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ZERO-TERM RANKS OF REAL MATRICES AND THEIR PRESERVERS

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Abstract. Zero-term rank of a matrix is the minimum number of lines (rows or columns) needed to cover all the zero entries of the given matrix. We characterize the linear operators that preserve zero-term rank of the $m \times n$ real matrices. We also obtain combinatorial equivalent condition for the zero-term rank of a real matrix.

Keywords: linear operator, zero-term rank, (P, Q, B)-operator

MSC 2000: 15A03, 15A04

1. Introduction and preliminaries

There are many papers on the research of linear operators on matrices that preserve certain matrix functions. But there are few papers on zero-term rank of real matrices. Recently Beasley, Song and Lee [2] obtained characterizations of zero-term rank preservers of matrices over anti-negative semirings.

In this article, we obtain characterizations of the linear operators that preserve zero-term rank of real matrices.

Let $M_{m,n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{R}$, the real numbers. Let $\mathbb{B} = \{0,1\}$ be the Boolean algebra. For a real matrix $A = [a_{ij}]$, let $\overline{A} = [\overline{a_{ij}}]$ denote the matrix with entries in $\mathbb{B}$ such that $\overline{a_{ij}} = 0$ if and only if $a_{ij} = 0$. Let E_{ij} be the $m \times n$ real matrix which has a 1 in the (i,j)-entry and is zero elsewhere. We call E_{ij} a *cell*. Let J denote the $m \times n$ matrix all of whose entries are 1. A matrix A is said to *dominate* matrix $B = [b_{ij}]$ if $a_{ij} = 0$ implies that $b_{ij} = 0$

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and we write $A \geqslant B$. If $A \geqslant B$ and there is some pair (i, j) such that $a_{ij} \neq 0$ but $b_{ij} = 0$, then we write $A > B$.

The *zero-term rank* [3] of a matrix A, $z(A)$, is the minimum number of lines (row or columns) needed to cover all the zero entries of A. Of course, the *term rank* [1] of A, $t(A)$, is defined similarly for all the nonzero entries of A.

Let $T\colon M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$ be a linear operator. Say that

(i) *T preserves zero-term rank k* if $z(T(A)) = k$ whenever $z(A) = k$ for all A in $M_{m,n}(\mathbb{R})$;

(ii) *T preserves zero-term rank* if it preserves zero-term rank k for every $k \leqslant \min\{m, n\}$.

Which linear operators over $M_{m,n}(\mathbb{R})$ preserve zero-term rank? The operations of (1) permuting rows, (2) permuting columns and (3) (if $m = n$) transposing the matrices in $M_{m,n}(\mathbb{R})$ are all linear, zero-term rank preserving operators on $M_{m,n}(\mathbb{R})$.

If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbb{R})$, all of whose entries are nonzero real numbers, then its *Schur product* $A \circ B = [a_{ij}b_{ij}]$ with A has the same zero-term rank as does A. The operator $A \mapsto A \circ B$ is linear. Similarly $A \mapsto B \circ A$ is linear, zero-term rank preserving operator. That these operations and their compositions are the only zero-term rank preservers is one of the consequence of Theorem 2.4 below.

Let $M_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. If $T\colon M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$ is a linear operator, define $\overline{T}\colon M_{m,n}(\mathbb{B}) \to M_{m,n}(\mathbb{B})$ by

$$\overline{T}(\overline{A}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})}$$

for any $A \in M_{m,n}(\mathbb{R})$.

A semiring $\mathbb{S}$ which has no zero-divisors and which has the property that for $a, b \in \mathbb{S}$, $a + b = 0$ implies that $a = b = 0$ is called *an anti-negative semiring.*

A linear operator $T\colon M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$ is called a *(P, Q, B)-operator* if there exist permutation matrices P and Q, and a matrix B, all of whose entries are nonzero, such that $T(A) = P(A \circ B)Q$ for all $A \in M_{m,n}(\mathbb{R})$ or if $m = n$, $T(A) = P(A \circ B)^t Q$ for all $A \in M_{m,n}(\mathbb{R})$.

In [1], Beasley and Pullman characterized the term rank preservers of matrices over semirings. And in [2], the linear operators that preserve zero-term rank over anti-negative semirings were shown to be (P, Q, B)-operators.

We now state the result for later reference.

Theorem 1.1 [2]. *If $\mathbb{S}$ is any anti-negative semiring, and T is a linear operator on the $m \times n$ matrices with entries in $\mathbb{S}$, then the following statements are equivalent:*
(i) *T preserves zero-term rank;*
(ii) *T preserves zero-term ranks 0 and 1;*
(iii) *T is a (P, Q, B)-operator.*

2. Linear operators that preserve zero-term rank of real matrices

In this section, we assume that T is a linear operator on $M_{m,n}(\mathbb{R})$ with $m > 1$, $n > 1$.

Let $\|A\|$ denote the number of nonzero entries of A. We begin with some lemmas.

Lemma 2.1. *If T preserves zero-term rank 1, then there exists $C \in M_{m,n}(\mathbb{R})$ such that $\|T(C)\| = mn$.*

Proof. Choose $C \in M_{m,n}(\mathbb{R})$ such that $T(C) \geqslant T(A)$ for all $A \in M_{m,n}(\mathbb{R})$. Suppose that $\|T(C)\| \neq mn$. Then, for some (s,t), $T(A) \circ E_{st} = 0$, for all $A \in M_{m,n}(\mathbb{R})$. By permuting rows and columns, we may assume that $(s,t) = (1,1)$. Also we assume that $\overline{C} = J$, so that $z(C) = 0$. Let E_{hk} be a cell such that $T(E_{hk})$ has a nonzero (p,q) entry with $p, q \geqslant 2$. If no such cell existed, then we obtain that $z(C - c_{ij}E_{ij}) = 1$ for every cell E_{ij} but

$$z(T(C - c_{ij}E_{ij})) = \min\{m, n\},$$

a contradiction. Now, for $T(E_{hk}) = D = (d_{ij})$, we have that

$$z\biggl(T\biggl(C - \frac{T(C)_{pq}}{d_{pq}}E_{hk}\biggr)\biggr) \geqslant 2, \quad \text{and} \quad z\biggl(C - \frac{T(C)_{pq}}{d_{pq}}E_{hk}\biggr) \leqslant 1.$$

Thus, we must have $z\Bigl(C - \frac{T(C)_{pq}}{d_{pq}}E_{hk}\Bigr) = 0$, since T preserves zero-term rank 1. Let $F = (f_{ij}) = C - \frac{T(C)_{pq}}{d_{pq}}E_{hk}$. If $T(E_{uv})_{pq} = 0$ for some cell E_{uv}, then $z(F - f_{uv}E_{uv}) = 1$, while $z(T(F - f_{uv}E_{uv})) = z(T(F) - f_{uv}T(E_{uv})) \geqslant 2$, which is a contradiction. Thus $T(E_{ij})_{pq} \neq 0$ for all cells E_{ij}.

If $T(E_{11}) = X = (x_{ij})$ and $T(E_{12}) = Y = (y_{ij})$, then

$$T\biggl(F - f_{11}E_{11} + \biggl(\frac{f_{11}x_{pq}}{y_{pq}}\biggr)E_{12}\biggr)$$

has zeros in the $(1,1)$ and (p,q) entries, and hence has zero term rank at least 2, while

$$z\biggl(F - f_{11}E_{11} + \biggl(\frac{f_{11}x_{pq}}{y_{pq}}\biggr)E_{12}\biggr) = 1,$$

a contradiction. Thus $\|T(C)\| = mn$. □

Lemma 2.2. *If T preserves zero-term rank 1, then T maps each cell to a nonzero multiple of some cell which induces a bijection on the set of indices $\{1,2,\dots,m\} \times \{1,2,\dots,n\}$.*

Proof. By Lemma 2.1, there exist $C \in M_{m,n}(\mathbb{R})$ such that $\|T(C)\| = mn$. Suppose that there is some cell E_{ij} such that $\|T(E_{ij})\| > 1$. If $\|T(E_{ij}\| \neq mn$, then there exists a pair (h,k) such that $(h,k) \neq (i,j)$ and for some nonzero real number r_{hk},

$$T(E_{ij} + r_{hk}E_{hk}) > T(E_{ij}).$$

Let $D_1 = E_{ij} + r_{hk}E_{hk}$. If $\|T(D_1)\| \neq mn$, then there is some cell E_{pq} such that for some nonzero real number r_{pq}, $T(D_1 + r_{pq}E_{pq}) > T(D_1)$. Continuing this process, we have a matrix $D = (d_{ij})$ such that $\|D\| < mn$ while $\|T(D)\| = mn$. Since $\|D\| < mn$, we may assume $d_{11} = 0$ without loss of generality. Let F be the $(0,1)$-matrix in $M_{m,n}(\mathbb{R})$ such that $f_{11} = 0$ and for $(i,j) \neq (1,1)$, $f_{ij} = 0$ if and only if $d_{ij} \neq 0$. Thus, for some sufficiently small positive real number r, we have

$$\|D + rF\| = mn - 1 \quad \text{and} \quad \|T(D + rF)\| = mn.$$

That is,

$$z(D + rF) = 1 \quad \text{and} \quad z(T(D + rF)) = 0.$$

This is a contradiction. If $\|T(E_{ij})\| = mn$, then we can take $D = E_{ij}$ in the above case and obtain the same contradiction. Thus $\|T(E_{ij})\| \leqslant 1$ for all cells E_{ij}. If $T(E_{ij}) = 0$ for some cell E_{ij}, then the fact that $\|T(C)\| = mn$ implies $\|T(E_{pq})\| \geqslant 2$ for some $(p \cdot q)$, which is a contradiction. That is, T is bijective on the set of indices $\{1,2,\dots,m\} \times \{1,2,\dots,n\}$. □

Theorem 2.3. *If T preserves zero-term rank 1, then T is a (P,Q,B)-operator.*

Proof. By Lemma 2.2, T is bijective on the set of indices $\{(i,j) \mid i = 1,\dots,m,\ j = 1,\dots,n\}$. Thus, for any A in $M_{mn}(\mathbb{R})$,

$$\overline{T(A)} = \overline{\sum_{i=1}^{m}\sum_{j=1}^{n} T(a_{ij}E_{ij})} = \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{T(a_{ij}E_{ij})} = \overline{T}(\overline{A}).$$

This shows that $\overline{T}$ preserves zero-term rank 1 since T does also. By Theorem 1.1, $\overline{T}$ is a (P,Q,B)-operator, where $B = J$. Thus, the mapping $\overline{A} \mapsto \overline{P^tT(A)Q^t}$ is the identity linear operator on $M_{m,n}(\mathbb{B})$. That is, $P^tT(E_{ij})Q^t = b_{ij}E_{ij}$ for each pair (i,j) (or perhaps $P^tT(E_{ij})Q^t = b_{ij}E_{ji}$ in the case $m = n$). Then, $T(C) = P(C \circ B)Q$ for all $C \in M_{m,n}(\mathbb{R})$ or $m = n$ and $T(C) = P(C \circ B)^tQ$ for all $C \in M_{m,n}(\mathbb{R})$. □

Now, we obtain the characterizations of the linear operators that preserve zero-term rank of real matrices.

Theorem 2.4. *For a linear operator $T\colon M_{m,n}(\mathbb{R}) \to M_{m,n}(\mathbb{R})$, the following are equivalent:*

(i) *T preserves zero-term rank;*

(ii) *T preserves zero-term rank 1;*

(iii) *T is a (P, Q, B)-operator.*

Proof. Obviously (i) implies (ii) and (iii) implies (i). By Theorem 2.3, we have that (ii) implies (iii). □

3. Combinatorial characterization of zero-term rank

In this section, we obtain an equivalent condition for the zero-term rank. A minimal covering of the zeros of A is called *proper* provided that it does not consist of all m rows of A or of all n columns of A.

Theorem 3.1. *Let A be an $m \times n$ real matrix. Then the zero-term rank of A is equal to the maximal number of zeros in A with no two of the zeros on a line.*

Proof. We prove this equality by induction on the number of lines in A. For the case that $m = 1$ or $n = 1$, the equality holds. Hence we take $m > 1$ and $n > 1$. Let $z(A) = p$ and q denote the maximal number of zeros in A with no two of the zeros on a line. Then the definition of zero-term rank implies that $q \leqslant p$. Hence it suffices to show that $q \geqslant p$. Consider two cases :

Case 1) Assume that A does not have a proper covering. Then we must have $p = \min\{m, n\}$. We permute the lines of A so that the permuted matrix B has a zero in the $(1, 1)$ position. We delete row 1 and column 1 of the permuted matrix B and denote the resulting matrix of size $m-1$ by $n-1$ by $B(1|1)$. The matrix $B(1|1)$ cannot have a covering composed of fewer than $p - 1 = \min\{m-1, n-1\}$ lines because such a covering of $B(1|1)$ plus the two deleted lines would yield a proper covering for A. We now apply the induction hypothesis to $B(1|1)$ and this allows us to conclude that $B(1|1)$ has $p-1$ zeros with no two of the zeros on a line. But then A has p zeros with no two of the zeros on a line and it follows that $q \geqslant p$.

Case 2) Assume that A has a proper covering composed of e rows and f columns where $p = e + f$. We permute lines of A so that these e rows and f columns occupy the left-upper positions of the permuted matrix B. Then B assumes the following form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

In this decomposition B_{22} is the $(m-e) \times (n-f)$ submatrix with all nonzero entries. The matrix B_{12} has e rows and cannot be covered by fewer than e lines and the matrix B_{21} has f columns and cannot be covered by fewer than f lines. This is the case because otherwise we contradict the fact that $p = e + f$ is the minimal number of lines in A that cover all of the zeros on A. We may apply the induction hypothesis to both A_1 and A_2 and this allows us to conclude that $q \geqslant p$. □

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