic of vocabulary $\{T\}$ (shortly: a sentence) we mean a formula $\mathbf{s}$ such that for every atomic subformula $\mathbf{a}$ of $\mathbf{s}$ it holds that every variable belonging to $\mathbf{a}$ is in the scope of the corresponding quantifier. (For further details and more explicit formulations, the reader is referred to [2], Chapter 0).

We will define a partial isomorphism from a ternary structure to a ternary structure as a special case of the partial isomorphism defined in [2], p. 15. Let (U_1, T_1) and (U_2, T_2) be ternary structures. By a partial isomorphism from (U_1, T_1) to (U_2, T_2) we mean an injective mapping α with the properties that $\mathrm{Def}(\alpha) \subseteq U_1$, $\mathrm{Im}(\alpha) \subseteq U_2$ and

$$T_1(u, v, w) \quad \text{if and only if} \quad T_2(\alpha(u), \alpha(v), \alpha(w))$$

for all $u, v, w \in \mathrm{Def}(\alpha)$.

Let (U_1, T_1) and (U_2, T_2) be ternary structures and let $n \geqslant 0$. We will write $(U_1, T_1) \cong_n (U_2, T_2)$ if there exist nonempty subsets $\mathbf{Q}_0, \ldots, \mathbf{Q}_n$ of the set of all partial isomorphisms from (U_1, T_1) to (U_2, Y_2) such that the following statements (I) and (II) hold:

(I) for every m, $0 < m \leqslant n$, $u \in U_1$ and $\alpha \in \mathbf{Q}_m$, there exists $\beta \in \mathbf{Q}_{m-1}$ with the properties that $\alpha \subseteq \beta$ and $u \in \mathrm{Def}(\beta)$.

(II) for every m, $0 < m \leqslant n$, $u \in U_2$ and $\alpha \in \mathbf{Q}_m$, there exists $\beta \in \mathbf{Q}_{m-1}$ with the properties that $\alpha \subseteq \beta$ and $u \in \mathrm{Im}(\beta)$.

The next theorem will be an important tool for us. Recall that by a sentence we mean a sentence of the first-order logic of vocabulary $\{T\}$.

Theorem 1. *Let (U_1, T_1) and (U_2, T_2) be ternary structures and let $n \geqslant 0$. Then (U_1, T_1) and (U_2, T_2) satisfy the same sentences $\mathbf{s}$ of $\mathrm{qr}(\mathbf{s}) \leqslant \mathrm{n}$ if and only if $(U_1, T_1) \cong_n (U_2, T_2)$.*

Theorem 1 is a special case of Fraïssé's Theorem. Its proof can be found in [2], Chapter 1, where also further important notions closely connected to this theorem appear.

Let (U_0, T_0) be a ternary structure. We denote by E_0 the set of all ordered pairs (u, v) of distinct elements of U_0 such that

$$\text{if} \quad T_0(u, w, v), \quad \text{then} \quad u = w \quad \text{or} \quad v = w \quad \text{for every } w \in U_0.$$

By the *underlying digraph* of (U_0, T_0) we mean the digraph D defined as follows: $V(D) = U_0$ and $E(D) = E_0$. It is clear that if (U_0, T_0) is a directed geodetic structure and D is its underlying digraph, then D is strong and $T_0 = \Gamma_D^{\mathrm{dir}}$.

We will construct a certain infinite sequence of ternary structures. We will need them for proving the main result of the present paper. If $e \geqslant 2$, then we denote $N_e = \{1, 2, \ldots, e\}$.

Let $g, h \geqslant 2$. We denote by $L_{g,h}$ the mapping of N_{g+h} into itself defined as follows:

$L_{g,h}(e) = h + e + 1$ for $1 \leqslant e < g$;

$L_{g,h}(g) = h$;

$L_{g,h}(g + 1) = h + 1$;

$L_{g,h}(g + f + 1) = f$ for $1 \leqslant f \leqslant h - 1$.

Clearly, $L_{g,h}$ is a bijection of N_{g+h} onto itself.

Let $g, h \geqslant 2$. We denote by $B_{g,h}$ the ternary relation on N_{g+h} defined as follows:

$$\begin{aligned}(1) \qquad & B_{g,h}(e_1, e_2, e_3) \quad \text{if and only if} \\ & (e_1 \leqslant e_2 \leqslant e_3) \quad \text{or} \quad (L_{g,h}(e_1) \leqslant L_{g,h}(e_2) \leqslant L_{g,h}(e_3)) \quad \text{or} \\ & (e_1 = g + 1,\ 1 \leqslant e_2 \leqslant g - 1,\ e_3 = g)\end{aligned}$$

for all $e_1, e_2, e_3 \in N_{g+h}$. We denote by $D_{g,h}$ the digraph defined as follows:

$$V(D_{g,h}) = N_{g+h}$$

and

$$E(D_{g,h}) = \{(1, 2), (2, 3), \ldots, (g + h - 1, g + h)\} \cup \{(g + 1, 1), (g + h, g)\}.$$

A diagram of $D_{5,7}$ is given in Fig. 1.

Clearly, $D_{g,h}$ is the underlying digraph of the ternary structure $(N_{g+h}, B_{g,h})$. We can see that $D_{g,h}$ is strong. As follows from (1), $B_{g,h} = \Gamma_{D_{g,h}}^{\mathrm{dir}}$ if and only if $g \leqslant h$.

Lemma 1. *Let $i, j \geqslant 2$. Then $(N_{i+j}, B_{i,j})$ is a directed geodetic structure if and only if $i \leqslant j$.*

P r o o f is obvious. □

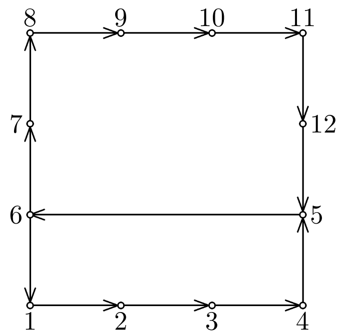

Fig. 1

Lemma 2. *Let $n \geqslant 1$, and let $i, j \geqslant 2^n$. Then*

$$(N_{i+j}, B_{i,j}) \cong_n (N_{i+j}, B_{j,i}).$$

P r o o f. Put $A_{[i]} = B_{i,j}$ and $A_{[j]} = B_{j,i}$. Let $k \in \{i, j\}$. We define $M_{[k],1} = \{1, 2, \dots, k+1\}$ and $M_{[k],2} = \{k, k+1, \dots, i+j\}$. We denote by $G_{[k],1}$ and $G_{[k],2}$ the graphs defined as follows: $V(G_{[k],1}) = M_{[k],1}$, $V(G_{[k],2}) = M_{[k],2}$,

$$E(G_{[k],1}) = \{\{1,2\}, \{2,3\}, \dots, \{k-1,k\}\} \cup \{\{k+1, 1\}\}$$

and

$$E(G_{[k],2}) = \{\{k+1, k+2\}, \{k+2, k+3\}, \dots, \{i+j-1, i+j\}\} \cup \{\{i+j, k\}\}.$$

Obviously, $G_{[k],1}$ and $G_{[k],2}$ are paths. We denote by $d_{[k],1}$ and $d_{[k],2}$ the distance function of $G_{[k],1}$ and $G_{[k],2}$, respectively.

We denote by $S_{[k]}$ the binary relation on N_{i+j} defined as follows:

$$S_{[k]}(e,f) \quad \text{if and only if there exists} \quad r \in \{1,2\} \text{ such that } e, f \in M_{[k],r}$$

for all $e, f \in N_{i+j}$. Obviously, $S_{[k]}(k, e)$ and $S_{[k]}(k+1, e)$ for all $e \in N_{i+j}$.

Moreover, for all distinct $e, f \in N_{i+j}$ such that $\{e, f\} \neq \{k, k+1\}$ we denote by $s_{[k]}(e,f)$ the unique $r \in \{1,2\}$ with the property that $\{e,f\} \subseteq M_{[k],r}$.

Let $m \in \{0, 1, \dots, n\}$. Put $d^m_{[k]}(e, e) = 0$ for every $e \in N_{i+j}$. For all distinct $f, g \in N_{i+j}$ such that $S_{[k]}(f, g)$ we define $d^m_{[k]}(f, g)$ as follows:

$$d^m_{[k]}(f,g) = \infty \quad \text{if} \quad \{f,g\} = \{k, k+1\},$$
$$d^m_{[k]}(f,g) = \infty \quad \text{if} \quad \{f,g\} \neq \{k, k+1\} \quad \text{and} \quad d_{[k],s_{[r]}(f,g)}(f,g) \geqslant 2^m$$

and

$$d^m_{[k]}(f,g) = d_{[k],s_{[k]}(f,g)}(f,g) \quad \text{if} \quad \{f,g\} \neq \{k, k+1\}$$
$$\text{and} \quad d_{[k],s_{[r]}(f,g)}(f,g) < 2^m.$$

Let $m \in \{1, \dots, n\}$. It is easy to see that

(2) $$\text{if} \quad d^m_{[i]}(e,f) = d^m_{[j]}(g,h), \quad \text{then} \quad d^{m-1}_{[i]}(e,f) = d^{m-1}_{[j]}(g,h)$$
$$\text{for all } e,f,g,h \in N_{i+j} \text{ such that } S_{[i]}(e,f) \text{ and } S_{[j]}(g,h).$$

We denote by $\mathbf{P}$ the set of all partial isomorphisms α of $(N_{i+j}, A_{[i]})$ into $(N_{i+j}, A_{[j]})$ such that $\{i, i+1\} \subseteq \mathrm{Def}(\alpha)$ and for all $e \in \mathrm{Def}(\alpha)$ it holds that

$$\alpha(e) \in M_{[j],1} \setminus \{j, j+1\} \quad \text{if} \quad e \in M_{[i],1} \setminus \{i, i+1\},$$
$$\alpha(e) \in M_{[j],2} \setminus \{j, j+1\} \quad \text{if} \quad e \in M_{[i],2} \setminus \{i, i+1\},$$
$$\alpha(e) = j \quad \text{if} \quad e = i$$

and

$$\alpha(e) = j+1 \quad \text{if} \quad e = i+1.$$

Define $\alpha_0 = \{(i,j), (i+1, j+1)\}$. Obviously, $\alpha_0 \in \mathbf{P}$.

Let $\alpha \in \mathbf{P}$, and let $e, f \in \mathrm{Def}(\alpha)$. It follows from the definition of $\mathbf{P}$ that if $S_{[i]}(e,f)$, then $S_{[j]}(\alpha(e), \alpha(f))$.

For every m, $0 \leqslant m \leqslant n$, we denote by $\mathbf{Q}_m$ the set of all $\alpha \in \mathbf{P}$ such that

(3) $$d^m_{[i]}(e,f) = d^m_{[j]}(\alpha(e), \alpha(f)) \quad \text{for all } e,f \in \mathrm{Def}(\alpha) \text{ such that } S_{[i]}(e,f).$$

Clearly, $\alpha_0 \in \mathbf{Q}_n$. It follows from (2) that

(4) $$\mathbf{Q}_n \subseteq \mathbf{Q}_{n-1} \subseteq \dots \subseteq \mathbf{Q}_0.$$

To finish the proof we need to show that the statements (I) and (II) hold for $\mathbf{Q}_0, \mathbf{Q}_1, \dots, \mathbf{Q}_n$.

Let $m \in \{1, \dots, n\}$, $\alpha \in \mathbf{Q}_m$, and let $g \in N_{i+j}$. First, assume that $g \in \mathrm{Def}(\alpha)$. By (4), $\alpha \in \mathbf{Q}_{m-1}$. We put $\beta = \alpha$. Now, assume that $g \notin \mathrm{Def}(\alpha)$. Then $g \notin \{i, i+1\}$. There exists exactly one $r \in \{1, 2\}$ such that $g \in V(G_{[i],r})$. It is clear that there exist $e, f \in V(G_{[i],r}) \cap \mathrm{Def}(\alpha)$ such that

e belongs to the $(i+1) - g$ path in $G_{[i],r}$,

f belongs to the $g - i$ path in $G_{[i],r}$,

g belongs to the $e - f$ path in $G_{[i],r}$

and no inner vertex of the $e - f$ path in $G_{[i],r}$ belongs to $\mathrm{Def}(\alpha)$. It is easy to see that $\alpha(e), \alpha(f) \in V(G_{[j],r})$ and $d^m_{[j],r}(\alpha(e), \alpha(f)) = d^m_{[i],r}(e, f)$. Let P denote the $\alpha(e) - \alpha(f)$ path in $G_{[j],r}$. Obviously, no inner vertex of P belongs to $\mathrm{Im}(\alpha)$.

We distinguish three cases.

Case 1. Assume that $d_{[i],r}(e, g) < 2^{m-1}$. Then there exists exactly one $h_1 \in V(G_{[j],r})$ such that h_1 belongs to P and $d_{[j],r}(\alpha(e), h_1) = d_{[i],r}(e, g)$; we put $h = h_1$.

Case 2. Assume that $d_{[i],r}(e, g) \geqslant 2^{m-1}$ and $d_{[i],r}(g, f) < 2^{m-1}$. Then there exists exactly one $h_2 \in V(G_{[j],r})$ with the properties that h_2 belongs to P and $d_{[j],r}(h_2, \alpha(f)) = d_{[i],r}(g, f)$; we put $h = h_2$.

Case 3. Assume that $d_{[i],r}(e, g) \geqslant 2^{m-1}$ and $d_{[i],r}(g, f) \geqslant 2^{m-1}$. Then $d_{[i],r}(e, f) \geqslant 2^m$. There exists exactly one $h_3 \in V(G_{[j],r})$ with the properties that h_3 belongs to the P and $d_{[i],r}(\alpha(e), h_3) = 2^{m-1}$; we put $h = h_3$.

Now put $\beta = \alpha \cup \{(g, h)\}$. Since (2) holds, it is easy to see that $\beta \in \mathbf{Q}_{m-1}$. Thus (I) holds. The fact that (II) also holds can be proved similarly.

Hence $(N_{i+j}, B_{i,j}) \cong_n (N_{i+j}, B_{j,i})$, which completes the proof. □

The next theorem gives the main result of the present paper:

Theorem 2. *There exists no sentence* s *of the first-order logic of vocabulary* $\{T\}$ *such that a connected ternary structure is a directed geodetic structure if and only if it satisfies* s.

P r o o f. Combining Lemmas 1 and 2 with Theorem 1, we get the result. □

Remark 1. Theorem 1 was used by the present author for proving another, very different, result on ternary structures in [7].

Remark 2. The idea of functions $d^m_{[k]}$ in the proof of Lemma 2 was inspired by one of the ideas in Example 1.3.5 of [2].

Remark 3. A preliminary version of the main result of this paper was presented by the author on Slovak and Czech conference GRAPHS 2000 held at Liptovský Trnovec (Slovakia), May 15–19, 2000 (organized by School of Finance of Matej Belo University, Banská Bystrica, and other institutions).

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Author's address: Univerzita Karlova v Praze, Filozofická fakulta, nám. J. Palacha 2, 116 38 Praha 1, ČR, e-mail: `Ladislav.Nebesky@ff.cuni.cz`.