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SUBALGEBRAS AND HOMOMORPHIC IMAGES OF ALGEBRAS HAVING THE CEP AND THE WCIP

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Abstract. In the present paper we consider algebras satisfying both the congruence extension property (briefly the CEP) and the weak congruence intersection property (WCIP for short). We prove that subalgebras of such algebras have these properties. We deduce that a lattice has the CEP and the WCIP if and only if it is a two-element chain. We also show that the class of all congruence modular algebras with the WCIP is closed under the formation of homomorphic images.

Keywords: CEP, WCIP, weak congruence, lattice

MSC 2000: 08A30

1. Preliminaries

Let $\mathbf{A} = (A, F)$ be an algebra. A weak congruence relation α on $\mathbf{A}$ ([7]) is a symmetric transitive binary relation on A satisfying the usual substitution property and a weak reflexivity: if c is a constant in $\mathbf{A}$, then $c\alpha c$. For a given algebra $\mathbf{A}$, let $\mathrm{Con}\,(\mathbf{A})$, $\mathrm{Sub}\,(\mathbf{A})$ and $C_W(\mathbf{A})$ be the sets of all congruences, subalgebras and weak congruences on $\mathbf{A}$, respectively.

It was proved in [7] that $(C_W(\mathbf{A}), \leqslant)$ is an algebraic lattice.

For $\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$, let $\Delta_B = B^2 \wedge \Delta$, where $\Delta = \{(x, x) \colon x \in A\}$ (diagonal). If $\alpha \in C_W(\mathbf{A})$ and $\alpha \leqslant B^2$, then we set $\alpha_B = \bigcap\{\beta \in \mathrm{Con}\,(\mathbf{B}) \colon \alpha \leqslant \beta\}$. Note that $\alpha_B = \alpha \vee \Delta_B$.

Now we define our basic concepts. An algebra $\mathbf{A}$ is said to have the CEP if every congruence of an arbitrary subalgebra of $\mathbf{A}$ is a restriction of a congruence of $\mathbf{A}$. We recall the following characterization of the CEP.

Lemma 1 (cf. [5], Lemma 8)**.** *An algebra* **A** *has the CEP if and only if for every* $\alpha \in C_W(\mathbf{A})$ *and* $\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$ *if* $\alpha \leqslant B^2$, *then* $\alpha_B = \alpha_A \wedge B^2$.

We will say that **A** has the WCIP ([3], [4] and [6]) if for all $\alpha \in C_W(\mathbf{A})$ and $\beta \in \mathrm{Con}(\mathbf{A})$, $(\alpha \wedge \beta)_A = \alpha_A \wedge \beta$. The WCIP is a weakened congruence intersection property (CIP). **A** has the CIP if for all $\alpha, \beta \in C_W(\mathbf{A})$, $(\alpha \wedge \beta)_A = \alpha_A \wedge \beta_A$. It is obvious that the CIP implies the WCIP, since $\beta_A = \beta$ for all $\beta \in \mathrm{Con}(\mathbf{A})$.

To the class of all algebras which satisfy the CEP and the WCIP belong, for example, unary algebras (see [6]). It was proved in [2] that if **G** is a Hamiltonian group (i.e., if all subgroups of **G** are normal), then **G** has the CEP and the CIP, and therefore it has also the WCIP. It is easy to see that every simple algebra has the WCIP. Hence each field satisfies the WCIP. Obviously, fields have the CEP.

2. Subalgebras of algebras with the CEP and the WCIP

In this part we prove that the class of all algebras which satisfy the CEP and the WCIP is closed under the formation of subalgebras.

Theorem 1. *If an algebra* **A** *has the CEP and the WCIP, then every subalgebra of* **A** *has these properties.*

Proof. Let **B** be a subalgebra of **A**. It is clear that **B** has the CEP. To prove that **B** possesses the WCIP it is sufficient to show that for every $\alpha \in C_W(\mathbf{A})$ and $\beta \in \mathrm{Con}(\mathbf{B})$,

$$(\alpha \wedge \beta)_B = \alpha_B \wedge \beta. \tag{1}$$

From Lemma 1 it follows that $\beta = \beta_A \wedge B^2$. It is easily seen that $\alpha \wedge \beta = \alpha \wedge \beta_A$. Since **A** has the WCIP we have $(\alpha \wedge \beta)_A = (\alpha \wedge \beta_A)_A = \alpha_A \wedge \beta_A$. Hence

$$(\alpha \wedge \beta)_A \wedge B^2 = \alpha_A \wedge \beta_A \wedge B^2. \tag{2}$$

Since **A** has the CEP, we can apply Lemma 1 arriving at $\alpha_B = \alpha_A \wedge B^2$ and $(\alpha \wedge \beta)_B = (\alpha \wedge \beta)_A \wedge B^2$. From this and (2) we obtain (1). Indeed, $(\alpha \wedge \beta)_B = (\alpha \wedge \beta)_A \wedge B^2 = \alpha_A \wedge \beta_A \wedge B^2 = \alpha_B \wedge \beta_A \wedge B^2 = \alpha_B \wedge \beta$. Thus **B** has the WCIP, and the proof is complete. □

Remark. The assumption of the CEP cannot be dropped in the previous theorem. Indeed, any simple lattice of more than three elements has the WCIP, as it is simple, but it contains a three-element chain as a sublattice, which fails the WCIP (see the proof of Proposition 1).

Proposition 1. *A lattice* $\mathbf{L} = (L, \wedge, \vee)$ *satisfies the CEP and the WCIP if and only if* $\mathbf{L}$ *is a two-element chain.*

P r o o f. Let $\mathbf{L}$ have the CEP and the WCIP. Suppose that $|L| > 2$. Then $\mathbf{L}$ contains a three-element chain $\mathbf{C_3}$. The chain $\mathbf{C_3}$ has the lattice of weak congruences, given in Figure 1.

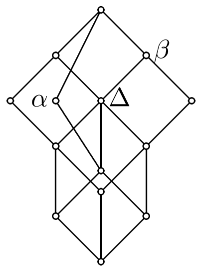

Fig. 1

It is easy to see that $(\alpha \wedge \beta) \vee \Delta \neq (\alpha \vee \Delta) \wedge \beta$, and therefore, $\mathbf{C_3}$ does not satisfy the WCIP, contrary to Theorem 1. The converse is obvious. $\square$

It is known (see [1]) that every distributive lattice has the CEP. Then Proposition 1 implies

Corollary 1. *A distributive lattice satisfies the WCIP if and only if it is a two-element chain.*

3. Factor algebras with the CEP and the WCIP

Let $\mathbf{A}$ be an algebra and $\theta \in \mathrm{Con}\,(\mathbf{A})$. For $B \subseteq A$ we define two sets: $B_\theta = \{b/\theta\colon\ b \in B\}$ and $B[\theta] = \{x \in A\colon\ x\theta b \text{ for some } b \in B\}$. Obviously, $\mathrm{Sub}(\mathbf{A}/\theta) = \{\mathbf{B}_\theta\colon\ \mathbf{B} \in \mathrm{Sub}(\mathbf{A}) \text{ and } B = B[\theta]\}$.

It is well-known that the lattice $\mathrm{Con}(\mathbf{A}/\theta)$ is isomorphic with the filter $[\theta)$ in $\mathrm{Con}\,(\mathbf{A})$. Therefore, for $\mathbf{B}_\theta \in \mathrm{Sub}(\mathbf{A}/\theta)$ we have that $\mathrm{Con}(\mathbf{B}_\theta) \cong [B^2 \wedge \theta, B^2]$, where $[B^2 \wedge \theta, B^2]$ is the interval sublattice of $C_W(\mathbf{A})$. Consequently, $C_W(\mathbf{A}/\theta)$ is (up to an isomorphism) $\bigcup\{[B^2 \wedge \theta, B^2]\colon\ \mathbf{B} \in \mathrm{Sub}(\mathbf{A}) \text{ and } B = B[\theta]\}$.

According to the above remark, we obtain the following

Lemma 2. *Let* $\mathbf{A}$ *be an algebra and let* $\theta \in \mathrm{Con}\,(\mathbf{A})$*. Then*

(i) $\mathbf{A}/\theta$ *has the CEP if and only if for every* $\mathbf{B} \in \mathrm{Sub}\,(\mathbf{A})$ *satisfying* $B = B[\theta]$ *and for every* $\alpha \in [B^2 \wedge \theta, B^2]$, $\alpha = (\alpha \vee \theta) \wedge B^2$.

(ii) $\mathbf{A}/\theta$ *has the WCIP if and only if for every* $\alpha \in [B^2 \wedge \theta, B^2]$ ($\mathbf{B} \in \mathrm{Sub}\,(\mathbf{A})$, $B = B[\theta]$) *and* $\beta \in [\theta)$, $(\alpha \wedge \beta) \vee \theta = (\alpha \vee \theta) \wedge \beta$.

Theorem 2. *The following assertions are equivalent for an algebra* $\mathbf{A}$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$*:*

(i) $\mathbf{A}/\theta$ *has the CEP and the WCIP.*

(ii) *For* $\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$ *such that* $B = B[\theta]$, $[B^2 \wedge \theta, B^2] \cong [\theta, B^2 \vee \theta]$ *under* $f_B\colon\ \beta \to \beta \vee \theta$.

Proof. (i) $\Longrightarrow$ (ii). We first prove that f_B is a homomorphism from $[B^2 \wedge \theta, B^2]$ into $[\theta, B^2 \vee \theta]$. It suffices to show that $(\alpha \wedge \beta) \vee \theta = (\alpha \wedge \theta) \vee (\beta \wedge \theta)$ for $\alpha, \beta \in [B^2 \wedge \theta, B^2]$. By the CEP, $\beta = (\beta \vee \theta) \wedge B^2$. Consequently, $(\alpha \wedge \beta) \vee \theta = [\alpha \wedge (\beta \vee \theta) \wedge B^2] \vee \theta = [\alpha \wedge (\beta \vee \theta)] \vee \theta = (\alpha \vee \theta) \wedge (\beta \vee \theta)$, since $\mathbf{A}/\theta$ has the WCIP. Let $\gamma \in [\theta, B^2 \vee \theta]$. By the WCIP, $f_B(B^2 \wedge \gamma) = (B^2 \wedge \gamma) \vee \theta = (B^2 \vee \theta) \wedge \gamma = \gamma$. Therefore, f_B is a homomorphism from $[B^2 \wedge \theta, B^2]$ onto $[\theta, B^2 \vee \theta]$. Now, if $\alpha \vee \theta = \beta \vee \theta$ ($\alpha, \beta \in [B^2 \wedge \theta, B^2]$), then $(\alpha \vee \theta) \wedge B^2 = (\beta \vee \theta) \wedge B^2$. The equality $\alpha = \beta$ follows from Lemma 2(i). Thus $[B^2 \wedge \theta, B^2] \cong [\theta, B^2 \vee \theta]$.

(ii) $\Longrightarrow$ (i). Let $\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$ satisfy $B = B[\theta]$, and let $\alpha \in [B^2 \wedge \theta, B^2]$. Since $(\alpha \vee \theta) \wedge B^2 \leqslant \alpha \vee \theta$, we have

$$[(\alpha \vee \theta) \wedge B^2] \vee \theta \leqslant \alpha \vee \theta. \tag{3}$$

On the other hand, $\alpha \leqslant (\alpha \vee \theta) \wedge B^2$ and therefore, $\alpha \vee \theta \leqslant [(\alpha \vee \theta) \wedge B^2] \vee \theta$. From this and (3) we obtain $[(\alpha \vee \theta) \wedge B^2] \vee \theta = \alpha \vee \theta$. As f_B is an injection from $[B^2 \wedge \theta, B^2]$ into $[\theta, B^2 \vee \theta]$, we get $(\alpha \vee \theta) \wedge B^2 = \alpha$. From Lemma 2(i) it follows that $\mathbf{A}/\theta$ has the CEP.

We now prove that $\mathbf{A}/\theta$ also possesses the WCIP. Let $\alpha \in [B^2 \wedge \theta, B^2]$ ($\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$ and $B = B[\theta]$) and $\beta \in [\theta)$. We consider two cases.

Case 1. $\beta \leqslant B^2 \vee \theta$.

Since $\beta \in [\theta, B^2 \vee \theta]$ and f_B is a surjection, there exists $\gamma \in [B^2 \wedge \theta, B^2]$ such that $\gamma \vee \theta = \beta$. As f_B is a homomorphism we have $(\alpha \wedge \beta) \vee \theta \geqslant (\alpha \wedge \gamma) \vee \theta = (\alpha \vee \theta) \wedge (\gamma \vee \theta) = (\alpha \vee \theta) \wedge \beta$. Since the inequality $(\alpha \wedge \beta) \vee \theta \leqslant (\alpha \vee \theta) \wedge \beta$ is obvious, we get $(\alpha \wedge \beta) \vee \theta = (\alpha \vee \theta) \wedge \beta$.

Case 2. $\beta \not\leqslant B^2 \vee \theta$.

Then $\beta_1 = (\alpha \vee \theta) \wedge \beta \in [\theta, B^2 \vee \theta]$ and by previous consideration $(\alpha \wedge \beta_1) \vee \theta = (\alpha \vee \theta) \wedge \beta_1$. Hence, $(\alpha \wedge \beta) \vee \theta = (\alpha \wedge \beta_1) \vee \theta = (\alpha \vee \theta) \wedge \beta_1 = (\alpha \vee \theta) \wedge \beta$. Thus, for every $\alpha \in [B^2 \wedge \theta, B^2]$ ($\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$ and $B = B[\theta]$) and $\beta \in [\theta)$, $(\alpha \wedge \beta) \vee \theta = (\alpha \vee \theta) \wedge \beta$. From Lemma 2(ii) it follows that $\mathbf{A}/\theta$ has the WCIP. The proof is complete. □

4. Homomorphic images of algebras with the WCIP

An element a of a lattice $\mathbf{L}$ is said to be modular, if for all $x, y \in L$, $a \leqslant y$ implies $a \vee (x \wedge y) = (a \vee x) \wedge y$.

Theorem 3. *Let an algebra* $\mathbf{A}$ *have the WCIP and let* $\theta \in \mathrm{Con}(\mathbf{A})$ *be a modular element of* $\mathrm{Con}(\mathbf{A})$. *Then* $\mathbf{A}/\theta$ *has the WCIP.*

P r o o f. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ such that $B = B[\theta]$. Let $\alpha \in [B^2 \wedge \theta, B^2]$ and $\beta \in [\theta)$. Since $\mathbf{A}$ has the WCIP, we have $(\alpha \wedge \beta) \vee \theta = (\alpha \wedge \beta)_A \vee \theta = (\alpha_A \wedge \beta) \vee \theta$. By assumption, θ is modular in $\mathrm{Con}(\mathbf{A})$. Then $(\alpha_A \wedge \beta) \vee \theta = (\alpha_A \vee \theta) \wedge \beta$. Consequently, $(\alpha \wedge \beta) \vee \theta = (\alpha_A \vee \theta) \wedge \beta = (\alpha \vee \theta) \wedge \beta$. From Lemma 2(ii) it follows that $\mathbf{A}/\theta$ has the WCIP. □

We recall that an algebra $\mathbf{A}$ is called congruence modular if $\mathrm{Con}\,(\mathbf{A})$ is a modular lattice. Since every homomorphic image of $\mathbf{A}$ is isomorphic with $\mathbf{A}/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$, Theorem 3 gives

Corollary 2. *If* $\mathbf{A}$ *is a congruence modular algebra, then the WCIP is hereditary for homomorphic images of* $\mathbf{A}$.

Congruence modular algebras include groups, rings, modules, quasigroups, Heyting algebras and lattices. In the case of groups, Corollary 2 implies

Corollary 3. *If a group* $\mathbf{G}$ *possesses the WCIP, then so does each homomorphic image of* $\mathbf{G}$.

We know (see Theorem 5 of [3]) that if an algebra $\mathbf{A}$ has the WCIP and $\mathrm{Con}(\mathbf{A})$ is modular, then the CEP is hereditary for homomorphic images of $\mathbf{A}$. Now, combining Theorem 1 with Corollary 2 we obtain

Corollary 4. *The class of all congruence modular algebras satisfying the CEP and the WCIP is closed under the formation of homomorphic images and subalgebras.*

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