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OSCILLATION THEOREMS FOR NEUTRAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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Abstract. In this paper we present some new oscillatory criteria for the n-th order neutral differential equations of the form

$$(x(t) \pm p(t)x[\tau(t)])^{(n)} + q(t)x[\sigma(t)] = 0.$$

The results obtained extend and improve a number of existing criteria.

Keywords: neutral equation, delayed argument

MSC 2000: 34C10

1. Introduction

In this paper we are concerned with the problem of oscillatory properties of n-th order neutral differential equations

$$(\mathrm{E}_n^{\pm}) \qquad (x(t) \pm p(t)x[\tau(t)])^{(n)} + q(t)x[\sigma(t)] = 0, \quad n \geqslant 2.$$

Throughout this paper the following hypotheses (H) are assumed to hold.

(H1) $\tau(t) \in C[t_0, \infty)$, $\tau(t) \leqslant t$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(H2) $p(t) \in C[t_0, \infty)$, $0 \leqslant p(t) < 1$;

(H3) $q(t) \in C[t_0, \infty)$, $q(t) > 0$,

(H4) $\sigma(t) \in C^1[t_0, \infty)$, $\sigma'(t) > 0$, $\sigma(t) \leqslant t$ and $\lim_{t\to\infty} \sigma(t) = \infty$.

In this paper, we restrict our attention only to the nontrivial solutions of Eq. (E_n^+), which exist on some ray $[T, \infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Eq. (E_n^+) is said to be oscillatory if all its solutions are oscillatory.

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In the last two decades some authors (see the attached references) have obtained sufficient conditions for oscillation of Eq. (E_n^+). However, the results established in this paper are based on conditions and techniques which are different from theirs. Our results here are new also for the corresponding delay differential equation (i.e. $p(t) \equiv 0$).

As is customary, all functional inequalities presented in this paper are assumed to hold eventually, that is to be satisfied for all sufficiently large t.

2. Main Results

We begin with the following identity, which holds for any n-times differentiable function $z(t)$.

$$(1) \quad z^{(i)}(t) = \sum_{j=i}^{k} (-1)^{j-i}(s-t)^{j-i} z^{(j)}(s) + (-1)^{k-i+1} \int_t^s \frac{(u-t)^{k-i}}{(k-i)!} z^{(k+1)}(u)\,\mathrm{d}u,$$

where $0 \leqslant i \leqslant k \leqslant n-1$. This identity is a generalization of Taylor's formula with remainder encountered in calculus. For convenience we introduce the following notation:

$$a_{n-1}(t) = (1 - p[\sigma(t)])q(t),$$
$$a_l(t) = \int_t^\infty \frac{(u-t)^{n-l-2}}{(n-l-2)!}(1 - p[\sigma(u)])q(u)\,\mathrm{d}u,$$

for all $l \in \{1, 2, \ldots, n-3\}$.

Theorem 1. *Assume that for all $l \in \{1, 2, \ldots, n-1\}$ such that $n + l$ is odd*

$$(2_l) \qquad \int^\infty \left(\sigma^l(t)a_l(t) - \frac{\lambda_l l^2 (l-1)!\,\sigma'(t)}{4\sigma(t)}\right) \mathrm{d}t = \infty, \quad \textit{for some } \lambda_l > 1.$$

Further assume that for n odd $p(t) \leqslant p < 1$. Then for n even Eq. (E_n^+) is oscillatory and for n odd every solution $x(t)$ of Eq. (E_n^+) oscillates or tends to zero as $t \to \infty$.

P r o o f. Assume that, to the contrary, $x(t)$ is a nonoscillatory solution of Eq. (E_n^+). Without loss of generality we may assume that $x(t) > 0$. (The case when $x(t) < 0$ can be proved by the same arguments). Set

$$z(t) = x(t) + p(t)x[\tau(t)].$$

Then $z(t) > x(t) > 0$ and

$$z^{(n)}(t) + q(t)x[\sigma(t)] = 0. \tag{3}$$

Thus $z^{(n)}(t) < 0$ and consequently $z'(t), z''(t), \dots, z^{(n-1)}(t)$ are of constant signs in some neighborhood of the infinity. One can easily conclude that there exists $l \in \{0, 1, \dots, n-1\}$ such that $n + l$ is odd and

$$z^{(i)}(t) > 0 \quad \text{for} \quad 0 \leqslant i \leqslant l, \tag{4}$$

$$(-1)^{l+i} z^{(i)}(t) > 0 \quad \text{for} \quad l \leqslant i \leqslant n-1. \tag{5}$$

Therefore, $z^{(n-1)}(t) > 0$. Now we consider the following two cases.

Case 1. Let $l \geqslant 1$. Then $z'(t) > 0$ and using the monotonicity of $z(t)$ one gets

$$x(t) = z(t) - p(t)x[\tau(t)] > z(t) - p(t)z[\tau(t)] > z(t)(1 - p(t)).$$

Combining the last inequalities together with (3) we are lead to

$$z^{(n)}(t) + (1 - p[\sigma(t)])q(t)z[\sigma(t)] \leqslant 0. \tag{6}$$

Assume that $l < n - 1$. Setting $i = l + 1$, $k = n - 1$ and $s > t$ in (2) and using (5) and (6), we have

$$z^{(l+1)}(t) \leqslant -\int_t^s \frac{(u-t)^{n-l-2}}{(n-l-2)!}(1 - p[\sigma(u)])q(u)z[\sigma(u)]\,\mathrm{d}u.$$

Taking into account the monotonicity of $z[\sigma(t)]$ and letting $s \to \infty$, we obtain

$$z^{(l+1)}(t) + a_l(t)z[\sigma(t)] \leqslant 0. \tag{7}$$

From (6) it is easy to see that (7) is true also for $l = n - 1$. Define

$$w_l(t) = \sigma^l(t)\frac{z^{(l)}(t)}{z[\sigma(t)]}. \tag{8}$$

Then $w_l(t) > 0$ and further

$$\begin{aligned} w_l'(t) = {} & l\sigma^{l-1}(t)\sigma'(t)\frac{z^{(l)}(t)}{z[\sigma(t)]} + \sigma^l(t)\frac{z^{(l+1)}(t)}{z[\sigma(t)]} \\ & - \sigma^l(t)\frac{z^{(l)}(t)}{z^2[\sigma(t)]}z'(\sigma(t))\sigma'(t). \end{aligned} \tag{9}$$

For $n > 2$ we let $i = 1$, $k = l - 1$, $s = t_0 < t$ in (2) and noting (4) one can see that for any $\lambda_l > 1$

$$
\begin{aligned}
z'(t) &\geqslant \int_{t_0}^{t} \frac{(t-u)^{l-2}}{(l-2)!} z^{(l)}(u)\,\mathrm{d}u \geqslant z^{(l)}(t)\frac{(t-t_0)^{l-1}}{(l-1)!} \\
&\geqslant \frac{1}{\lambda_l (l-1)!}\, t^{l-1} z^{(l)}(t),
\end{aligned} \tag{10}
$$

holds eventually. Note that (10) is satisfied also for $n = 2$. In this case $l = 1$ and $\lambda_l = 1$. It follows from (10) that

$$
z'[\sigma(t)] \geqslant \frac{1}{\lambda_l (l-1)!}\, \sigma^{l-1}(t) z^{(l)}[\sigma(t)] \geqslant \frac{1}{\lambda_l (l-1)!}\, \sigma^{l-1}(t) z^{(l)}(t),
$$

which in view of (9) and (7) leads to

$$
\begin{aligned}
w_l'(t) \leqslant &-\sigma^l(t) a_l(t) - \frac{\sigma^{2l-1}(t)\sigma'(t)}{\lambda_l (l-1)!}\left(\frac{z^{(l)}(t)}{z[\sigma(t)]}\right)^2 \\
&+ l\sigma^{l-1}(t)\sigma'(t)\frac{z^{(l)}(t)}{z[\sigma(t)]} \\
= &-\sigma^l(t) a_l(t) + \frac{l^2 \lambda_l (l-1)!\,\sigma'(t)}{4\sigma(t)} \\
&- \frac{\sigma^{2l-1}(t)\sigma'(t)}{\lambda_l (l-1)!}\left(\frac{z^{(l)}(t)}{z[\sigma(t)]} - \frac{l\lambda_l (l-1)!}{2\sigma^l(t)}\right)^2 \\
\leqslant &-\sigma^l(t) a_l(t) + \frac{l^2 \lambda_l (l-1)!\,\sigma'(t)}{4\sigma(t)}.
\end{aligned}
$$

Integrating from t_1 to t, we get

$$
w_l(t) \leqslant w_l(t_1) - \int_{t_1}^{t} \left[\sigma^l(s) a_l(s) - \frac{l^2 \lambda_l (l-1)!\,\sigma'(s)}{4\sigma(s)}\right] \mathrm{d}s.
$$

Letting $t \to \infty$ we get $w_l(t) \to -\infty$. This contradicts the positivity of $w_l(t)$ and we conclude that Case 1 is impossible.

Case 2. Let $l = 0$. Note that this case is possible only when n is odd. Therefore, for n even the proof of our theorem is complete. To finish the proof we shall show that $\lim_{t\to\infty} x(t) = 0$. Since $z(t) > x(t) > 0$, it is sufficient to verify that $\lim_{t\to\infty} z(t) = 0$. On the other hand, (4)–(5) with $l = 0$ imply that $\lim_{t\to\infty} z(t)$ exists and is nonnegative and finite. Aiming at a contradiction we assume that $\lim_{t\to\infty} z(t) = c > 0$. Then $z(t) > c$, eventually. Choose $0 < \varepsilon < c(1-p)/p$. Evidently $z[\sigma(t)] < c + \varepsilon$, for all large t. It is easy to verify that

$$
x(t) > z(t) - p(t) z[\tau(t)] > c - p(c+\varepsilon) > c_1 z(t),
$$

where $0 < c_1 = (c - p(c+\varepsilon))/(c+\varepsilon)$. Then (3) implies

$$z^{(n)}(t) + c_1 q(t) z[\sigma(t)] \leqslant 0. \tag{11}$$

Setting $i = 0$, $k = n-1$ and $s > t = t_1$ in (2) and using (5), one gets

$$z(t_1) \geqslant -\int_{t_1}^{s} \frac{(u-t_1)^{n-1}}{(n-1)!} z^{(n)}(u)\,\mathrm{d}u. \tag{12}$$

Substituting (11) into (12), using $z[\sigma(t)] \geqslant c$ and then letting $s \to \infty$, we obtain

$$z(t_1) \geqslant c_1 c \int_{t_1}^{\infty} \frac{(u-t_1)^{n-1}}{(n-1)!} q(u)\,\mathrm{d}u,$$

which implies

$$\int_{t_1}^{\infty} u^{n-1} q(u)\,\mathrm{d}u < \infty. \tag{13}$$

But in view of (2_{n-1}) we have

$$\infty = \int_{t_1}^{\infty} \sigma^{n-1}(u)(1 - p[\sigma(u)])q(u)\,\mathrm{d}u \leqslant \int_{t_1}^{\infty} u^{n-1} q(u)\,\mathrm{d}u,$$

which contradicts (13). Consequently, $\lim\limits_{t\to\infty} z(t) = 0$. The proof is now complete. □

For the third order neutral equation the previous theorem provides the following criterion.

Corollary 1. *Assume that for some* $\lambda > 1$

$$\int^{\infty} \left(\sigma^2(t)(1 - p[\sigma(u)])q(t) - \frac{\lambda \sigma'(t)}{\sigma(t)} \right) \mathrm{d}t = \infty.$$

Then every solution $x(t)$ *of Eq.* (E_3^+) *oscillates or tends to zero as* $t \to \infty$.

Remark 1. We note that for $n = 2$, $\sigma(t) = t$ and $p(t) \equiv 0$, condition (2_1) of Theorem 1 reduces to

$$\int^{\infty} \left(t q(t) - \frac{1}{4t} \right) \mathrm{d}t = \infty,$$

which is the well known Kiguradze and Chanturia oscillation criterion [3] for the corresponding second order differential equation

$$x'' + q(t)x = 0.$$

Remark 2. For Eq. (E_2^+) Theorem 1 improves Theorem 2 in [2] where the condition $\int^{\infty} q(s)\,\mathrm{d}s = \infty$ is required.

Corollary 2. *Assume that for all $l \in \{1, 2, \dots, n-1\}$ such that $n + l$ is odd*

$$(14_l) \qquad \liminf_{t\to\infty} \frac{\sigma^{l+1}(t)a_l(t)}{\sigma'(t)} > \frac{l^2(l-1)!}{4}.$$

Then for n even Eq. (E_n^+) is oscillatory and for n odd every solution $x(t)$ of Eq. (E_n^+) oscillates or tends to zero as $t \to \infty$.

Proof. Note that (14_l) implies (2_l). □

Remark 3. Recently Parhi and Mohanty in [12] presented another oscillation criterion for Eq. (E_n^+). This criterion extends some other known results. Our results here generalize those in [5], [7], [8] and [12].

Example 1. We consider the third order differential equation

$$(15) \qquad (x(t) + px[\tau(t)])''' + \frac{b}{t^3}x[\beta t] = 0,$$

with $b > 0$, $0 < \beta < 1$, $0 < p < 1$. Corollary 2 implies that all nonoscillatory solutions of (15) tend to zero as $t \to \infty$ provided that

$$a > \frac{1}{\beta^2(1-p)}.$$

On the other hand Theorem 2.1 in [12] requires

$$a > \frac{8}{\mathrm{e}(-\ln\beta)\beta^2(1-p)}.$$

Now we turn our attention to oscillatory properties of Eq. (E_n^-). We shall consider the following functions:

$$b_{n-1}(t) = q(t)$$
$$b_l(t) = \int_t^\infty \frac{(u-t)^{n-l-2}}{(n-l-2)!} q(u)\,\mathrm{d}u,$$

for all $l \in \{1, 2, \dots, n-3\}$.

Theorem 2. *Let $0 \leqslant p(t) \leqslant p < 1$. Assume that for every $l \in \{1,2,\dots,n-1\}$ such that $n+l$ is odd*

$$(16_l)\qquad \int^{\infty}\left(\sigma^l(t)b_l(t) - \frac{\lambda_l l^2(l-1)!\,\sigma'(t)}{4\sigma(t)}\right)\mathrm{d}t = \infty, \quad \textit{for some } \lambda_l > 1.$$

Then every solution $x(t)$ of Eq. (E_n^-) oscillates or tends to zero as $t \to \infty$.

P r o o f. Let $x(t)$ be an eventually positive solution of Eq. (E_n^-). Setting

$$(17)\qquad z(t) = x(t) - p(t)x[\tau(t)]$$

we obtain $z(t) < x(t)$ and (3). Since $z^{(n)}(t) < 0$ then $z^{(i)}(t)$, for $i = 0, 1 \dots, n-1$ are of constant sign eventually.

We claim that $x(t)$ is bounded. To prove this assume, to the contrary, that $x(t)$ is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim\limits_{m\to\infty} t_m = \infty$, moreover $\lim\limits_{m\to\infty} x(t_m) = \infty$ and $x(t_m) = \max\{x(s); t_0 \leqslant s \leqslant t_m\}$. Since $\tau(t) \to \infty$ as $t \to \infty$, we can choose a large m such that $\tau(t_m) > t_0$. As $\tau(t) \leqslant t$, we have

$$\begin{aligned} x(\tau(t_m)) &\leqslant \max\{x(s); t_0 \leqslant s \leqslant \tau(t_m)\} \\ &\leqslant \max\{x(s); t_0 \leqslant s \leqslant t_m\} \leqslant x(t_m). \end{aligned}$$

Therefore for all large m

$$z(t_m) \geqslant x(t_m) - px[\tau(t_m)] \geqslant (1-p)x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$. Since $z(t)$, $z'(t)$ are of constant sign this yields $z(t) > 0$, $z'(t) > 0$. By the well known lemma of Kiguradze it is easy to check that there exists $l \in \{1,2,\dots,n-1\}$ such that $n+l$ is odd and (4)–(5) hold. In view of (3) we see that

$$z^{(n)}(t) + q(t)z[\sigma(t)] \leqslant 0.$$

Proceeding similarly as in the Case 1 of the proof of Theorem 1 we obtain

$$z^{(l+1)}(t) + b_l(t)z[\sigma(t)] \leqslant 0.$$

We define the function $w_l(t)$ as in (8). Following all steps of the proof of Theorem 1, Case 1 we arrive to a contradiction with (16_l) and so we can conclude that $x(t)$ is bounded. Consequently, in view of (17) $z(t)$ is bounded and hence

$$(18)\qquad (-1)^{n+j}z^{(j)}(t) < 0, \quad \text{for } j = 1,2,\dots,n-1.$$

We distinguish the following two cases.

Case 1. Let $z(t) > 0$. Then for n even (18) implies $z'(t) > 0$ and this situation has been shown to lead to a contradiction with (16_l) above.

For n odd, (18) implies that $l = 0$. Thus $z(t)$ is positive and decreasing, therefore there exists a finite $\lim_{t\to\infty} z(t) = c \geqslant 0$. If $c > 0$, then (3) yields

$$z^{(n)}(t) + q(t)z(\sigma(t)) \leqslant 0. \tag{19}$$

Setting $i = 0$, $k = n-1$ and $s > t = t_1$ in (2) we get (12). Taking into account (19) we have in view of (12) that

$$z(t_1) \geqslant c\int_{t_1}^{\infty} \frac{(u-t_1)^{n-1}}{(n-1)!}\, q(u)\,\mathrm{d}u. \tag{20}$$

Then (16_{n-1}) yields

$$\infty = \int_{t_2}^{\infty} \sigma^{n-1}(u)q(u)\,\mathrm{d}u \leqslant \int_{t_2}^{\infty} u^{n-1}q(u)\,\mathrm{d}u.$$

This contradicts (20) and consequently $\lim_{t\to\infty} z(t) = 0$. On the other hand the boundedness of $x(t)$ yields $\lim_{t\to\infty}\sup x(t) = a$, $0 \leqslant a < \infty$. Then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a$. If $a > 0$, choosing $\varepsilon = a(1-p)/(2p)$ we see that $x[\tau(t)] < a + \varepsilon$, eventually. Moreover

$$0 = \lim_{k\to\infty} z(t_k) \geqslant \lim_{k\to\infty} (x(t_k) - p(a+\varepsilon)) = \frac{a}{2}(1-p) > 0. \tag{21}$$

Thus $a = 0$ and that is $\lim_{t\to\infty} x(t) = 0$.

Case 2. Let $z(t) < 0$. For n even, it follows form (18) that $z'(t) > 0$ which implies that $\lim_{t\to\infty} z(t) = c \leqslant 0$. Denote $\limsup_{t\to\infty} x(t) = a$. If $a > 0$ then considering a sequence $\{t_k\}$ as above and proceeding exactly as above we are led to

$$0 \geqslant c = \lim_{k\to\infty} z(t_k) \geqslant \lim_{k\to\infty} (x(t_k) - p(a+\varepsilon)) = \frac{a}{2}(1-p) > 0.$$

Then $a = 0$ and $\lim_{t\to\infty} x(t) = 0$ and moreover (17) implies $\lim_{t\to\infty} z(t) = 0$.

For n odd we have $z'(t) < 0$ which yields $\lim_{t\to\infty} z(t) = -c < 0$.

This again yields $\lim_{t\to\infty} x(t) = 0$, while, on the other hand, it follows from the inequality $z(t) \geqslant x(t) - px(\tau(t))$ that $\lim_{t\to\infty} z(t) \geqslant 0$, a contradiction. The proof is complete. □

Corollary 3. *Let $0 \leqslant p(t) \leqslant p < 1$. Assume that for every $l \in \{1, 2, \dots, n-1\}$ such that $n + l$ is odd*

$$(21_l) \qquad \limsup_{t\to\infty} \frac{\sigma^{l+1}(t) b_l(t)}{\sigma'(t)} > \frac{l^2(l-1)!}{4}.$$

Then every solution $x(t)$ of Eq. (E_n^-) oscillates or tends to zero as $t \to \infty$.

Proof. Note that (21_l) implies (16_l). □

It is useful to notice the following result which immediately follows from the proof of Theorem 2. This corollary can be used in the comparison theory of neutral differential equations.

Corollary 4. *Let all the assumptions of Theorem 2 hold. Let $x(t)$ be an eventually positive solution of Eq. (E_n^-). Let $z(t)$ be defined by (17). Then*

(i) *for n even we have*

$$(22) \quad \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} z^{(j)}(t) = 0, \quad (-1)^{j+1} z^{(j)}(t) > 0, \quad j = 0, 1, \dots, n-1,$$

(ii) *for n odd we have*

$$(23) \quad \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} z^{(j)}(t) = 0, \quad (-1)^{j} z^{(j)}(t) > 0, \quad j = 0, 1, \dots, n-1.$$

Remark 4. It is evident from the proofs of Theorems 1 and 2 that we can let $\lambda_1 = 1$ in (21_l), (16_l), respectively.

Example 2. Let us consider the second order neutral differential equation

$$(24) \qquad (x(t) - 0{,}5x(t-1))'' + \frac{\mathrm{e} - 2}{2\mathrm{e}} x(t-1) = 0.$$

Then by Corollary 3 every nonoscillatory solution $x(t)$ of (24) satisfies (22). One such solution is $x(t) = \mathrm{e}^{-t}$.

Employing additional conditions imposed on the coefficients of Eq. (E_n^-) the conclusion of Theorem 2 (Corollary 3) can be strenghtened as follows.

Corollary 5. *Assume that n is even. Let all the assumptions of Theorem 2 (Corollary 3) hold. Then if $p(t)$ oscillates, then Eq.* (E_n^-) *is oscillatory.*

P r o o f. Let $x(t)$ be a positive solution of (E_n^-), then by Corollary 4, $z(t) < 0$. If $\{t_k\}$ is a sequence of zeros of $p(t)$ then

$$0 > z(t_k) = x(t_k) - p(t_k)x(\tau(t_k)) > 0,$$

a contradiction. □

Example 3. We consider the fourth order neutral differential equation

$$\left(x(t) - \frac{1-\sin t}{3}x[\tau(t)]\right)^{(\mathrm{IV})} + \frac{a}{t^4}x(\beta t) = 0, \quad 0 < \beta < 1. \tag{25}$$

Then by Corollary 5, Eq. (25) is oscillatory provided that

$$a > \frac{9}{2\beta^3}.$$

On the other hand, Parhi and Mohanty's result [12] guarantees oscillation of (25) if

$$a > \frac{2^9}{\beta^3 \mathrm{e}(-\ln\beta)}.$$

On the other hand, the results presented in [8] cannot be applied to Eq. (28) as the required condition $\int^\infty q(s)\,\mathrm{d}s = \infty$ is not satisfied for (25).

In the following we are concerned with the investigation of oscillation of the special case of (E_n^-) with n odd, that is we shall assume that $\sigma(t) = t - \sigma$, $\tau(t) = t - \tau$, $p(t) = p$, with $\sigma > 0$, $\tau > 0$, $p \in (0,1)$.

Corollary 6. *Assume that n is odd. Let the hypotheses of Theorem 2 hold. Furthermore assume that*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} q(s)(s-t)^{n-1}\,\mathrm{d}s > (1-p)(n-1)!. \tag{26}$$

Then Eq. (E_n^-) *is oscillatory.*

P r o o f. Let $x(t)$ be an eventually positive solution of (E_n^-). Then it follows from Corollary 4 that (23) holds. On the other hand the condition (26) (see [8]) implies that Eq. (E_n^-) has no solution satisfying (23). The proof is complete. □

As we mentioned above our results here generalize and extend a number of existing oscillation criteria. Moreover our results are new even for the corresponding delay differential equations, that is for $p(t) \equiv 0$.

We remark that it is only routine work to extend our results to equations with several delays of the form

$$(x(t) \pm p(t)x[\tau(t)])^{(n)} + \sum_{i=0}^{k} q_i(t)x[\sigma_k(t)] = 0.$$

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