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# NORMAL VIETORIS IMPLIES COMPACTNESS: A SHORT PROOF

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*Abstract.* One of the most celebrated results in the theory of hyperspaces says that if the Vietoris topology on the family of all nonempty closed subsets of a given space is normal, then the space is compact (Ivanova-Keesling-Velichko). The known proofs use cardinality arguments and are long. In this paper we present a short proof using known results concerning Hausdorff uniformities.

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Suppose $(X,\tau)$ is a $T_1$ space and $CL(X)$, the family of all nonempty closed subsets of $X$, is assigned the Vietoris topology $\tau_V$. Suppose $(CL(X),\tau_V)$ is normal. One of the most spectacular results in Hyperspaces due to Ivanova, Keesling and Velichko ([4], [6] and [8]) implies that $(X,\tau)$ is compact. In this paper we provide an alternative short proof using some recent results in Hyperspaces.

We use the notation

$V^+ = \{E \in CL(X)\colon\ E \subset V\}$,

$V^- = \{E \in CL(X)\colon\ E \cap V \neq \emptyset\}$,

for $\mathcal{A} \subset \tau$, $\mathcal{A}^- = \bigcap\{V^-\colon\ V \in \mathcal{A}\}$.

The Vietoris topology $\tau_V$ is generated by sets of the form $\{V^+\colon\ V \in \tau\}$ and $\mathcal{A}^-$ where $\mathcal{A} \subset \tau$ is finite ([1]).

Let $\mathcal{U}$ be a compatible uniformity on $X$ ([3]). For each $U \in \mathcal{U}$, let $\hat{U} = \{(A,B)\colon A,B \in CL(X),\ A \subseteq U[B]$ and $B \subseteq U[A]\}$. Then, $\{\hat{U}\colon\ U \in \mathcal{U}\}$ is a base for a uniformity $\mathbf{U}_H$ on $CL(X)$ called the Hausdorff uniformity associated with $\mathcal{U}$ ([7], [2]).

We note the following:

(a) Since $X$ is embedded in $(CL(X), \tau_V)$ as a closed subset, $(X, \tau)$ itself is normal.

(b) Each real valued continuous function $f$ on a space gives rise to a continuous pseudometric $d_f(x, y) = |f(x) - f(y)|$.

(c) The finest totally bounded uniformity $\mathcal{U}_0$ on the normal space $X$ is generated by pseudometrics arising from all the members of $C^\star(X)$ (the set of all continuous functions $f$ from $X$ to the real interval $[0, 1]$). Moreover, the Hausdorff uniformity $\mathbf{U}_{0H}$ on $CL(X)$ associated with $\mathcal{U}_0$ is compatible with the Vietoris topology $\tau_V$ ([2]).

(d) If $\mathcal{F}$ is a nonconvergent ultrafilter, then each $F \in \mathcal{F}$ has more than one point (otherwise it would be a principal ultrafilter; a contradiction).

Proof. Suppose $(CL(X), \tau_V)$ is normal but not compact. Then it has a nonconvergent ultrafilter $\mathcal{F}$ which is Cauchy with respect to $\mathbf{U}_{0H}$. Choose distinct elements $\{x_F, y_F\}$ from each element $F \in \mathcal{F}$. Then $\{(x_F, y_F)\colon F \in \mathcal{F}\}$ is a Cauchy net with respect to $\mathbf{U}_{0H}$. Obviously $A = \{x_F\colon F \in \mathcal{F}\}$ and $B = \{y_F\colon F \in \mathcal{F}\}$ are disjoint closed sets in $X$ and so there is a continuous function $f\colon X \to [0, 1]$ with $f(A) = 0$ and $f(B) = 1$. This shows that the net $\{(x_F, y_F)\colon F \in \mathcal{F}\}$ is not *small* (see [3]) with respect to the entourage in $\mathbf{U}_{0H}$ corresponding to the pseudometric $d_f$ on $X$; a contradiction.

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