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BANACH AND STATISTICAL CORES OF BOUNDED SEQUENCES

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Abstract. In this paper, we are mainly concerned with characterizing matrices that map every bounded sequence into one whose Banach core is a subset of the statistical core of the original sequence.

Keywords: almost convergent sequence, statistically convergent sequence, core of a sequence

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1. Introduction

If \( T = (t_{nk}) \) is an infinite matrix with real entries, and if \( x = (x_k) \) is a sequence of real numbers, then \( Tx \) denotes the transformed sequence whose \( n \)-th term is given by

\[
(Tx)_n = \sum_{k=1}^{\infty} t_{nk}x_k.
\]

In order to investigate the effect of such transformations upon the derived set, Knopp [14] introduced the idea of the core (\( K \)-core) of a sequence and proved the well-known Core Theorem. That theorem asserts that \( K \)-core\( \{Tx\} \subseteq K \)-core\( \{x\} \), whenever \( Tx \) exists for the nonnegative regular matrix \( T \). Some variants of the Core Theorem may be found in [4], [19], [23], [26].

Considering the method of almost convergence Loone [17] and Das [4] introduced the Banach core (\( B \)-core) of a bounded sequence and proved some analogues of the assertions for the \( K \)-core (see also [12], [23], [26], [27]).

In [10], [11], the notion of statistical core of a sequence is introduced and a statistical core theorem is proved.

Section 2 of the present paper presents a result which is complementary to [17] and [23], while Section 3 deals with characterizing matrices that map every bounded sequence into one whose \( B \)-core is a subset of the statistical core of the original
sequence. Before proceeding further we recall some notation and terminology. By $l^\infty$ and $c$ we denote the spaces of all bounded and convergent real sequences, respectively.

Let $T = (t_{nk})$ be an infinite matrix, and let $X$ and $Y$ be two sequence spaces. If $Tx$ exists for each $x \in X$ and $Tx \in Y$ then we say that $T$ maps $X$ into $Y$. The set of matrices that map $X$ into $Y$ is denoted by $(X, Y)$. The set of matrices that map $X$ into $Y$ and leave the limit or sum invariant is denoted by $(X, Y; p)$.

For example, if $T \in (c, c; p)$, then $\lim Tx = \lim x$ for every $x \in c$. In this case $T$ is called regular (see [3], [24]). If it is regular and satisfies $\lim_n \sum_k |t_{nk} - t_{n,k+1}| = 0$, then $T$ is called strongly regular [24].

2. $B$-core and absolute equivalence

This section is complementary to [23] and [17]. It is well-known [18], [24] that the functional

$$q(x) = \inf_{n_1, n_2, \ldots, n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}$$

is sublinear on $l^\infty$. We also consider the following functionals on $l^\infty$:

$$L(x) = \limsup x_n,$$

$$l^*(x) = \liminf_n \frac{1}{n+1} \sum_{r=i}^{i+n} x_r,$$

$$L^*(x) = \limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} x_r.$$

It follows from the Corollary to Theorem 1 of [5] that $q(x) = L^*(x)$. If $q(x) = -q(-x) = s$, then $x$ is called almost convergent to $s$ [18], and in this case we write $F$-lim $x = s$. By $F$ we denote the set of all almost convergent sequences.

The Banach core ($B$-core) of a bounded sequence $x$ is defined to be the closed interval $[-q(-x), q(x)]$ (see Loone [17], Das [4]). Since $q(x) \leq L(x)$ for every $x \in l^\infty$, it follows that $B$-core$\{x\} \subseteq K$-core$\{x\}$ where $K$-core$\{x\}$ is the Knopp core and it is given by $K$-core$\{x\} = [\liminf x, \limsup x]$. It is shown in [23], [17] that

$$K$-core$\{Ax\} \subseteq B$-core$\{x\}$ (for every $x \in l^\infty$$)

if and only if $A$ is strongly regular and $\lim_n \sum_k |a_{nk}| = 1$.

Now we have the following
**Theorem 1.** Let $x \in l^\infty$ and let $A$ be a strongly regular matrix. Then
\[ K\text{-core}\{Ax\} \subseteq B\text{-core}\{x\} \] if and only if $A$ is absolutely equivalent to a non-negative strongly regular matrix $B$ for all bounded sequences.

**Proof.** Sufficiency. Since $A$ is absolutely equivalent to a nonnegative strongly regular matrix $B$, we have

(1) \[ \lim_n \{(Ax)_n - (Bx)_n\} = 0 \quad \text{(for every } x \in l^\infty). \]

Now Theorem 6.5.I of Cooke [3] implies that

(2) \[ K\text{-core}\{Ax\} \subseteq K\text{-core}\{x\}, \quad \text{(for every } x \in l^\infty). \]

Since $B$ is a non-negative strongly regular matrix, it follows from Theorem 3 of [23] that, for every $x \in l^\infty$,

(3) \[ K\text{-core}\{Bx\} \subseteq B\text{-core}\{x\}. \]

Since (1) holds, Theorem 6.3.II of Cooke [3] implies that

(4) \[ K\text{-core}\{Ax\} = K\text{-core}\{Bx\}. \]

Now (3) and (4) imply $K\text{-core}\{Ax\} \subseteq B\text{-core}\{x\}$.

Necessity. Let $x \in l^\infty$ and let $A$ be a strongly regular matrix. By hypothesis,

(5) \[ K\text{-core}\{Ax\} \subseteq B\text{-core}\{x\} \subseteq K\text{-core}\{x\}. \]

Now, there is a non-negative regular matrix $B$ such that $A$ and $B$ are absolutely equivalent on $l^\infty$ (see Theorem 6.5.I of [3]). So, by Theorem 5.4.I of Cooke [3], we have

(6) \[ \lim_n \sum_k |b_{nk} - a_{nk}| = 0. \]

It remains to show that

(7) \[ \lim_n \sum_k |b_{nk} - b_{n,k+1}| = 0. \]

To see this, we first write

\[ \sum_k |b_{nk} - b_{n,k+1}| \leq \sum_k |b_{nk} - a_{nk}| + \sum_k |a_{n,k+1} - b_{n,k+1}| + \sum_k |a_{nk} - a_{n,k+1}| \]

\[ = c^1_n + c^2_n + c^3_n, \text{ say.} \]
By (6), \( c_1^n \to 0 \ (n \to \infty) \). By the strong regularity of \( A \), \( c_3^n \to 0 \ (n \to \infty) \), and by the absolute equivalence

\[
c_2^n = \sum_k |a_{n,k+1} - b_{n,k+1}| \leq \sum_k |a_{nk} - b_{nk}| \to 0 \quad (n \to \infty),
\]

hence (7) holds. This proves the theorem.

\[\square\]

3. Statistical and Banach cores

If \( K \subseteq \mathbb{N} \) then let \( K_n := \{ k \in K : k \leq n \} \); and \( |K_n| \) will denote the cardinality of \( K_n \). The natural density \([22]\) of \( K \) is given by \( \delta(K) := \lim \frac{n}{n}^{-1}|K_n| \), if it exists.

In [9] a statistical cluster point of a sequence \( x \) is defined as a number \( \gamma \) such that for every \( \varepsilon > 0 \) the set \( \{ k \in \mathbb{N} : |x_k - \gamma| < \varepsilon \} \) does not have density zero. In [10] the sequence \( x \) is defined to be statistically bounded if \( x \) has a bounded subsequence of density one; and the statistical core of such an \( x \) of real values is the closed interval \( \text{st-lim inf } x, \text{st-lim sup } x \), where \( \text{st-lim inf } x \) and \( \text{st-lim sup } x \) are the least and greatest statistical cluster points of \( x \) (see [6], [10], [11], [16]). Recall [10] that, for a sequence \( x \) the number \( \beta \) is the \( \text{st-lim sup } x \) if and only if for every \( \varepsilon > 0 \),

\[
\delta\{k : x_k > \beta - \varepsilon\} \neq 0 \quad \text{and} \quad \delta\{k : x_k > \beta + \varepsilon\} = 0.
\]

The dual statement for \( \text{st-lim inf } x \) is as follows: The number \( \alpha \) is the \( \text{st-lim inf } x \) if and only if for every \( \varepsilon > 0 \),

\[
\delta\{k : x_k < \alpha + \varepsilon\} \neq 0 \quad \text{and} \quad \delta\{k : x_k < \alpha - \varepsilon\} = 0.
\]

A statistically bounded sequence \( x \) is statistically convergent if and only if \( \text{st-lim sup } x = \text{st-lim inf } x \) [10]. Some results on statistical convergence may be found in [2], [8], [9], [10], [20], [21], [25].

In this section we are mainly concerned with characterizing matrices that map every bounded sequence into one whose \( B \)-core is a subset of the statistical core of the original sequence. The final result follows a result of Choudhary [1] in giving conditions on matrices \( T \) and \( H \) so that the Banach core of \( Tx \) is contained in the statistical core of \( Hx \).

We note that statistical convergence and almost convergence are incomparable [21].

By \( \text{st}(b) \) we denote the set of all bounded statistically convergent sequences. It follows from Theorem 4.1 of [15] that \( T \in (\text{st}(b), F; p) \) if and only if \( T \in (c, F; p) \) and \( T^{[K]} \in (l^\infty, F) \) for every \( K \) of density zero where \( T^{[K]} = (d_{nk}) \) is given by \( d_{nk} = t_{nk} \) if \( k \in K \) and \( d_{nk} = 0 \) otherwise.

By [13] and [7], this is equivalent to the following
Proposition 2. \( T \in (\text{st}(b), F; p) \) if and only if

(i) \( \sup_n \sum_k |t_{nk}| < \infty \),
(ii) \( F\text{-}\lim t_{nk} = 0 \) for every \( k \),
(iii) \( F\text{-}\lim n \sum_k t_{nk} = 1 \), and
(iv) \( \lim_r \frac{1}{r+1} \sum_{i=1}^r \sum_{k \in K} |t_{n+i,k}| = 0 \), uniformly in \( n \) for every \( K \) of density zero.

Now we have

Theorem 3. Let \( T: l^\infty \to l^\infty \) and \( \beta(x) := \text{st}\text{-}\lim sup x \). Then

(8) \( L^*(Tx) \leq \beta(x) \) (for every \( x \in l^\infty \)),

if and only if

(a) \( T \in (\text{st}(b), F; p) \),
(b) \( \lim \frac{1}{r+1} \sum_{i=1}^r \sum_{k \in K} |t_{n+i,k}| = 1 \), uniformly in \( n \).

Proof. Assume (8) holds and \( x \in l^\infty \). Then \( Tx \in l^\infty \); and also we have

\[ \beta(-x) \leq -L^*(-Tx) \leq L^*(Tx) \leq \beta(x). \]

If \( x \in \text{st}(b) \), then \( \beta(x) = -\beta(-x) \), hence \( T \) maps \( \text{st}(b) \) into \( F \) and \( F\text{-}\lim Tx = \text{st}\text{-}\lim x \), which proves (a). To prove (b), we first observe that Proposition 2 implies the conditions of Lemma 2 of [4]. Hence, from that Lemma, there is a bounded sequence \( x \) such that \( \|x\|_\infty := \sup_k |x_k| \leq 1 \) and

(9) \( \lim sup_n \sum_k b_{nk}(i)x_k = \lim sup_n \sum_i \sum_k |b_{nk}(i)| \)

where \( b_{nk}(i) = \frac{1}{n+1} \sum_{r=i}^{i+n} t_{rk} \).

Hence, by Proposition 2,

\[ 1 = \lim inf_n \sum_k b_{nk}(i) \leq \lim inf_n \sum_i \sum_k |b_{nk}(i)| \]
\[ \leq \lim sup_n \sum_i \sum_k |b_{nk}(i)| \]
\[ = \lim sup_n \sum_i \sum_k b_{nk}(i)x_k, \text{ by } (9) \]
\[ \leq \beta(x), \text{ by hypothesis} \]
\[ \leq \|x\|_\infty \leq 1 \]

from which we get (b).
Conversely assume (a) and (b) hold, and let $x \in l^\infty$. Then $Tx \in l^\infty$ and $\beta(x)$ is finite. Given $\varepsilon > 0$, let $E := \{k: x_k > \beta(x) + \varepsilon\}$. Hence $\delta(E) = 0$, and if $k \notin E$ then $x_k \leq \beta(x) + \varepsilon$. For any real number $z$ we write $z^+ := \max\{z, 0\}$ and $z^- := \max\{-z, 0\}$, whence

$$|z| = z^+ + z^-, \quad z = z^+ - z^-, \quad |z| - z = 2z^-.$$ 

Letting

$$b_{rk}(i) := \frac{1}{r + 1} \sum_{n=i}^{i+r} t_{nk},$$

$$(b_{rk}(i))^+ := \frac{1}{r + 1} \sum_{n=i}^{i+r} t_{nk}^+,$$

$$(b_{rk}(i))^- := \frac{1}{r + 1} \sum_{n=i}^{i+r} t_{nk}^-,$$

then for a fixed positive integer $m$ we write

$$\frac{1}{r + 1} \sum_{n=i}^{i+r} (Tx)_n = \sum_{k<m} b_{rk}(i)x_k + \sum_{k\geq m} (b_{rk}(i))^+ x_k$$

$$+ \sum_{k\geq m} (b_{rk}(i))^- x_k - \sum_{k\geq m} (b_{rk}(i))^- x_k$$

$$\leq \|x\|_{\infty} \sum_{k<m} |b_{rk}(i)| + (\beta(x) + \varepsilon) \sum_{k\geq m} |b_{rk}(i)|$$

$$+ \|x\|_{\infty} \sum_{k\geq m} |b_{rk}(i)| + \|x\|_{\infty} \sum_{k\geq m} (|b_{rk}(i)| - b_{rk}(i)).$$

On applying the operator $\limsup_i \sup_r$ and considering Proposition 2, we get

$$L^*(Tx) \leq \beta(x) + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary we conclude that (8) holds, whence the result. \qed

Similarly we could get $\alpha(x) \leq l^*(Tx)$, and hence we have the following result.

**Theorem 4.** If $T: l^\infty \to l^\infty$, then

$$\mathcal{B}\text{-core}\{Tx\} \subseteq \text{st-core}\{x\} \quad \text{for every } x \in l^\infty$$

if and only if conditions (a) and (b) of Theorem 3 hold.
Theorem 5. Let $H$ be a triangular matrix with non-zero diagonal entries, and denote its triangular inverse by $H^{-1}$. For an arbitrary matrix $T$, in order that, whenever $Hx \in l^\infty$, $Tx$ should exist and be bounded and satisfy

$$
(10) \quad \mathcal{B}\text{-core}\{Tx\} \subseteq \text{st-core}\{Hx\},
$$

it is necessary and sufficient that

(i) $C := TH^{-1}$ exists;

(ii) $C \in (\text{st}(b), f; p)$;

(iii) $\lim_{r \to \infty} \sum_{k=1}^{\infty} \left| \frac{1}{r+1} \sum_{i=1}^{r} c_{n+i,k} \right| = 1$;

(iv) for any fixed $n$,

$$
\lim_{\nu \to \infty} \sum_{k=0}^{\nu} \left| \sum_{j=\nu+1}^{\infty} t_{nj} h_{jk}^{-1} \right| = 0.
$$

Proof. Necessity. If $(Tx)_n$ exists for each $n$ whenever $Hx \in l^\infty$, then by Lemma 2 of Choudhary [1], (i) and (iv) hold. By the same Lemma, we also have $Tx = Cy$ where $y = Hx$. By hypothesis $Tx \in l^\infty$ hence $Cy \in l^\infty$. Now (10) implies that $\mathcal{B}\text{-core}\{Cy\} \subseteq \text{st-core}\{y\}$. By Theorem 4, we get (ii) and (iii).

Sufficiency. Conditions (i)–(iv) imply the conditions of Lemma 2 of Choudhary [1]; so, it follows from that Lemma that $Cy \in l^\infty$, and hence $Tx \in l^\infty$. Now Theorem 4 yields that $\mathcal{B}\text{-core}\{Cy\} \subseteq \text{st-core}\{y\}$, and since $y = Hx$ and $Cy = Tx$, we have $\mathcal{B}\text{-core}\{Tx\} \subseteq \text{st-core}\{Hx\}$, whence the result. \qed

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