

Zhiguo Luo; Jian Hua Shen

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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 1, 79–93

Persistent URL: <http://dml.cz/dmlcz/127866>

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OSCILLATION AND NONOSCILLATION OF NEUTRAL
DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE
COEFFICIENTS

ZHIGUO LUO and JIANHUA SHEN, Changsha

(Received April 24, 2001)

Abstract. In this paper, oscillation and nonoscillation criteria are established for neutral differential equations with positive and negative coefficients. Our criteria improve and extend many results known in the literature.

Keywords: oscillation, neutral differential equations, positive and negative coefficients

MSC 2000: 34K15, 34K40, 34C10

1. INTRODUCTION

Consider the neutral delay differential equation with positive and negative coefficients

$$(1.1) \quad [x(t) - R(t)x(t-r)]' + \sum_{i=1}^m P_i(t)x(t-\tau_i) - \sum_{j=1}^n Q_j(t)x(t-\sigma_j) = 0, \quad t \geq t_0,$$

where $P_i, Q_j, R \in C([t_0, \infty), \mathbb{R}^+)$, $r \in (0, \infty)$ and $\tau_i, \sigma_j \in \mathbb{R}^+$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

When $m = n = 1$, Eq. (1.1) reduces to

$$(1.2) \quad [x(t) - R(t)x(t-r)]' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0, \quad t \geq t_0,$$

where $P, Q, R \in C([t_0, \infty), \mathbb{R}^+)$, $r \in (0, \infty)$ and $\tau, \sigma \in \mathbb{R}^+$. In recent years, the oscillation of Eq. (1.2) has been investigated by many authors. See, for example, [2],

This work was supported by the National Nature Science Foundation of P. R. of China (10071018).

[4]–[6], [9], [10], [12], [16] and the references cited therein. However, to the best of our knowledge, there is little in the way of results for the oscillation and nonoscillation of neutral differential equations with positive and negative coefficients with more than one delay.

Our aim in this paper is to establish oscillation and also nonoscillation criteria for Eq. (1.1). Our results improve and extend many results known in the literature.

The following assumptions will be used throughout the paper without further notice.

(A₁) There exist a positive integer number $p \leq m$ and a partition of the set $\{1, 2, \dots, n\}$ into p disjoint subsets J_1, J_2, \dots, J_p such that $j \in J_i$ implies that $\sigma_j \leq \tau_i$;

(A₂) $H_i(t) := P_i(t) - \sum_{k \in J_i} Q_k(t - \tau_i + \sigma_k) \geq 0$ ($\neq 0$) for $i = 1, 2, \dots, p$, $H_i(t) := P_i(t)$ for $i = p + 1, \dots, m$;

(A₃) $\varrho = \max\{r, \tau_i, \sigma_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\delta = \min\{r, \tau_i, \sigma_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

A function $x(t) \in C([t_1 - \varrho, \infty), R)$ is said to be a solution of Equation (1.1) for some $t_1 \geq t_0$ if $x(t) - R(t)x(t - r)$ is continuously differentiable on $[t_1, \infty)$ and satisfies (1.1) for $t > t_1$.

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

For convenience, we will assume that all inequalities concerning the values of functions are satisfied eventually for all large t .

2. LEMMAS

We need the following lemmas for the proofs of our main results.

Lemma 2.1. *Assume that*

$$(2.1) \quad R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s) ds \leq 1.$$

Let $x(t)$ be an eventually positive solution of the differential inequality

$$(2.2) \quad [x(t) - R(t)x(t - r)]' + \sum_{i=1}^m P_i(t)x(t - \tau_i) - \sum_{j=1}^n Q_j(t)x(t - \sigma_j) \leq 0$$

and set

$$(2.3) \quad z(t) = x(t) - R(t)x(t - r) - \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s - \sigma_k) ds.$$

Then

$$(2.4) \quad z'(t) \leq 0, \quad z(t) > 0, \quad \text{and} \quad z'(t) + \sum_{i=1}^m H_i(t)z(t - \tau_i) \leq 0.$$

Proof. Assume that $t_1 \geq t_0 + \varrho$ is such that $x(t)$ is positive for $t \geq t_1$. Then by (2.2) and (2.3), we get

$$(2.5) \quad \begin{aligned} z'(t) = & - \sum_{i=1}^p P_i(t)x(t - \tau_i) + \sum_{i=1}^p \sum_{k \in J_i} Q_k(t - \tau_i + \sigma_k)x(t - \tau_i) \\ & - \sum_{i=p+1}^m P_i(t)x(t - \tau_i). \end{aligned}$$

In view of $x(t) \geq z(t)$, (2.5) yields

$$z'(t) + \sum_{i=1}^m H_i(t)z(t - \tau_i) \leq 0.$$

Now we prove $z(t) > 0$. For otherwise, there would exist a $t_2 \geq t_1$ such that $z(t_2) \leq 0$. Then eventually $z(t) < 0$ because $z'(t) \leq 0$ and so there exist $t_3 \geq t_2$ and $\mu > 0$ such that $z(t) \leq -\mu$ for $t \geq t_3$. Hence

$$\begin{aligned} x(t) & \leq -\mu + R(t)x(t - r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^t Q_k(s)x(s - \sigma_k) ds \\ & \leq -\mu + \left(R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^t Q_k(s) ds \right) \max_{t - \varrho \leq s \leq t} x(s) \\ & \leq -\mu + \max_{t - \varrho \leq s \leq t} x(s). \end{aligned}$$

Lemma 1.5.4 in [6] implies that $x(t)$ cannot be a nonnegative function on $[t_3, \infty)$, thus contradicting $x(t) > 0$. The proof is complete. \square

Lemma 2.2. Assume that

$$(2.6) \quad R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^t Q_k(s) ds \geq 1.$$

Let $x(t)$ be an eventually positive solution of (2.2) and let $z(t)$ be defined by (2.3). Then the oscillation of all solutions of the second order ordinary differential equation

$$(2.7) \quad y''(t) + \varrho^{-1} \sum_{i=1}^m H_i(t)y(t) = 0, \quad t \geq t_0$$

implies that $z'(t) \leq 0$ and $z(t) < 0$ eventually.

Proof. From (2.5) we have

$$(2.8) \quad z'(t) \leq - \sum_{i=1}^m H_i(t)x(t - \tau_i) \leq 0.$$

Therefore, if $z(t) < 0$ does not hold eventually, then $z(t) > 0$ eventually. Let $t_1 > t_0 + \varrho$ be such that $x(t - \varrho) > 0$, $z(t) > 0$ for $t \geq t_1$. Set $M = 2^{-1} \min\{x(t) : t_1 = \varrho \leq t \leq t_1\}$. Then $x(t) > M$ for $t_1 - \varrho \leq t \leq t_1$. We claim that

$$(2.9) \quad x(t) > M, \quad t \geq t_1.$$

If (2.9) does not hold, then there exists a $t^* > t_1$ such that $x(t) > M$ for $t_1 - \varrho \leq t < t^*$ and $x(t^*) = M$. By (2.3) and (2.6) we get

$$\begin{aligned} M = x(t^*) &= z(t^*) + R(t)x(t - r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^{t^*} Q_k(s)x(s - \sigma_k) ds \\ &> \left(R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^{t^*} Q_k(s) ds \right) M \geq M. \end{aligned}$$

This is a contradiction and so (2.9) holds. Let $\lim_{t \rightarrow \infty} z(t) = a$. There exist two possible cases:

Case I. $a = 0$. There exists a $T_1 > t_1$ such that $z(t) < M/2$ for $t \geq T_1$. Then for any $\bar{t} > T_1$, we have

$$\frac{1}{\varrho} \int_{\bar{t}}^{\bar{t} + \varrho} z(s) ds \leq M < x(t), \quad t \in [\bar{t}, \bar{t} + \varrho].$$

Case II. $a > 0$. Then $z(t) \geq a$ for $t \geq t_1$. From (2.3) and (2.9) we get

$$x(t) \geq a + R(t)x(t - r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t - \tau_i + \sigma_k}^t Q_k(s)x(s - \sigma_k) ds \geq a + M, \quad t \geq t_1.$$

By induction, it is easy to see that $x(t) \geq ka + M$ for $t \geq t_1 + (k - 1)\varrho$ and so $\lim_{t \rightarrow \infty} x(t) = \infty$, which implies that there exists a $T > T_1$ such that

$$\frac{1}{\varrho} \int_T^{T + \varrho} z(s) ds \leq 2z(T) < x(t), \quad t \in [T, T + \varrho].$$

Combining the cases I and II we see that

$$x(t) > \frac{1}{\varrho} \int_T^{T + \varrho} z(s) ds, \quad t \in [T, T + \varrho].$$

Now we prove that

$$(2.10) \quad x(t) > \frac{1}{\varrho} \int_T^{t+\varrho} z(s) \, ds, \quad t \geq T + \varrho.$$

Otherwise, there would exist a $t^* > T + \varrho$ such that

$$\begin{aligned} x(t^*) &= \frac{1}{\varrho} \int_T^{t^*+\varrho} z(s) \, ds, \\ x(t) &> \frac{1}{\varrho} \int_T^{t+\varrho} z(s) \, ds \quad \text{for } t \in (T + \varrho, t^*). \end{aligned}$$

Then, from (2.3) and (2.6), we have

$$\begin{aligned} \frac{1}{\varrho} \int_T^{t^*+\varrho} z(s) \, ds &= z(t^*) + R(t^*)x(t^* - r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s)x(s - \sigma_k) \, ds \\ &> \frac{1}{\varrho} \int_{t^*}^{t^*+\varrho} z(s) \, ds + \left(R(t^*) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t^*-\tau_i+\sigma_k}^{t^*} Q_k(s) \, ds \right) \frac{1}{\varrho} \int_T^{t^*} z(s) \, ds \\ &\geq \frac{1}{\varrho} \int_T^{t^*+\varrho} z(s) \, ds. \end{aligned}$$

This is a contradiction and so (2.10) holds. Thus, for $t > T + \varrho$, we obtain

$$(2.11) \quad x(t - \tau_i) > \frac{1}{\varrho} \int_T^t z(s) \, ds.$$

Substituting (2.11) into (2.8) leads to

$$z'(t) + \sum_{i=1}^m H_i(t) \left(\frac{1}{\varrho} \int_T^t z(s) \, ds \right) \leq 0, \quad t > T + \varrho.$$

Set

$$y(t) = \int_T^t z(s) \, ds, \quad t > T + \varrho.$$

Then $y'(t) = z(t)$, $y''(t) = z'(t)$ and

$$y''(t) + \frac{1}{\varrho} \sum_{i=1}^m H_i(t)y(t) \leq 0, \quad t > T + \varrho.$$

By Lemma 2.4 in [11], Eq. (2.7) has an eventually positive solution. This is a contradiction and the proof is complete. \square

Lemma 2.3. Assume that

$$(2.12) \quad R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s) ds \equiv 1.$$

Then the fact that the inequality (2.2) has an eventually positive solution $x(t)$ implies that Eq. (1.1) has a solution $\bar{x}(t)$ which satisfies $0 < \bar{x}(t) \leq x(t)$ eventually.

Proof. Let $z(t)$ be defined by (2.3). By Lemma 2.1 there exists a $t_1 > t_0$ such that $x(t - \varrho) > 0$, $z(t) > 0$ and $z'(t) \leq 0$ for $t \geq t_1$. Set $M = 2^{-1} \min\{x(t) : t_1 = \varrho \leq t \leq t_1\}$. Then $x(t) > M$ for $t \geq t_1 - \varrho$. From (2.3) and (2.4) we have

$$(2.13) \quad x(t) \geq R(t)x(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s-\sigma_k) ds \\ + \int_t^\infty \sum_{i=1}^m H_i(s)x(s-\tau_i) ds, \quad t \geq t_1.$$

Define a sequence of functions $\{x_v(t)\}$ by $x_0(t) = x(t)$ and for $v = 1, 2, \dots$ by

$$(2.14) \quad x_v(t) = R(t)x_{v-1}(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x_{v-1}(s-\sigma_k) ds \\ + \int_t^\infty \sum_{i=1}^m H_i(s)x_{v-1}(s-\tau_i) ds, \quad t \geq t_1 + \varrho, \\ x_v(t) = M + \frac{x_v(t_1 + \varrho) - M}{x(t_1 + \varrho) - M} (x(t) - M), \quad t_1 \leq t < t_1 + \varrho.$$

Then, from (2.13) and (2.14), we have for $t \geq t_1 + \varrho$

$$x_0(t) = x(t) \geq x_1(t) \\ = R(t)x(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s-\sigma_k) ds \\ + \int_t^\infty \sum_{i=1}^m H_i(s)x(s-\tau_i) ds \\ \geq \left(R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s) ds \right) M = M.$$

For $t_1 \leq t < t_1 + \varrho$ we have

$$x_0(t) = x(t) \geq M + \frac{x_1(t_1 + \varrho) - M}{x(t_1 + \varrho) - M} (x(t) - M) = x_1(t) \geq M.$$

Thus, $x_0(t) \geq x_1(t) \geq M$ for $t \geq t_1$. By induction, one can easily prove that

$$x_v(t) \geq x_{v+1}(t) \geq M, \quad t \geq t_1, \quad v = 1, 2, \dots$$

Therefore, $\{x_v(t)\}$ has a pointwise limit function $\bar{x}(t)$ with $0 < M \leq \lim_{v \rightarrow \infty} x_v(t) = \bar{x}(t) \leq x(t)$ for $t \geq t_1$. By the Monotone Convergence Theorem we have

$$\begin{aligned} \bar{x}(t) &= R(t)\bar{x}(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)\bar{x}(s-\sigma_k) ds \\ &+ \int_t^\infty \sum_{i=1}^m H_i(s)\bar{x}(s-\tau_i) ds, \quad t \geq t_1 + \varrho. \end{aligned}$$

This implies that

$$[\bar{x}(t) - R(t)\bar{x}(t-r)]' + \sum_{i=1}^m P_i(t)\bar{x}(t-\tau_i) - \sum_{j=1}^n Q_j(t)\bar{x}(t-\sigma_j) = 0, \quad t \geq t_1 + \varrho.$$

The proof is complete. □

Lemma 2.4. *Assume that (2.12) holds with $\delta > 0$. Then Eq. (1.1) has an eventually positive solution if the second order ordinary differential equation*

$$(2.15) \quad y''(t) + \delta^{-1} \sum_{i=1}^m H_i(t)y(t) = 0, \quad t \geq t_0$$

has an eventually positive solution.

Proof. Let $y(t)$ be an eventually positive solution of (2.15). Then there exists a $t_1 > t_0$ such that $y(t) > 0$, $y''(t) \leq 0$ and $y'(t) > 0$ for $t \geq t_1$. Define a function $x(t)$ by

$$\begin{aligned} x(t) &= \delta^{-1}y(t_1), \quad t_1 \leq t \leq t_1 + \varrho - \delta, \\ x(t) &= \delta^{-1}[y(t_1) + (t - t_1 - \varrho + \delta)y'(t_1 + \varrho)], \quad t_1 + \varrho - \delta \leq t \leq t_1 + \varrho, \end{aligned}$$

and

$$\begin{aligned} x(t) &= y'(t) + R(t)x(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s-\sigma_k) ds, \\ &t_1 + \varrho + l\delta < t \leq t_1 + \varrho + (l+1)\delta, \quad l = 0, 1, \dots \end{aligned}$$

Then $x(t)$ is continuous and positive for $t \geq t_1$, and

$$(2.16) \quad y'(t) = x(t) - R(t)x(t-r) - \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s-\sigma_k) ds, \quad t \geq t_1.$$

Since $y'(t) > 0$ and $y''(t) \leq 0$, we have for $t_1 + \varrho - \delta \leq t \leq t_1 + \varrho$

$$y(t) - y(t_1) = y'(\xi)(t - t_1) \geq y'(t_1 + \varrho)(t - t_1) \geq (t - t_1 - \varrho + \delta)y'(t_1 + \varrho),$$

and so

$$x(t) \leq \frac{1}{\delta}y(t), \quad t_1 \leq t \leq t_1 + \varrho.$$

For $t_1 + \varrho \leq t \leq t_1 + \varrho + \delta$, we have

$$\begin{aligned} x(t) &= y'(t) + R(t)x(t-r) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s)x(s-\sigma_k) ds \\ &\leq \frac{1}{\delta}(y(t) - y(t-\delta)) + \left(R(t) + \sum_{i=1}^p \sum_{k \in J_i} \int_{t-\tau_i+\sigma_k}^t Q_k(s) ds \right) \frac{1}{\delta}y(t-\delta) \\ &= \frac{1}{\delta}y(t). \end{aligned}$$

By induction, one can prove in general that for $l = 0, 1, \dots$

$$x(t) \leq \frac{1}{\delta}y(t), \quad t_1 + \varrho + l\delta < t \leq t_1 + \varrho + (l+1)\delta.$$

Therefore

$$x(t) \leq \frac{1}{\delta}y(t), \quad t \geq t_1$$

and so

$$(2.17) \quad x(t - \tau_i) \leq \frac{1}{\delta}y(t - \tau_i) < \frac{1}{\delta}y(t), \quad t \geq t_1 + \varrho, \quad i = 1, 2, \dots, m.$$

Substituting (2.16) and (2.17) into (2.15) we obtain

$$[x(t) - R(t)x(t-r)]' + \sum_{i=1}^m P_i(t)x(t - \tau_i) - \sum_{j=1}^n Q_j(t)x(t - \sigma_j) \leq 0.$$

By Lemma 2.3, Eq. (1.1) has an eventually positive solution. The proof is complete. \square

Lemma 2.5 ([1], [7]). Consider the ordinary differential equation

$$(2.18) \quad y''(t) + p(t)y(t) = 0, \quad t \geq t_0,$$

where $p(t) \in C([t_0, \infty), \mathbb{R}^+)$. Then

(i) All solutions of (2.18) oscillate if

$$\liminf t \int_t^\infty p(s) ds > \frac{1}{4}.$$

(ii) Eq. (2.18) has an eventually positive solution if

$$t \int_t^\infty p(s) ds \leq \frac{1}{4} \quad \text{for large } t.$$

3. RESULTS AND PROOFS

Theorem 3.1. Assume that (2.1) holds, $\tau_p = \max\{\tau_1, \tau_2, \dots, \tau_m\}$ and

$$\limsup_{t \rightarrow \infty} \int_t^{t+\tau_p} H_p(s) ds > 0.$$

If

$$(3.1) \quad \int_{t_0}^\infty \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] dt = \infty,$$

then all solutions of (1.1) oscillate.

Proof. On the contrary, assume that (1.1) has an eventually positive solution $x(t)$ and let $z(t)$ be defined by (2.3). It follows from Lemma 2.1 that (2.4) holds. From Corollary 3.2.2 in [6], we have that the delay differential equation

$$(3.2) \quad y'(t) + \sum_{i=1}^m H_i(t)y(t - \tau_i) = 0$$

has an eventually positive solution $y(t)$. Let $\lambda(t) = -y'(t)/y(t)$. Then $\lambda(t) \geq 0$ and it satisfies

$$(3.3) \quad \lambda(t) = \sum_{i=1}^m H_i(t) \exp\left(\int_{t-\tau_i}^t \lambda(s) ds\right)$$

or

$$\lambda(t) \sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds = \sum_{i=1}^m H_i(t) \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \cdot \exp\left(\int_{t-\tau_i}^t \lambda(s) ds\right).$$

One can easily show that

$$(3.4) \quad \varphi(u)ue^x \geq \varphi(u)x + \varphi(u) \ln(eu + 1 - \operatorname{sgn} u) \quad \text{for } u \geq 0 \text{ and } x \in \mathbb{R},$$

where $\varphi(0) = 0$ and $\varphi(u) \geq 0$ for $u > 0$.

Employing inequality (3.4) on the right-hand side of (3.3) we get

$$\begin{aligned} \lambda(t) \sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds &\geq \sum_{i=1}^m H_i(t) \int_{t-\tau_i}^t \lambda(s) ds \\ &+ \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] \end{aligned}$$

or

$$(3.5) \quad \begin{aligned} \lambda(t) \sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds - \sum_{i=1}^m H_i(t) \int_{t-\tau_i}^t \lambda(s) ds \\ \geq \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right. \\ \left. + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right]. \end{aligned}$$

Then for $N > T$

$$(3.6) \quad \begin{aligned} \int_T^N \lambda(t) \sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds dt - \int_T^N \sum_{i=1}^m H_i(t) \int_{t-\tau_i}^t \lambda(s) ds dt \\ \geq \int_T^N \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right. \\ \left. + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] dt. \end{aligned}$$

By interchanging the order of integration, we find that

$$(3.7) \quad \int_T^N H_i(t) \int_{t-\tau_i}^t \lambda(s) ds dt \geq \int_T^{N-\tau_i} \int_s^{s+\tau_i} H_i(t) \lambda(s) dt ds \\ = \int_T^{N-\tau_i} \lambda(t) \int_t^{t+\tau_i} H_i(s) ds dt.$$

From (3.6) and (3.7) it follows that

$$(3.8) \quad \sum_{i=1}^m \int_{N-\tau_i}^N \lambda(t) \int_t^{t+\tau_i} H_i(s) ds dt \\ \geq \int_T^N \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right. \\ \left. + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] dt.$$

On the other hand, since (3.2) has an eventually positive solution, by Lemma 2 in [8] we have

$$(3.9) \quad \int_t^{t+\tau_i} H_i(s) ds < 1, \quad i = 1, 2, \dots, m$$

eventually. Then by (3.8) and (3.9) we obtain

$$\sum_{i=1}^m \int_{N-\tau_i}^N \lambda(t) dt \geq \int_T^N \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right. \\ \left. + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] dt$$

or

$$\sum_{i=1}^m \ln \frac{y(N-\tau_i)}{y(N)} \geq \int_T^N \sum_{i=1}^m H_i(t) \ln \left[e \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right. \\ \left. + 1 - \operatorname{sgn} \left(\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds \right) \right] dt.$$

By the assumption

$$\lim_{t \rightarrow \infty} \prod_{i=1}^m \frac{y(t-\tau_i)}{y(t)} = \infty.$$

This implies

$$(3.10) \quad \lim_{t \rightarrow \infty} \frac{y(t - \tau_p)}{y(t)} = \infty.$$

However, by Lemma 1 in [8] we have

$$\liminf_{t \rightarrow \infty} \frac{y(t - \tau_p)}{y(t)} < \infty.$$

This contradicts (3.10) and completes the proof. \square

Remark 3.1. We note that when $R(t) \equiv 0$, Theorem 3.1 improves Theorem 3.2 in [3] because the condition $\sum_{i=1}^m \int_t^{t+\tau_i} H_i(s) ds > 0$ is no longer required.

Theorem 3.2. Assume that (2.12) holds and that

$$(3.11) \quad \liminf_{t \rightarrow \infty} t \int_t^{\infty} \sum_{i=1}^m H_i(s) ds > \frac{\varrho}{4}.$$

Then all solutions of (1.1) oscillate.

Proof. Suppose that Eq. (1.1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (2.3). Then by Lemma 2.1 we have $z(t) > 0$ eventually. On the other hand, by Lemma 2.5, (3.11) implies that all solutions of Eq. (2.7) oscillate. By Lemma 2.2, it follows that $z(t) < 0$. This contradiction completes the proof. \square

Theorem 3.3. Assume that (2.6) and (3.11) hold and that

$$(3.12) \quad R(t - \tau_i)H_i(t) \leq hH_i(t - r), \quad i = 1, 2, \dots, m.$$

Also suppose that $H_i(t)/Q_j(t - \tau_i + \sigma_j)$ is nonincreasing and satisfies

$$(2.13) \quad H_i(t)Q_j(t - \tau_i) \leq h_jH_i(t - \sigma_j), \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n,$$

where h, h_j ($j = 1, 2, \dots, n$) are nonnegative constants satisfying

$$(3.14) \quad h + \sum_{i=1}^p \sum_{k \in J_i} h_k(\tau_i - \sigma_k) = 1.$$

Then every solution of (1.1) oscillates.

Proof. Assume the contrary. Eq. (1.1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (2.3). Then by Lemma 2.2 we have $z(t) < 0$ eventually. From (2.8), (3.12) and (3.13) we have

$$\begin{aligned}
z'(t) &\leq - \sum_{i=1}^m H_i(t)x(t - \tau_i) \\
&= - \sum_{i=1}^m H_i(t)[z(t - \tau_i) + R(t - \tau_i)x(t - r - \tau_i)] \\
&\quad + \sum_{l=1}^p \sum_{k \in J_l} \int_{t-\tau_l+\sigma_k}^t Q_k(s - \tau_i)x(s - \tau_i - \sigma_k) ds \\
&\geq - \sum_{i=1}^m H_i(t)z(t - \tau_i) - h \sum_{i=1}^m H_i(t - r)x(t - r - \tau_i) \\
&\quad - \sum_{l=1}^p \sum_{k \in J_l} \sum_{i=1}^m h_k \frac{H_i(t - \sigma_k)}{Q_k(t - \tau_i)} \int_{t-\tau_l+\sigma_k}^t Q_k(s - \tau_i)x(s - \tau_i - \sigma_k) ds \\
&\geq - \sum_{i=1}^m H_i(t)z(t - \tau_i) + hz'(t - r) \\
&\quad - \sum_{l=1}^p \sum_{k \in J_l} h_k \sum_{i=1}^m \int_{t-\tau_l+\sigma_k}^t H_i(s - \sigma_k)x(s - \tau_i - \sigma_k) ds \\
&= - \sum_{i=1}^m H_i(t)z(t - \tau_i) + hz'(t - r) + \sum_{l=1}^p \sum_{k \in J_l} h_k \int_{t-\tau_l+\sigma_k}^t z'(s - \sigma_k) ds \\
&= - \sum_{i=1}^m H_i(t)z(t - \tau_i) + hz'(t - r) + \sum_{j=1}^n h_j z(t - \sigma_j) - \sum_{l=1}^p \sum_{k \in J_l} h_k z(t - \tau_l).
\end{aligned}$$

Define $\bar{P}_i(t)$ by

$$\begin{aligned}
\bar{P}_i(t) &= H_i(t) + \sum_{k \in J_i} h_k, \quad i = 1, 2, \dots, p, \\
\bar{P}_i(t) &= H_i(t), \quad i = p + 1, p + 2, \dots, m.
\end{aligned}$$

We obtain

$$[z(t) - hz(t - r)]' + \sum_{i=1}^m \bar{P}_i(t)z(t - \tau_i) - \sum_{j=1}^n h_j z(t - \sigma_j) \geq 0.$$

This implies that $-z(t)$ is a positive solution of the inequality

$$[y(t) - hy(t - r)]' + \sum_{i=1}^m \bar{P}_i(t)y(t - \tau_i) - \sum_{j=1}^n h_j y(t - \sigma_j) \leq 0,$$

which yields a contradiction by Lemmas 2.1 and 2.2. The proof is complete. \square

Next we give a criterion for nonoscillation.

Theorem 3.4. *Assume that (2.12) holds with $\delta > 0$ and that*

$$(3.17) \quad t \int_t^\infty \sum_{i=1}^m H_i(s) ds \leq \frac{\delta}{4} \quad \text{for large } t.$$

Then Eq. (1.1) has an eventually positive solution.

Proof. The conclusion of Theorem 3.4 is an immediate consequence of Lemma 2.4 and Lemma 2.5. \square

Example ([14]). Consider the equation

$$(3.18) \quad [x(t) - (1 - \alpha)x(t - r)]' + (\alpha + t^{-\beta})x(t - \tau) - \alpha x(t - \sigma) = 0, \quad t \geq 1,$$

where $0 \leq \alpha < 1$, $-\infty < \beta \leq 2$, $\tau = \sigma + 1$, $\sigma > 0$ and $r > 0$. All conditions of Theorem 3.2 are satisfied when $-\infty < \beta < 2$ or $\beta = 2$ and $\varrho < 4$. Thus, all solutions of (3.18) oscillate when $-\infty < \beta < 2$ or $\beta = 2$ and $\varrho < 4$. On the other hand, by Theorem 3.4, Eq. (3.18) has an eventually positive solution when $\beta > 2$ or $\beta = 2$ and $\delta \geq 4$.

Remark 3.2. It should be noted that condition (11) in [14] is not satisfied for Eq. (3.18) when $3/2 < \beta \leq 2$. Thus our condition (3.11) is better than condition (11) in [14], and so Theorem 3.2 and Theorem 3.3 improve and extend Theorem 1 and Theorem 3 in [14], respectively.

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Authors' address: Department of Mathematics, Hunan Normal University, Changsha 410081, P. R. China, e-mail: luozg@mail.hunnu.edu.cn.