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RELATIONS BETWEEN SOME DIMENSIONS OF SEMIMODULAR LATTICES

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Abstract. The aim of this paper is to present relations between Goldie, hollow and Kurosh-Ore dimensions of semimodular lattices. Relations between Goldie and Kurosh-Ore dimensions of modular lattices were studied by Grzeszczuk, Okiński and Puczyłowski.

Keywords: semimodular lattice, Goldie dimension, hollow dimension, Kurosh-Ore dimension

MSC 2000: 06C10

1. Preliminaries

Let L be a lattice of finite length. We will denote by L^* the dual of L. For elements $a, b \in L$ $(a \leqslant b)$ we define the *interval* $[a, b]$ to be the set of all $c \in L$ such that $a \leqslant c \leqslant b$. We say that b *covers* a if $a < b$ and $[a, b] = \{a, b\}$; in this case we write $a \prec b$. If $p \in L$ covers 0, then p is an *atom* of L. Let $A(L)$ be the set of all atoms of L. Define a lattice L to be *upper semimodular* (briefly: *semimodular*) if it satisfies the following condition:

$$a \wedge b \prec a \ \text{ implies } \ b \prec a \vee b.$$

L is *lower semimodular* if its dual lattice is semimodular.

Let $T \subseteq L - \{0\}$. T is called *join independent* if for every finite subset $S \subseteq T$ and each element $t \in T - S$, $t \wedge \bigvee S = 0$. The *Goldie dimension* $d_{\mathrm{G}}(L)$ of L is defined (see [1]) as

$$d_{\mathrm{G}}(L) = \max\{|T| \colon T \text{ is a join independent subset of } L\}.$$

The Goldie dimension of the lattice L^* is called the *hollow dimension* and denoted by $d_{\mathrm{H}}(L)$ (see [2]). We have $d_{\mathrm{H}}(L) = d_{\mathrm{G}}(L^*)$.

An element $m \in L - \{1\}$ is *meet irreducible* if $m = x \wedge y$ implies that $m = x$ or $m = y$. Dually, an element $u \in L - \{0\}$ is *join irreducible* if $u = x \vee y$ implies that $u = x$ or $u = y$. By $M(L)$ (resp. $J(L)$) we denote the set of all meet irreducible (resp. join irreducible) elements of the lattice L. A subset T of L is said to be *meet irredundant* (resp. *join irredundant*) if for each element $t \in T$, $\bigwedge(T - \{t\}) \nleqslant t$ (resp. $t \nleqslant \bigvee(T - \{t\})$).

If $a = x_1 \wedge x_2 \wedge \ldots \wedge x_m$ for $a \in L$ and $x_1, x_2, \ldots, x_m \in M(L)$, then we say that $x_1 \wedge x_2 \wedge \ldots \wedge x_m$ is a $\wedge$-*decomposition* of a. A $\wedge$-decomposition $x_1 \wedge x_2 \wedge \ldots \wedge x_m$ of a is called *irredundant* if the set $\{x_1, x_2, \ldots, x_m\}$ is meet irredundant. Dually, if $a = x_1 \vee x_2 \vee \ldots \vee x_m$ and $x_1, x_2, \ldots, x_m \in J(L)$, then we say that $x_1 \vee x_2 \vee \ldots \vee x_m$ is a $\vee$-*decomposition* of a. This $\vee$-decomposition of a is *irredundant* if the set $\{x_1, x_2, \ldots, x_m\}$ is join irredundant.

The following classical result is referred to as the Kurosh-Ore Theorem:

Theorem. *If L is a modular lattice and if $a = x_1 \wedge x_2 \wedge \ldots \wedge x_m = y_1 \wedge y_2 \wedge \ldots \wedge y_n$ are two irredundant $\wedge$-decomposition of $a \in L$, then $m = n$. Dually, the number of join irreducible elements in any irredundant finite $\vee$-decomposition of a is unique.*

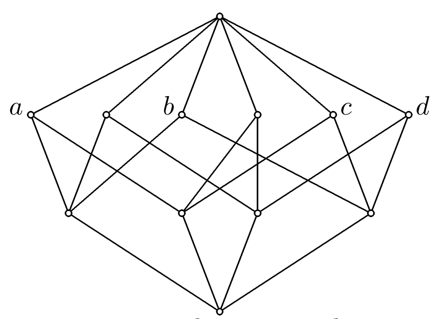

Figure 1.

The lattice of Fig. 1 shows that for semimodular lattices, the Kurosh-Ore Theorem does not hold.

We say that the *Kurosh-Ore dimension* (for $\wedge$-decompositions) of L equals n, and write $d_\wedge(L) = n$ if there exists a meet irredundant subset $\{a_1, \ldots, a_n\}$ of $M(L)$ such that $0 = a_1 \wedge \ldots \wedge a_n$ and for every irredundant $\wedge$-decomposition $0 = \bigwedge T$ of 0, $|T| \leqslant n$. By dualizing we get the concept of Kurosh-Ore dimension for $\vee$-decompositions. We have $d_\vee(L) = n$ if and only if $d_\wedge(L^*) = n$. Obviously,

$$d_\wedge(L) = 1 \Leftrightarrow 0 \in M(L) \text{ and } d_\vee(L) = 1 \Leftrightarrow 1 \in J(L).$$

2. Results

Let L be a semimodular lattice of finite length and let $x \in L$ The height of $[0,x]$ will be denoted by $h(x)$ and called the *height* of x ($h(x) = |C| - 1$, where C is a maximal chain in $[0,x]$). Write $h(L) = h(1)$. It is easy to see that the following three lemmas hold.

Lemma 1. *Let L be a semimodular lattice of finite length. If $\{b_1, \dots, b_n\}$ is a join irredundant subset of L, then $h(b_1 \vee \dots \vee b_n) \geqslant n$.*

Lemma 2. *Let L be a lattice of finite length. If $d_{\mathrm{G}}(L) = n$, then there exists a join independent set of n atoms of L.*

Lemma 3 ([4], Theorem 1.9.3). *If L is a semimodular lattice and 1 is a join of a finite join independent set, containing, say, n atoms, then $h(L) = n$.*

Theorem 1. *If L is a semimodular lattice of finite length, then $d_\wedge(L) = d_{\mathrm{G}}(L)$.*

Proof. Let $d_\wedge(L) = n$ and let $0 = a_1 \wedge a_2 \wedge \dots \wedge a_n$ be an irredundant $\wedge$-decomposition of 0. Set $b_i = \bigwedge\{a_j : j \neq i\}$ for $i \in I = \{1, 2, \dots, n\}$. Since the set $\{a_1, a_2, \dots, a_n\}$ is meet irredundant, we conclude that $\{b_1, b_2, \dots, b_n\} \subseteq L - \{0\}$. Observe that

$$b_i \wedge \bigvee\{b_j : j \neq i\} = 0$$

for each $i \in I$. Indeed,

$$b_i \wedge \bigvee\{b_j : j \neq i\} \leqslant b_i \wedge a_i = a_1 \wedge a_2 \wedge \dots \wedge a_n = 0.$$

Therefore, $\{b_1, b_2, \dots, b_n\}$ is a join independent subset of L. Hence $d_{\mathrm{G}}(L) \geqslant n$.

Suppose that $d_{\mathrm{G}}(L) > n$. By Lemma 2, there is a join independent set $\{p_1, p_2, \dots, p_n\} \subseteq A(L)$ with $k > n$. For $1 \leqslant i \leqslant k$, we put $c_i = \bigvee\{p_j : j \neq i\}$. We prove that

$$c_1 \wedge c_2 \wedge \dots \wedge c_k = 0. \tag{1}$$

Assume that $c_1 \wedge c_2 \wedge \dots \wedge c_k > 0$, and let q be an atom of L such that $q \leqslant c_1 \wedge c_2 \wedge \dots \wedge c_k$. Obviously,

$$q \nleqslant p_2 \text{ and } q \leqslant c_1 = p_2 \vee p_3 \vee \dots \vee p_k.$$

Therefore,

$$q \leqslant p_2 \vee p_3 \vee \dots \vee p_{i+1} \text{ and } q \nleqslant p_2 \vee p_3 \vee \dots \vee p_i$$

for some $2 \leqslant i < k$. We have $p_{i+1} \wedge (p_2 \vee p_3 \vee \ldots \vee p_i) = 0 \prec p_{i+1}$ and hence, by semimodularity,

$$p_2 \vee \ldots \vee p_i \prec p_2 \vee \ldots \vee p_i \vee p_{i+1}.$$

Consequently, $q \vee p_2 \vee \ldots \vee p_i = p_2 \vee \ldots \vee p_i \vee p_{i+1}$. Then $p_{i+1} \leqslant q \vee p_2 \vee \ldots \vee p_i \leqslant c_{i+1}$, a contradiction. Thus (1) holds.

Let $1 \leqslant j \leqslant k$. It follows that

$$c_1 \wedge \ldots \wedge c_{j-1} \wedge c_{j+1} \wedge \ldots \wedge c_k \nleqslant c_j,$$

since otherwise $p_j \leqslant c_j$, contradicting our assumption that $\{p_1, p_2, \ldots, p_k\}$ is a join independent subset of L. Therefore, the set $\{c_1, c_2, \ldots, c_k\}$ is meet irredundant. Take a $\wedge$-decomposition $c_i = \bigwedge T_i$ of c_i. For $1 \leqslant i \leqslant k$, let T_i' be a subset of T_i such that $T = T_1' \cup T_2' \cup \ldots \cup T_k'$ is a meet irredundant set and $0 = \bigwedge T$. Since the set $\{c_1, c_2, \ldots, c_k\}$ is meet irredundant, we conclude that $|T| > k > n$. Thus $d_\wedge(L) > n$, a contradiction. From this we see that $d_{\mathrm{G}}(L) = n$. □

Theorem 2. *Let L be a semimodular lattice of finite length. Then the following conditions are equivalent:*

(i) 1 *is a join of atoms.*

(ii) $d_{\mathrm{G}}(L) = d_\vee(L) = h(L)$.

Proof. (i) $\Rightarrow$ (ii). Let 1 be a join of a finite join independent set, containing, say, n atoms. Then $d_{\mathrm{G}}(L) \geqslant n = h(L)$ (see Lemma 3). Let $d_{\mathrm{G}}(L) = k$. By Lemma 2, there exists a join independent set $\{p_1, p_2, \ldots, p_k\}$ of k atoms of L. From (i) it follows that there are atoms $q_1, q_2, \ldots, q_m$ such that

$$1 = p_1 \vee p_2 \vee \ldots \vee p_k \vee q_1 \vee q_2 \vee \ldots \vee q_m$$

and the set $\{p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_m\}$ is join irredundant. By the definition of $d_\vee(L)$, $d_\vee(L) \geqslant m + k \geqslant k$, i.e., $d_{\mathrm{G}}(L) \leqslant d_\vee(L)$. From Lemma 1 we conclude that $d_\vee(L) \leqslant h(L)$. Thus we have (ii).

(ii) $\Rightarrow$ (i). Let $d_{\mathrm{G}}(L) = d_\vee(L) = h(L) = n$. By Lemma 2, there exists a join independent set $\{a_1, a_2, \ldots, a_n\}$ of n atoms of L. It follows that

$$1 = a_1 \vee a_2 \vee \ldots \vee a_n,$$

since otherwise $h(L) > h(a_1 \vee a_2 \vee \ldots \vee a_n) \geqslant n$, contradicting our assumption that $h(L) = n$. □

An immediate consequence of Theorems 1 and 2 is

Corollary 1. *Let L be a semimodular lattice of finite length. If 1 is a join of atoms, then $d_\wedge(L) = d_\vee(L) = d_{\mathrm{G}}(L) = h(L)$.*

Recall that a lattice L is *atomistic* if every element of L is a join of atoms (note that 0 is the join of the empty set of atoms). A *geometric* lattice is a finite semimodular atomistic lattice.

From Corollary 1 we have

Corollary 2. *If L is a geometric lattice, then $d_\wedge(L) = d_\vee(L) = d_{\mathrm{G}}(L) = h(L)$.*

The dual of Theorem 1 yields

Corollary 3. *If L is a lower semimodular lattice of finite length, then $d_\vee(L) = d_{\mathrm{H}}(L)$.*

Combining Corollary 1 and Corollary 2 we get

Corollary 4. *Let L be an atomistic modular lattice of finite length. Then $d_\wedge(L) = d_\vee(L) = d_{\mathrm{G}}(L) = d_{\mathrm{H}}(L) = h(L)$.*

In particular, we have

Corollary 5. *If L is a modular geometric lattice, then $d_\wedge(L) = d_\vee(L) = d_{\mathrm{G}}(L) = d_{\mathrm{H}}(L) = h(L)$.*

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