Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 1, 95–102

Persistent URL: http://dml.cz/dmlcz/127867

Czechoslovak Mathematical Journal, 54 (129) (2004), 95–102

STATISTICAL CLUSTER POINTS OF SEQUENCES IN FINITE DIMENSIONAL SPACES

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(Received May 14, 2001)

Abstract. In this paper we study the set of statistical cluster points of sequences in m-dimensional spaces. We show that some properties of the set of statistical cluster points of the real number sequences remain in force for the sequences in m-dimensional spaces too. We also define a notion of Γ-statistical convergence. A sequence x is Γ-statistically convergent to a set C if C is a minimal closed set such that for every $\varepsilon > 0$ the set $\{k\colon \varrho(C, x_k) \geqslant \varepsilon\}$ has density zero. It is shown that every statistically bounded sequence is Γ-statistically convergent. Moreover if a sequence is Γ-statistically convergent then the limit set is a set of statistical cluster points.

Keywords: compact sets, natural density, statistically bounded sequence, statistical cluster point

MSC 2000: 40A05, 11B05

1. Introduction and background

The notion of statistical convergence of sequences of real numbers was introduced by Fast [2] in a short note. Later this notion has been studied by Šalát [9], Fridy [3], [4], Connor [1] and so on. Recently, Pehlivan and Mamedov [7] have proved that all optimal paths have the same unique statistical cluster point which is also a statistical limit point. In [8] these concepts were used in Turnpike theory as an application.

The idea of statistical convergence is closely related to the concept of natural density or asymptotic density of subsets of the positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Note that the natural density $\delta(K)$ of a set $K \subset \mathbb{N}$ is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n\colon\ k \in K\}|$$

where the vertical bars indicate the number of elements in the enclosed set. If x is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we write that x_k satisfies P for almost all k (a.a.k). The sequence x is statistically convergent to the point L if $\delta\{k\colon |x_k - L| \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$. Statistical convergence has been studied in Banach spaces by Kolk [6].

Let $K = \{k(j)\colon k(1) < k(2) < k(3) < \ldots\} \subset \mathbb{N}$ and $\{x\}_K = \{x_{k(j)}\}$ be a subsequence of x. If the set K has density zero (i.e. $\delta(K) = 0$) the subsequence $\{x\}_K$ of the sequence x is called a *thin* subsequence. If the set K does not have density zero the subsequence $\{x\}_K$ is called a *nonthin* subsequence of x. The statement $\delta(K) \neq 0$ means that either $\delta(K) > 0$ or $\delta(K)$ is not defined (i.e. K does not have natural density).

In [4] Fridy introduced the concept of statistical limit points and statistical cluster points of real number sequences and gave some properties of the sets of statistical limit and cluster points of x. Recall that the number η is a statistical limit point of the number sequence x provided that there is a nonthin subsequence of x that converges to η. Note that γ is a statistical cluster point if a set $\{k\colon |x_k - \gamma| < \varepsilon\}$ does not have density zero for every $\varepsilon > 0$. It was established that the set of statistical limit points of a bounded sequence may be empty while the set of statistical cluster points is nonempty and compact.

In Section 2 we give some properties of the set of statistical cluster points in $\mathbb{R}^m$. In Section 3 we contribute to the study of Γ-statistical convergence. It is proved that the sequence x is Γ-statistically convergent if and only if $\delta\{k\colon \varrho(\Gamma_x, x_k) \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$. In this case $\Gamma_x = C$.

It should be noted that results obtained below can not be extended to any metric spaces. Note that the generalization to m-dimensional case here also could not be based only on the idea of coordinatewise convergence.

2. Statistical cluster points in $\mathbb{R}^m$

In this section we first introduce some properties of the set of statistical cluster points in m-dimensional spaces. Let $\mathbb{R}^m$ be m-dimensional space with norm $\|\cdot\|$. Consider a sequence $x = (x_k)$, $x_k \in \mathbb{R}^m$, $k \in \mathbb{N}$, and a point $\xi \in \mathbb{R}^m$.

Definition 1. The sequence x is statistically convergent to ξ if for every $\varepsilon > 0$

$$\delta\{k\colon \|x_k - \xi\| \geqslant \varepsilon\} = 0.$$

Definition 2. The point ξ is called a statistical cluster point (s.c.p.) if for every $\varepsilon > 0$

$$\delta\{k\colon \|x_k - \xi\| < \varepsilon\} \neq 0.$$

Clearly in this case

$$\limsup_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ \|x_k - \xi\| < \varepsilon\}| > 0.$$

By Γ_x we denote the set of s.c.p. of the sequence x.

Let $\varrho(A, \xi)$ stand for the distance from a point ξ to the closed set A: $\varrho(A, \xi) = \min\limits_{y\in A} \|y - \xi\|$. Let $S_\varepsilon(A) = \{y \in \mathbb{R}^m\colon\ \varrho(A, y) < \varepsilon\}$ be the open ε-neighbourhood of A.

Lemma 1. *Let $A \subset \mathbb{R}^m$ be a compact set and $A \cap \Gamma_x = \emptyset$. Then the set $\{k\colon\ x_k \in A\}$ has density zero; i.e. $\delta\{k\colon\ x_k \in A\} = 0$.*

P r o o f. By the condition every point $\xi \in A$ is not a s.c.p., that is for every point $\xi \in A$ there is a positive number $\varepsilon = \varepsilon(\xi) > 0$ such that $\delta\{k\colon\ \|x_k - \xi\| < \varepsilon\} = 0$.

Let $S_\varepsilon(\xi) = \{y \in \mathbb{R}^m\colon\ \|y - \xi\| < \varepsilon\}$. The open sets $S_\varepsilon(\xi)$, $\xi \in A$ form an open covering of A. But A is a compact set and so there exists a finite subcover of A, say $S_i = S_{\varepsilon_i}(\xi_i)$, $i = 1, 2, \ldots, p$. Clearly $A \subset \bigcup\limits_i S_i$ and $\delta\{k\colon\ \|x_k - \xi_i\| < \varepsilon_i\} = 0$ for every i. We can write

$$|\{k \leqslant n\colon\ x_k \in A\}| \leqslant \sum_{i=1}^{p} |\{k \leqslant n\colon\ \|x_k - \xi_i\| < \varepsilon_i\}|$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ x_k \in A\}| \leqslant \sum_{i=1}^{p} \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n\colon\ \|x_k - \xi_i\| < \varepsilon_i\}| = 0.$$

This implies that $\delta\{k\colon\ x_k \in A\} = 0$ and the Lemma is proved. □

Note that if the set A is open or unbounded, Lemma 1 need not be true. In fact for the sequence $x = (0, 1, 0, 2, 0, 3, \ldots)$ the set of s.c.p. is $\Gamma_x = \{0\}$. In this case for the unbounded closed set $A = [1, \infty)$ we have $A \cap \Gamma_x = \emptyset$, but $\delta\{k\colon\ x_k \in A\} = 1/2 \neq 0$. For a sequence $x_k = 1/k$ we have $\Gamma_x = \{0\}$. In this case for a bounded and open set $A = (0, 1)$, where $A \cap \Gamma_x = \emptyset$, we obtain $\delta\{k\colon\ x_k \in A\} = 1 \neq 0$.

Theorem 1. *If the sequence x has a bounded nonthin subsequence, then the set Γ_x is a nonempty and closed set.*

P r o o f. Let $\{x\}_K$ be a bounded nonthin subsequence of x; i.e. $\delta(K) \neq 0$ and there exists a compact set A such that $x_k \in A$ for every $k \in K$. If Γ_x is empty then $A \cap \Gamma_x = \emptyset$. By Lemma 1 we have $\delta\{k\colon\ x_k \in A\} = 0$. But $|\{k \leqslant n\colon\ k \in K\}| \leqslant |\{k \leqslant n\colon\ x_k \in A\}|$ and therefore $\delta(K) = 0$. This is a contradiction. □

We proved that Γ_x is nonempty. It is not diffucult to show that it is also a closed set.

And now we give a definition of statistically bounded sequences which for real number sequences was introduced in [5].

Definition 3. The sequence x is called statistically bounded if there exists a compact set B such that the set $\{k\colon\ x_k \notin B\}$ has density zero; i.e. $\delta\{k\colon\ x_k \notin B\} = 0$.

Note that in this case we have $\delta\{k\colon\ x_k \in B\} = 1$. It is also clear that every bounded sequence is statistically bounded.

Corollary 1. *If x is a statistically bounded sequence then the set Γ_x is nonempty and compact.*

P r o o f. Let B be a compact set such that $\delta\{k\colon\ x_k \notin B\} = 0$. Then $\delta\{k\colon\ x_k \in B\} = 1 > 0$. In other words the set B contains a nonthin subsequence of x, and so from Theorem 1 we obtain that Γ_x is nonempty and closed.

Now we show that $\Gamma_x \subset B$ which implies that Γ_x is bounded and therefore is compact. To the contrary, suppose that $\xi \in \Gamma_x$ is such that $\xi \notin B$. As B is compact there exists a number $\varepsilon > 0$ such that ε-neighbourhood of the point ξ has an empty intersection with B. In this case we have $\{k\colon\ \|x_k - \xi\| < \varepsilon\} \subset \{k\colon\ x_k \notin B\}$ and therefore the set $\delta\{k\colon\ \|x_k - \xi\| < \varepsilon\} = 0$ has density zero contrary to the assumption $\xi \in \Gamma_x$. □

Theorem 2. *Let x be a statistically bounded sequence. Then for every $\varepsilon > 0$ the set $\{k\colon\ \varrho(\Gamma_x, x_k) \geqslant \varepsilon\}$ has density zero; i.e.*

$$\delta\{k\colon\ \varrho(\Gamma_x, x_k) \geqslant \varepsilon\} = 0. \tag{1}$$

P r o o f. Let B be a compact set such that $\delta\{k\colon\ x_k \notin B\} = 0$. From Corollary 1 the set Γ_x is nonempty and clearly $\Gamma_x \subset B$.

Suppose that the equality (1) does not hold. In this case there exists a number $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leqslant n\colon\ \varrho(\Gamma_x, x_k) \geqslant \varepsilon\}| > 0.$$

Define $S_\varepsilon(\Gamma_x) = \{y\colon\ \varrho(\Gamma_x, y) < \varepsilon\}$ and let $A = B \setminus S_\varepsilon(\Gamma_x)$.

Clearly A is a compact set and contains nonthin subsequence of x. Then by Lemma 1 $A \cap \Gamma_x \neq \emptyset$; i.e. A contains a s.c.p. This is a contradiction. □

Note 1. If the sequence x is not bounded Theorem 2 need not be true. For example if $x = \{1, 0, 2, 0, 3, 0, 4, 0, \ldots\}$ we have $\Gamma_x = \{0\}$, but

$$\delta\{k\colon\ \varrho(\Gamma_x, x_k) \geqslant \varepsilon\} = 1/2 > 0.$$

Now we only formulate a generalized version of Theorem 2 in [4]. This theorem and its proof extend to the m-dimensional case.

Theorem 3. *Let $x = (x_k)$ be a given sequence. There exists a sequence $y = (y_k)$ such that*

(i) *$L_y = \Gamma_x$; where L_y is a set of ordinary limit points of the sequence y;*
(ii) *$y_k = x_k$ for a.a. k; that is:*

$$\delta\{k\colon\ y_k \neq x_k\} = 0.$$

3. Γ-statistical convergence

Now let $C \subset \mathbb{R}^m$ be a closed set for which the following property holds

$$\delta\{k\colon\ \varrho(C, x_k) \geqslant \varepsilon\} = 0 \quad \text{for every } \varepsilon > 0. \tag{2}$$

We say that the set C is minimal closed set satisfying (2) if for every closed set $C' \subset C$, where $C \setminus C' \neq \emptyset$ there is a number $\varepsilon' > 0$ such that

$$\delta\{k\colon\ \varrho(C', x_k) \geqslant \varepsilon'\} \neq 0.$$

Definition 4. The sequence x is Γ-statistically convergent to the set C if C is a minimal nonempty closed set satisfying (2).

The following results, which are easy consequences of the definition.

Lemma 2. *If the sequence x is Γ-statistically convergent then the limit set is unique.*

Lemma 3. *If the limit set in Lemma 2 consists of one point then the sequence is statistically convergent to this point.*

Consider the problem: under what conditions a sequence x is Γ-statistically convergent; i.e. when there exists a minimal closed set among all closed sets C satisfying (2).

Note that a sequence x need not be Γ-statistically convergent. We explain this fact. Let C_α be the system of all closed sets satisfying (2). Clearly this system is nonempty, for example if we take $C = \mathbb{R}^m$ the condition holds. Consider the intersection $C = \bigcap\limits_\alpha C_\alpha$. Is this set C a minimal closed set satisfying (2)? This is not true in general. Moreover the set C may be empty. For example if $x = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ we can take $C_\alpha = (-\infty, -\alpha] \cup [\alpha; \infty)$ (with $\alpha > 0$) and clearly in this case $C = \emptyset$, and the sequence x is not Γ-statistically convergent.

The next result shows that if x is statistically bounded then $C = \bigcap\limits_\alpha C_\alpha = \Gamma_x$ and therefore C is nonempty.

Theorem 4. *If x is a statistically bounded sequence then it is Γ-statistically convergent to the set Γ_x.*

P r o o f. From Theorem 2 we observe that Γ_x is a nonempty compact set and for this set the condition (2) holds. Suppose that it is not minimal; i.e. there exists a closed set C satisfying (2) such that $C \subset \Gamma_x$ and $\Gamma_x \setminus C \neq \emptyset$. In this case there is a point $\xi \in \Gamma_x$ such that $\xi \notin C$. Then there exists a number $\varepsilon > 0$ such that $S_\varepsilon(\xi) \cap S_\varepsilon(C) = \emptyset$. Since ξ is a s.c.p. by Definition 2

$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leqslant n\colon\ x_k \in S_\varepsilon(\xi)\}| > 0.$$

Then from $\{k\colon\ x_k \in S_\varepsilon(\xi)\} \subset \{k\colon\ x_k \notin S_\varepsilon(C)\}$ we have

$$\limsup_{n\to\infty} \frac{1}{n} |\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C)\}| > 0$$

which contradicts (2).

Let x be Γ-statistically convergent to the set C. Theorem 4 shows that if this sequence is statistically bounded then $C = \Gamma_x$. Now we consider sequences which are not statistically bounded. Such a sequence also may be Γ-statistically convergent. For example the sequence $x_k = p$, where $k = 2^{p-1}(2q+1)$; i.e. $p-1$ is the number of factors of 2 in the prime factorization of k (see example 3 in [4]), is not statistically bounded but is Γ-statistically convergent to the set $C = \{1, 2, 3, \ldots\}$. □

From the proof of Theorem 4 we conclude the following immediately.

Corollary 2. *Let Γ_x be a set of s.c.p. of the sequence x (which is not statistically bounded in general). If $\delta\{k\colon\ \varrho(\Gamma_x, x_k) \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$ then the sequence x is Γ-statistically convergent to Γ_x.*

The next result shows that if x is Γ-statistically convergent then this limit set may be only the set of s.c.p. Γ_x.

Theorem 5. *If x is Γ-statistically convergent to the set C then $C = \Gamma_x$.*

Proof. First we show that $\Gamma_x \subset C$. Let there be a point $\xi \in \Gamma_x$ such that $\xi \notin C$. As the set C is closed there is a number $\varepsilon > 0$ for which $S_\varepsilon(\xi) \cap S_\varepsilon(C) = \emptyset$. In this case $\{k\colon\ x_k \notin S_\varepsilon(C)\} \supset \{k\colon\ x_k \in S_\varepsilon(\xi)\}$ and therefore from (2) we have $\delta\{k\colon\ x_k \in S_\varepsilon(\xi)\} = 0$ which contradics $\xi \in \Gamma_x$.

And now we show that $C \subset \Gamma_x$. Suppose this is not true, i.e. there is a point $\xi \in C$ such that $\xi \notin \Gamma_x$. By Definition 2 there is a number $\varepsilon' > 0$ such that $\delta\{k\colon\ x_k \in S_\varepsilon(\xi)\} = 0$ for every $\varepsilon \leqslant \varepsilon'$.

Consider two cases.

1) Let the point ξ be an isolated point of C. Then there is a number $\varepsilon \leqslant \varepsilon'$ such that $S_\varepsilon(\xi) \cap S_\varepsilon(C \setminus \{\xi\}) = \emptyset$. In this case $S_\varepsilon(C) = S_\varepsilon(\xi) \cup S_\varepsilon(C \setminus \{\xi\})$ and therefore $|\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C \setminus \{\xi\})\}| = |\{k \leqslant n\colon\ x_k \in S_\varepsilon(\xi)\}| + |\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C)\}|$. Then using (2) we have $\delta\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C \setminus \{\xi\})\} = 0$ for every sufficiently small $\varepsilon > 0$. This means that the set $C \setminus \{\xi\}$ also satisfies (2), i.e. C is not minimal set. This is a contradiction.

2) Suppose ξ is a limit point of the set C: there is a sequence $\xi_m \in C$ such that $\xi_m \to \xi$ and $\xi_j \neq \xi_i$ if $j \neq i$.

Let $\varepsilon > 0$. Choose any $\xi' = \xi_m$ such that $\|\xi - \xi'\| = 2\delta$ and $4\delta < \varepsilon$. We claim that $S_\delta(C) \subset S_\varepsilon(C \setminus S_\delta(\xi))$.

Let $x \in S_\delta(C)$ and $x' \in C$ be such that $\|x - x'\| < \delta$.

a) If $x' \notin S_\delta(\xi)$ then $x' \in C \setminus S_\delta(\xi)$ and therefore $x \in S_\delta(C \setminus S_\delta(\xi)) \subset S_\varepsilon(C \setminus S_\delta(\xi))$.

b) Let $x' \in S_\delta(\xi)$, i.e. $\|x' - \xi\| < \delta$. Then

$$\|x - \xi'\| \leqslant \|x - x'\| + \|x' - \xi\| + \|\xi - \xi'\| < 4\delta < \varepsilon.$$

But $\|\xi' - \xi\| = 2\delta$ and so $\xi' \in C \setminus S_\delta(\xi)$. Hence we have $x \in S_\varepsilon(C \setminus S_\delta(\xi))$.

Therefore the inclusion $S_\delta(C) \subset S_\varepsilon(C \setminus S_\delta(\xi))$ is proved. From this inclusion we have

$$|\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C \setminus S_\delta(\xi))\}| \leqslant |\{k \leqslant n\colon\ x_k \notin S_\delta(C)\}|$$

which implies $\delta\{k \leqslant n\colon\ x_k \notin S_\varepsilon(C \setminus S_\delta(\xi))\} = 0$. Then the set C is not a minimal set satisfying (2). This is a contradiction. □

The final result of this section is given in the following

Corollary 3. *The sequence x is Γ-statistically convergent if and only if $\delta\{k\colon \varrho(\Gamma_x, x_k) \geqslant \varepsilon\} = 0$ for every $\varepsilon > 0$.*

Acknowledgement. The authors are grateful to Professor J. A. Fridy for valuable comments which improved the presentation of the paper.

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