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ON UNIFORMLY LOCALLY COMPACT QUASI-UNIFORM HYPERSPACES

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Abstract. We characterize those Tychonoff quasi-uniform spaces (X, \mathcal{U}) for which the Hausdorff-Bourbaki quasi-uniformity is uniformly locally compact on the family $\mathcal{K}_0(X)$ of nonempty compact subsets of X . We deduce, among other results, that the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity of a Tychonoff space X is uniformly locally compact on $\mathcal{K}_0(X)$ if and only if X is paracompact and locally compact. We also introduce the notion of a co-uniformly locally compact quasi-uniform space and show that a Hausdorff topological space is σ -compact if and only if its (lower) semi-continuous quasi-uniformity is co-uniformly locally compact. A characterization of those Hausdorff quasi-uniform spaces (X, \mathcal{U}) for which the Hausdorff-Bourbaki quasi-uniformity is co-uniformly locally compact on $\mathcal{K}_0(X)$ is obtained.

Keywords: Hausdorff-Bourbaki quasi-uniformity, hyperspace, locally compact, cofinally complete, uniformly locally compact, co-uniformly locally compact

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper the letters \mathbb{R} , \mathbb{Z} and \mathbb{N} will denote the set of real numbers, the set of integer numbers and the set of positive integer numbers, respectively.

Our basic reference for quasi-uniform and quasi-metric spaces is [9], for general topology it is [6], and for hyperspaces it is [1].

Let us recall that a quasi-uniformity on a set X is a filter \mathcal{U} on $X \times X$ such that

- (i) for each $U \in \mathcal{U}$, $\{(x, x) : x \in X\} \subseteq U$;
- (ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

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A quasi-uniform space is a pair (X, \mathcal{U}) such that X is a (nonempty) set and \mathcal{U} is a quasi-uniformity on X .

Each quasi-uniformity \mathcal{U} on X generates a topology $\mathcal{T}(\mathcal{U}) = \{G \subseteq X: \text{for each } x \in G \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq G\}$, where, as usual, $U(x) = \{y \in X: (x, y) \in U\}$.

If \mathcal{U} is a quasi-uniformity on X then the filter $\mathcal{U}^{-1} = \{U^{-1}: U \in \mathcal{U}\}$ is also a quasi-uniformity on X , called the conjugate of \mathcal{U} (as usual, $U^{-1} = \{(x, y): (y, x) \in U\}$). Furthermore, the filter $\mathcal{U}^s := \mathcal{U} \vee \mathcal{U}^{-1}$ is the coarsest uniformity on X finer than \mathcal{U} .

Given a topological space X we denote by $\mathcal{P}_0(X)$ (resp. $\mathcal{CL}_0(X)$, $\mathcal{K}_0(X)$), the family of all nonempty subsets (resp. nonempty closed subsets, nonempty compact subsets) of X . If (X, \mathcal{U}) is a quasi-uniform space, $\mathcal{CL}_0(X)$ (resp. $\mathcal{K}_0(X)$) denotes the family of all nonempty $\mathcal{T}(\mathcal{U})$ -closed subsets (resp. nonempty $\mathcal{T}(\mathcal{U})$ -compact subsets) of X .

The Hausdorff-Bourbaki quasi-uniformity of a quasi-uniform space (X, \mathcal{U}) is defined as the quasi-uniformity \mathcal{U}_* on $\mathcal{P}_0(X)$ which has as a base the family of sets of the form

$$U_H = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X): B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\},$$

whenever $U \in \mathcal{U}$ ([2], [14]).

The restriction of \mathcal{U}_* to $\mathcal{CL}_0(X)$ and to $\mathcal{K}_0(X)$ is also denoted by \mathcal{U}_* if no confusion arises.

A filter \mathcal{F} on a (quasi-)uniform space (X, \mathcal{U}) is weakly Cauchy ([5], [8], [9]) provided that for each $U \in \mathcal{U}$, $\bigcap_{F \in \mathcal{F}} U^{-1}(F) \neq \emptyset$. A (quasi-)uniformity \mathcal{U} on a set X is called cofinally complete ([7], [13]) if every weakly Cauchy filter on (X, \mathcal{U}) has a cluster point. In this case, we say that (X, \mathcal{U}) is a cofinally complete quasi-uniform space.

It is well known that a Tychonoff space is paracompact if and only if its fine uniformity is cofinally complete ([5], [7], [8]).

In our context a topological space X will be called locally compact if each point of X has a neighborhood whose closure is compact.

A (quasi-)uniformity \mathcal{U} on a set X is said to be uniformly locally compact ([8], [9]) if there is $U \in \mathcal{U}$ such that for each $x \in X$, $\overline{U(x)}$ is compact. In this case, we say that (X, \mathcal{U}) is a uniformly locally compact (quasi-)uniform space.

It is well known ([7], [8], [9]) that a Tychonoff (quasi-)uniform space is uniformly locally compact if and only if it is locally compact and cofinally complete, and, hence, the fine uniformity of a Tychonoff space X is uniformly locally compact if and only if X is paracompact and locally compact.

Several authors have discussed the preservation of local compactness and uniform local compactness by the Vietoris topology and the Hausdorff-Bourbaki (quasi-)uniformity, respectively. Thus, Michael proved in [15] (see also [4]) that a Tychonoff space X is locally compact if and only if the Vietoris topology of X is locally compact on $\mathcal{K}_0(X)$. In [3] Burdick characterized uniform local compactness of $(\mathcal{C}\mathcal{L}_0(X), \mathcal{U}_*)$ in the case that \mathcal{U} is a uniformity on X and showed that the Hausdorff-Bourbaki uniformity of the Euclidean uniformity on \mathbb{R} is not locally compact on $\mathcal{C}\mathcal{L}_0(X)$. However, its restriction to $\mathcal{K}_0(\mathbb{R})$ is uniformly locally compact because it was proved in [13] that a uniform space (X, \mathcal{U}) is uniformly locally compact if and only if $(\mathcal{K}_0(X), \mathcal{U}_*)$ is uniformly locally compact. In [13], p. 140, a uniformly locally compact quasi-uniform (actually, quasi-metric) space (X, \mathcal{U}) is constructed such that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is not cofinally complete and, thus, not uniformly locally compact.

These results and examples suggest the problem of characterizing those (Tychonoff) quasi-uniform spaces (X, \mathcal{U}) for which the Hausdorff-Bourbaki quasi-uniformity is uniformly locally compact on $\mathcal{K}_0(X)$. Here we solve this problem and deduce conditions under which the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity and of the (lower) semicontinuous quasi-uniformity, respectively, is uniformly locally compact on $\mathcal{K}_0(X)$. In particular, we prove that the Hausdorff-Bourbaki quasi-uniformity of the locally finite quasi-uniformity of a Tychonoff space X is uniformly locally compact on $\mathcal{K}_0(X)$ if and only if X is paracompact and locally compact. The quasi-metric case is also discussed. Finally, we introduce the notion of a co-uniformly locally compact quasi-uniform space and show that a Hausdorff space is σ -compact if and only if its semicontinuous quasi-uniformity is co-uniformly locally compact. A characterization of Hausdorff quasi-uniform spaces (X, \mathcal{U}) for which the Hausdorff-Bourbaki quasi-uniformity is co-uniformly locally compact on $\mathcal{K}_0(X)$ is obtained.

2. UNIFORM LOCAL COMPACTNESS OF $(\mathcal{K}_0(X), \mathcal{U}_*)$

The following lemmas will be useful in establishing our main result.

Lemma 1 ([8], [9]). *A quasi-uniform space is uniformly locally compact if and only if it is locally compact and cofinally complete.*

Let us recall that a quasi-uniformity \mathcal{U} on a set X is precompact ([9]) provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$. \mathcal{U} is said to be totally bounded ([9]) if \mathcal{U}^s is a totally bounded uniformity on X . Of course, \mathcal{U} is totally bounded if and only if \mathcal{U}^{-1} is so. It is well known that every totally bounded quasi-uniformity is precompact but the converse does not hold in general.

Lemma 2. *Let (X, \mathcal{U}) be a compact quasi-uniform space such that for each $K \in \mathcal{K}_0(X)$, $\mathcal{U}^{-1}|K$ is precompact. Then \mathcal{U}^{-1} is hereditarily precompact.*

Proof. Let A be a nonempty subset of X . Since (X, \mathcal{U}) is compact, the closure \overline{A} of A in $(X, \mathcal{F}(\mathcal{U}))$ belongs to $\mathcal{K}_0(X)$. So $\mathcal{U}^{-1}|\overline{A}$ is precompact. It immediately follows that $\mathcal{U}^{-1}|A$ is precompact. \square

Following [19], a filter \mathcal{F} on a quasi-uniform space (X, \mathcal{U}) is left K -Cauchy provided that for each $U \in \mathcal{U}$ there is $F_U \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F_U$. (X, \mathcal{U}) is said to be left K -complete if every left K -Cauchy filter is convergent ([19]).

Since every left K -Cauchy filter converges to all of its cluster points ([19]) and each left K -Cauchy filter is weakly Cauchy, every cofinally complete quasi-uniform space is left K -complete.

Lemma 3 ([11]). *Let (X, \mathcal{U}) be a T_1 quasi-uniform space and let A be a precompact subspace of (X, \mathcal{U}) . If $(\mathcal{K}_0(X), \mathcal{U}_*)$ is left K -complete, hence $\mathcal{U}^{-1}|A$ is precompact.*

Lemma 4. *Let (X, \mathcal{U}) be a T_1 quasi-uniform space. Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact if and only if (X, \mathcal{U}) is compact and \mathcal{U}^{-1} is hereditarily precompact.*

Proof. It is proved in [12] that if (X, \mathcal{U}) is a T_1 quasi-uniform space, then $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact if and only if (X, \mathcal{U}) is compact and \mathcal{U}^{-1} is hereditarily precompact, and it is proved in [11] that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact if and only if $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact. Combining these results we obtain the conclusion. \square

Theorem 1. *Let (X, \mathcal{U}) be a Tychonoff quasi-uniform space. Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is uniformly locally compact if and only if (X, \mathcal{U}) is uniformly locally compact and for each $K \in \mathcal{K}_0(X)$, $\mathcal{U}^{-1}|K$ is precompact.*

Proof. Suppose that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is uniformly locally compact. By [13], Proposition 3.1 and Remark 3.2, (X, \mathcal{U}) is uniformly locally compact. Now let $K \in \mathcal{K}_0(X)$. Then K is a precompact subspace of (X, \mathcal{U}) . Since, by Lemma 1, $(\mathcal{K}_0(X), \mathcal{U}_*)$ is cofinally complete and every cofinally complete quasi-uniform space is left K -complete, it follows from Lemma 3 that $\mathcal{U}^{-1}|K$ is precompact.

Conversely, by assumption there is $U \in \mathcal{U}$ such that for each $x \in X$, $\overline{U(x)}$ is compact. Let $V \in \mathcal{U}$ be such that $V^3 \subseteq U$. We shall prove that for each $K \in \mathcal{K}_0(X)$, $\overline{V_H(K)}$ is compact in $(\mathcal{K}_0(X), \mathcal{U}_*)$.

Let $K \in \mathcal{K}_0(X)$. We first show that $\overline{V(K)}$ is a compact subset of (X, \mathcal{U}) . Indeed, let $(z_\lambda)_{\lambda \in \Lambda}$ be a net in $\overline{V(K)}$. Then there is a finite subset $\{x_1, \dots, x_n\}$ of K such that $K \subseteq \bigcup_{i=1}^n V(x_i)$. On the other hand, for each $\lambda \in \Lambda$ there is $a_\lambda \in K$ with

$z_\lambda \in \overline{V^2(a_\lambda)}$. (In fact, fix $\lambda \in \Lambda$. Since $z_\lambda \in \overline{V(K)}$, for each $W \in \mathcal{U}$ there exists $b_W \in K$ such that $W(z_\lambda) \cap V(b_W) \neq \emptyset$; by compactness of K , the net $(b_W)_{W \in \mathcal{U}}$ has a cluster point $a_\lambda \in K$; so $z_\lambda \in \overline{V^2(a_\lambda)}$.)

Since $a_\lambda \in V(x_i)$ for some $i \in \{1, \dots, n\}$, we deduce that $V^2(a_\lambda) \subseteq V^3(x_i)$ and thus $\{z_\lambda: \lambda \in \Lambda\} \subseteq \bigcup_{i=1}^n \overline{V^3(x_i)}$. Hence, the net $(z_\lambda)_{\lambda \in \Lambda}$ is contained in the compact set $\bigcup_{i=1}^n \overline{U(x_i)}$, so it has a cluster point which obviously belongs to $\overline{V(K)}$. We conclude that $\overline{V(K)}$ is a compact subset of (X, \mathcal{U}) .

By our hypothesis, $\mathcal{U}^{-1}|K'$ is precompact for each $K' \in \mathcal{X}_0(X)$ and, in particular, for each $K' \in \mathcal{X}_0(\overline{V(K)})$. Therefore, by Lemma 2, $\mathcal{U}^{-1}| \overline{V(K)}$ is hereditarily precompact and by Lemma 4, $(\mathcal{X}_0(\overline{V(K)}), \mathcal{U}_*)$ is compact.

Finally, let $(K_\lambda)_{\lambda \in \Lambda}$ be a net in $\overline{V_H(K)}$. Then $K_\lambda \subseteq \overline{V(K)}$ and thus $K_\lambda \in \mathcal{X}_0(\overline{V(K)})$ for all $\lambda \in \Lambda$. Hence, there is $C \in \mathcal{X}_0(\overline{V(K)})$ which is a cluster point of $(K_\lambda)_{\lambda \in \Lambda}$ with respect to $\mathcal{T}(\mathcal{U}_*)$. Obviously $C \in \mathcal{X}_0(X)$ and $C \in \overline{V_H(K)}$. We have shown that $\overline{V_H(K)}$ is compact in $(\mathcal{X}_0(X), \mathcal{U}_*)$ for all $K \in \mathcal{X}_0(X)$. Hence $(\mathcal{X}_0(X), \mathcal{U}_*)$ is uniformly locally compact. \square

Corollary 1 ([13]). *Let (X, \mathcal{U}) be a Tychonoff uniform space. Then $(\mathcal{X}_0(X), \mathcal{U}_*)$ is uniformly locally compact if and only if (X, \mathcal{U}) is uniformly locally compact.*

Next we apply Theorem 1 to study uniform local compactness on $\mathcal{X}_0(X)$ of the Hausdorff-Bourbaki quasi-uniformity corresponding to some canonical quasi-uniformities of a Tychonoff space X .

Following [9] we denote by \mathcal{P} , \mathcal{LF} , \mathcal{PF} and \mathcal{SC} the Pervin quasi-uniformity, the locally finite quasi-uniformity, the point finite quasi-uniformity and the (lower) semicontinuous quasi-uniformity, respectively, of a (topological) space X .

Let us recall that a cover \mathcal{G} of a topological space X is said to be compact finite if each compact subset of X meets only finitely many members of \mathcal{G} .

Let \mathcal{A} be the collection of all compact finite open covers \mathcal{G} of a topological space X . For each $\mathcal{G} \in \mathcal{A}$ let $U_{\mathcal{G}} = \bigcup_{x \in X} (\{x\} \times \bigcap \{G \in \mathcal{G}: x \in G\})$. Then $\{U_{\mathcal{G}}: \mathcal{G} \in \mathcal{A}\}$ is a subbase for a transitive quasi-uniformity \mathcal{CF} for X , called the *compact finite* quasi-uniformity of X .

It is clear that $\mathcal{P} \subseteq \mathcal{LF} \subseteq \mathcal{CF} \subseteq \mathcal{PF}$.

On the other hand, observe that if X is a locally compact space then $\mathcal{LF} = \mathcal{CF}$.

Lemma 5. *Let X be a space. Then $\mathcal{CF}|K$ is totally bounded for all $K \in \mathcal{X}_0(X)$.*

Proof. Let \mathcal{A} be the collection of all finite open covers \mathcal{G} of X . For each $\mathcal{G} \in \mathcal{A}$ let $U_{\mathcal{G}} = \bigcup_{x \in X} (\{x\} \times \bigcap \{G \in \mathcal{G}: x \in G\})$. It is well known ([9]) that $\{U_{\mathcal{G}}: \mathcal{G} \in \mathcal{A}\}$

is a subbase for the Pervin quasi-uniformity \mathcal{P} . So we immediately deduce that for each $K \in \mathcal{K}_0(X)$, $\mathcal{CF}|K = \mathcal{P}|K$. Then the result follows from the well-known facts that \mathcal{P} is totally bounded and that total boundedness is a hereditary property. \square

Proposition 1. *Let X be a Tychonoff space. Then $(\mathcal{K}_0(X), \mathcal{CF}_*)$ is uniformly locally compact if and only if (X, \mathcal{CF}) is uniformly locally compact.*

Proof. It is an immediate consequence of Theorem 1 and Lemma 5. \square

Proposition 2. *Let X be a space. Then $\mathcal{PF} = \mathcal{CF}$ if and only if $\mathcal{PF}^{-1}|K$ is precompact for all $K \in \mathcal{K}_0(X)$.*

Proof. Suppose that $\mathcal{PF}^{-1}|K$ is precompact for all $K \in \mathcal{K}_0(X)$. Let $K \in \mathcal{K}_0(X)$, \mathcal{G} a point finite open cover of X and $U \in \mathcal{PF}$ be such that $U(x) = \bigcap \{G \in \mathcal{G} : x \in G\}$ for all $x \in X$. Then there exists a finite subset F of K such that $K \subseteq U^{-1}(F)$. Let $G_0 \in \mathcal{G}$ be such that $G_0 \cap K \neq \emptyset$, then there exists $x \in F$ such that $G_0 \cap U^{-1}(x) \neq \emptyset$. Let $y \in G_0 \cap U^{-1}(x)$. Then $x \in U(y) = \bigcap \{G \in \mathcal{G} : y \in G\}$, so $x \in G_0$. It follows that $F \cap G_0 \neq \emptyset$. Since \mathcal{G} is point finite and F is finite, \mathcal{G} is compact finite. We conclude that $\mathcal{PF} = \mathcal{CF}$.

The converse follows from Lemma 5. \square

Corollary 2. *Let X be a Tychonoff space such that $\mathcal{PF} \neq \mathcal{CF}$. Then $(\mathcal{K}_0(X), \mathcal{PF}_*)$ is not uniformly locally compact.*

Denote by \mathcal{FN} the fine uniformity of a Tychonoff space X . It then follows from Corollary 1 and results cited in Section 1 that $(\mathcal{K}_0(X), \mathcal{FN}_*)$ is uniformly locally compact if and only if X is paracompact and locally compact.

Here, we obtain the following result.

Proposition 3. *Let X be a Tychonoff space. Then $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is uniformly locally compact if and only if X is paracompact and locally compact.*

Proof. Suppose that $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is uniformly locally compact. By Theorem 1, (X, \mathcal{LF}) is uniformly locally compact. Therefore X is locally compact and \mathcal{LF} is cofinally complete by Lemma 1. Hence X is paracompact by the third corollary of Theorem 2.2 in [8] that states that \mathcal{LF} is cofinally complete if and only if X is paracompact.

Conversely, if X is a paracompact locally compact space, then \mathcal{LF} is cofinally complete, so by Lemma 1, \mathcal{LF} is a uniformly locally compact quasi-uniformity. Furthermore, since $\mathcal{LF} = \mathcal{CF}$, it follows from Lemma 5 that for each $K \in \mathcal{K}_0(X)$, $\mathcal{LF}^{-1}|K$ is precompact. Therefore $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is uniformly locally compact by Theorem 1. \square

Remark 1. a) Note that the proof of Proposition 3 shows that if X is a (Tychonoff) paracompact locally compact space, then $(\mathcal{K}_0(X), \mathcal{L}\mathcal{F}_*)$ is uniformly locally compact and $\mathcal{L}\mathcal{F} = \mathcal{C}\mathcal{F}$.

b) We also have (compare Proposition 1) that for a Tychonoff space X , $(\mathcal{K}_0(X), \mathcal{L}\mathcal{F}_*)$ is uniformly locally compact if and only if $(X, \mathcal{L}\mathcal{F})$ is uniformly locally compact.

c) Since a space is metacompact if and only if $\mathcal{P}\mathcal{F}$ is cofinally complete ([8]), the existence of metacompact locally compact spaces that are not paracompact shows that uniform local compactness of $(X, \mathcal{P}\mathcal{F})$ does not imply uniform local compactness of $(\mathcal{K}_0(X), \mathcal{P}\mathcal{F}_*)$ in general.

Lemma 6. *Let X be a locally compact Tychonoff space, and suppose that X is not discrete. Then there exists a compact subspace K of X such that $\mathcal{S}\mathcal{C}^{-1}|_K$ is not precompact.*

Proof. Let x be a non-isolated point of X and let K be a compact neighborhood of x . Then there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of open neighborhoods of x such that $G_n \subseteq K$ and $\overline{G_{n+1}} \subseteq G_n$ with $G_n \neq G_{n+1}$ for each $n \in \mathbb{N}$. Let $G = \bigcap_{n \in \mathbb{N}} G_n = \bigcap_{n \in \mathbb{N}} \overline{G_n}$, which is clearly a nonempty compact set, and let $A_{-n} = G_n \setminus G$ and $A_n = X$ for each $n \in \mathbb{N}$. It is clear that A_n is open and $A_n \subseteq A_{n+1}$ whenever $n \in \mathbb{Z}$, $\bigcup_{n \in \mathbb{Z}} A_n = X$ and $\bigcap_{n \in \mathbb{Z}} A_n = \emptyset$, and hence $\mathcal{A} = \{A_n : n \in \mathbb{Z}\}$ is an open spectrum. Let $U \in \mathcal{S}\mathcal{C}$ be such that $U(x) = \bigcap \{A \in \mathcal{A} : x \in A\}$ for all $x \in X$ (see [9], Theorem 2.12). It is easy to check that, for any finite subset F of X , $U^{-1}(F) = X \setminus A_n$ for some $n \in \mathbb{Z}$, and hence $\mathcal{S}\mathcal{C}^{-1}|_K$ is not precompact, since $A_n \cap K \neq \emptyset$ for each $n \in \mathbb{Z}$. □

Lemma 7 ([8], [9]). *A Tychonoff space is Lindelöf if and only if $\mathcal{S}\mathcal{C}$ is cofinally complete.*

Proposition 4. *Let X be a Tychonoff space. Then $(\mathcal{K}_0(X), \mathcal{S}\mathcal{C}_*)$ is uniformly locally compact if and only if X is discrete and countable.*

Proof. Suppose that X is discrete and countable. By Lemmas 1 and 7, $(X, \mathcal{S}\mathcal{C})$ is uniformly locally compact. Clearly $\mathcal{S}\mathcal{C}^{-1}|_K$ is precompact for all $K \in \mathcal{K}_0(X)$. Then $(\mathcal{K}_0(X), \mathcal{S}\mathcal{C}_*)$ is uniformly locally compact by Theorem 1.

Conversely, it follows from Theorem 1 that $(X, \mathcal{S}\mathcal{C})$ is uniformly locally compact and $\mathcal{S}\mathcal{C}^{-1}|_K$ is precompact for all $K \in \mathcal{K}_0(X)$. Therefore X is discrete and Lindelöf by Lemmas 6 and 7, so it is discrete and countable. □

Example 1. From Propositions 3 and 4 it follows that $(\mathcal{K}_0(\mathbb{R}), \mathcal{L}\mathcal{F}_*)$ is uniformly locally compact while $(\mathcal{K}_0(\mathbb{R}), \mathcal{S}\mathcal{C}_*)$ is not.

Example 2. The result that $(\mathcal{K}_0(\mathbb{R}), \mathcal{L}\mathcal{F}_*)$ is uniformly locally compact can be extended to any Tychonoff topological group X of pointwise countable type whose left uniformity is cofinally complete. In fact, it was proved in [20] that such a topological group is locally compact and since every Tychonoff topological group of pointwise countable type is paracompact, it follows from Proposition 3 that $(\mathcal{K}_0(X), \mathcal{L}\mathcal{F}_*)$ is uniformly locally compact.

By \mathcal{T}_V we shall denote the Vietoris topology of a topological space X .

Remark 2. It is proved in [18] that for a quasi-uniform space (X, \mathcal{U}) , one has $\mathcal{T}_V = \mathcal{T}(\mathcal{U}_*)$ on $\mathcal{K}_0(X)$ if and only if $\mathcal{U}^{-1}|K$ is precompact for all $K \in \mathcal{K}_0(X)$.

Hence, by Theorem 1 and Proposition 3, the Hausdorff-Bourbaki quasi-uniformity on $\mathcal{K}_0(X)$ of the locally finite quasi-uniformity of any (Tychonoff) paracompact locally compact space X is compatible with the Vietoris topology of X .

Let (X, d) be a bounded quasi-pseudo-metric space. Then the Hausdorff-Bourbaki quasi-pseudo-metric d_* on $\mathcal{P}_0(X)$ is defined by

$$d_*(A, B) = \max \left\{ \sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B) \right\}$$

whenever $A, B \in \mathcal{P}_0(X)$ ([2], [14]).

The quasi-pseudo-metric d_* generates on $\mathcal{P}_0(X)$ the Hausdorff-Bourbaki quasi-uniformity of the quasi-uniformity \mathcal{U}_d , where \mathcal{U}_d is the quasi-uniformity on X generated by d (see [9], page 3), i.e. $\mathcal{U}_{d_*} = (\mathcal{U}_d)_*$.

Let us recall ([16]) that a sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-pseudo-metric space (X, d) is right K -Cauchy provided that for each $\varepsilon > 0$ there is n_ε such that $d(x_m, x_n) < \varepsilon$ whenever $m \geq n \geq n_\varepsilon$.

A quasi-metric space (X, d) is said to be uniformly locally compact if (X, \mathcal{U}_d) is a uniformly locally compact quasi-uniform space.

Thus Theorem 1 can be restated for quasi-metric spaces, as follows.

Theorem 2. *Let (X, d) be a Tychonoff quasi-metric space. Then $(\mathcal{K}_0(X), d_*)$ is uniformly locally compact if and only if (X, d) is uniformly locally compact and each convergent sequence in X has a right K -Cauchy subsequence.*

Proof. Let (X, d) be a Tychonoff quasi-metric space such that $(\mathcal{K}_0(X), d_*)$ is uniformly locally compact. By Theorem 1, (X, d) is uniformly locally compact and for each $K \in \mathcal{K}_0(X)$, $(\mathcal{U}_d)^{-1}|K$ is precompact. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X which is $\mathcal{T}(\mathcal{U}_d)$ -convergent to a point $x \in X$. Let $Y = \{x\} \cup \{x_n : n \in \mathbb{N}\}$. Then $(Y, \mathcal{U}_d|Y)$

is a compact quasi-uniform space and for each $K \in \mathcal{K}_0(Y)$, $(\mathcal{U}_d)^{-1}|K$ is precompact because such a K also belongs to $\mathcal{K}_0(X)$. Thus $(\mathcal{U}_d)^{-1}|Y$ is hereditarily precompact by Lemma 2. Hence, the sequence $(x_n)_{n \in \mathbb{N}}$ has a right K -Cauchy subsequence by [10], Theorem 3.

Conversely, suppose that there is $K \in \mathcal{K}_0(X)$ such that $(\mathcal{U}_d)^{-1}|K$ is not precompact. Then there exist a sequence $(x_n)_{n \in \mathbb{N}}$ in K and a $U \in \mathcal{U}_d$ such that $x_{n+1} \notin \bigcup_{i=1}^n U^{-1}(x_i)$ for all $n \in \mathbb{N}$. Since the sequence $(x_n)_{n \in \mathbb{N}}$ has a cluster point (in K), it follows from our hypothesis that $(x_n)_{n \in \mathbb{N}}$ has a right K -Cauchy subsequence, which contradicts that $x_{n+1} \notin \bigcup_{i=1}^n U^{-1}(x_i)$ for all $n \in \mathbb{N}$. We conclude that for each $K \in \mathcal{K}_0(X)$, $(\mathcal{U}_d)^{-1}|K$ is precompact. Hence $(\mathcal{K}_0(X), d_*)$ is uniformly locally compact by Theorem 1. \square

Remark 3. Let (X, \mathcal{U}) be a Tychonoff uniform space. Since $(\mathcal{K}_0(X), \mathcal{U}_*)$ is (uniformly) locally compact if and only if (X, \mathcal{U}) is (uniformly) locally compact ([4], [15] and Corollary 1 above), it seems natural to conjecture, in the light of Lemma 1, that cofinal completeness is also preserved by the Hausdorff-Bourbaki uniformity on $\mathcal{K}_0(X)$, and thus, Corollary 1 follows in an easy and elegant way, as a factorization. Unfortunately, this is not the case, not even for metric spaces as is shown in [13], Remark 3.6. Furthermore, for a metric space (X, d) , cofinal completeness of $(\mathcal{K}_0(X), d_*)$ is equivalent to its uniform local compactness and thus to uniform local compactness of (X, d) ([13], Proposition 3.4 and Corollary 3.5). In this direction, it seems interesting to note that if X is a metrizable topological group which admits a compatible cofinally complete metric d , then X is locally compact ([17]) and, hence, $(\mathcal{K}_0(X), d_*)$ is uniformly locally compact.

3. CO-UNIFORM LOCAL COMPACTNESS OF $(\mathcal{K}_0(X), \mathcal{U}_*)$

In [21] the third author introduced and studied the notion of a D -co-Lebesgue quasi-uniform space. In particular, it is proved that the fine transitive quasi-uniformity of a space X is D -co-Lebesgue if and only if X is metacompact, and that a quasi-uniform space is D -co-Lebesgue if and only if it is cofinally co-complete, where a quasi-uniform space (X, \mathcal{U}) is called cofinally co-complete if every weakly Cauchy filter on (X, \mathcal{U}^{-1}) has a cluster point in $(X, \mathcal{F}(\mathcal{U}))$.

In this context it seems natural to propose an appropriate notion of “co-uniform local compactness”. Thus, we say that a quasi-uniformity \mathcal{U} on a set X is *co-uniformly locally compact* if there exists $U \in \mathcal{U}$ such that for each $x \in X$, $\overline{U^{-1}(x)}$ is compact. In this case, the quasi-uniform space (X, \mathcal{U}) is said to be *co-uniformly locally compact*. (Here, closure means closure in $(X, \mathcal{F}(\mathcal{U}))$.)

In Theorem 3 below we shall show that the characterization of uniform local compactness of $(\mathcal{X}_0(X), \mathcal{U}_*)$ obtained in Theorem 1 admits an analogous result for co-uniformly locally compact Hausdorff quasi-uniform spaces.

Similarly to the proofs of Proposition 5.32 and Theorem 5.33 of [9] we can show the following result.

Proposition 5. *Let (X, \mathcal{U}) be a quasi-uniform space. Then*

1. *(X, \mathcal{U}) is cofinally co-complete if and only if every directed open cover of (X, \mathcal{U}) is a quasi-uniform cover in (X, \mathcal{U}^{-1}) .*
2. *If (X, \mathcal{U}) is locally compact and cofinally co-complete, then (X, \mathcal{U}) is co-uniformly locally compact.*
3. *If (X, \mathcal{U}) is co-uniformly locally compact, then (X, \mathcal{U}) is cofinally co-complete.*

Our next proposition shows that a co-uniformly locally compact quasi-uniform space need not be locally compact. We shall use the following observation.

Remark 4. It is well known ([9], Corollary 2.15) that if X is a space, then for each $U \in \mathcal{S}\mathcal{C}$ there is a countable subset D of X such that $U(D) = X$. However, it can be easily proved that for each $U \in \mathcal{S}\mathcal{C}$ there is a countable subset D of X such that $(U \cap U^{-1})(D) = X$. This fact will be used in the proof of Proposition 6 below.

Proposition 6. *A Hausdorff space is σ -compact if and only if $\mathcal{S}\mathcal{C}$ is co-uniformly locally compact.*

Proof. Suppose that X is a Hausdorff σ -compact space. Then there is an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of X such that $X = \bigcup_{n=1}^{\infty} K_n$. Define $A_{-n} = X \setminus K_n$ and $A_n = X$ for each $n \in \mathbb{N}$. Thus $\{A_n : n \in \mathbb{Z}\}$ is an open spectrum. So $U = \bigcup_{n \in \mathbb{Z}} (A_n \setminus A_{n-1}) \times A_n$ is an element of $\mathcal{S}\mathcal{C}$ (see [9], Theorem 2.12). Now let $x \in X$. There is $n \in \mathbb{N}$ such that $x \in A_{-n} \setminus A_{-(n+1)}$. Therefore $U^{-1}(x) = X \setminus A_{-(n+1)} = K_{n+1}$. We conclude that $\mathcal{S}\mathcal{C}$ is co-uniformly locally compact.

Conversely, let X be a (not necessarily Hausdorff) space such that $\mathcal{S}\mathcal{C}$ is co-uniformly locally compact. Let $U \in \mathcal{S}\mathcal{C}$ such that $\overline{U^{-1}(x)}$ is compact for all $x \in X$. By Remark 4, there is a countable subset D of X such that $(U \cap U^{-1})(D) = X$. Hence $X = \bigcup_{x \in D} U^{-1}(x) = \bigcup_{x \in D} \overline{U^{-1}(x)}$. Therefore X is σ -compact. \square

However, we obtain the following result.

Proposition 7. *Let X be a space such that $\mathcal{L}\mathcal{F}$ is co-uniformly locally compact. Then X is locally compact.*

Proof. Let \mathcal{G} be a locally finite open cover of X such that $\overline{U^{-1}(x)}$ is compact for all $x \in X$, where $U = \bigcup_{x \in X} \{x\} \times \bigcap \{G \in \mathcal{G} : x \in G\}$. Let $x \in X$ and let A be an open neighborhood of x such that $A \subseteq U(x)$ and A meets only a finite number of elements of \mathcal{G} , namely G_1, \dots, G_n .

For each finite subset J of $\{1, \dots, n\}$ such that $A \cap \left(\bigcap_{j \in J} G_j\right) \neq \emptyset$, let $x_J \in A \cap \left(\bigcap_{j \in J} G_j\right)$. We show that $A \subseteq \bigcup_J U^{-1}(x_J)$. Indeed, let $y \in A$ and $J_y = \{j \in \{1, \dots, n\} : y \in G_j\}$. Then $x_{J_y} \in \bigcap_{j \in J_y} G_j$, so $y \in U^{-1}(x_{J_y})$. Hence $A \subseteq \bigcup_J U^{-1}(x_J)$, and thus \overline{A} is a compact neighborhood of x . \square

Let us recall ([9]) that a quasi-uniformity \mathcal{U} on a set X is said to be point symmetric if $\mathcal{T}(\mathcal{U}) \subseteq \mathcal{T}(\mathcal{U}^{-1})$.

Lemma 8. *Let (X, \mathcal{U}) be a cofinally co-complete T_1 quasi-uniform space. Then \mathcal{U} is point symmetric.*

Proof. Let \mathcal{F} be a filter on X which is $\mathcal{T}(\mathcal{U}^{-1})$ -convergent to a point $x \in X$. Then \mathcal{F} is a weakly Cauchy filter on (X, \mathcal{U}^{-1}) and thus it has a $\mathcal{T}(\mathcal{U})$ -cluster point $y \in X$. Since (X, \mathcal{U}) is T_1 , it follows that $x = y$, so \mathcal{F} is $\mathcal{T}(\mathcal{U})$ -convergent to x . We conclude that \mathcal{U} is point symmetric. \square

Lemma 9. *Let (X, \mathcal{U}) be a quasi-uniform space such that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is point symmetric. Then $\mathcal{U}^{-1}|K$ is precompact for each $K \in \mathcal{K}_0(X)$.*

Proof. Let $K \in \mathcal{K}_0(X)$ and $U \in \mathcal{U}$. Then, there exists $V \in \mathcal{U}$ such that $V_H^{-1}(K) \subseteq U_H(K)$. Since K is compact there exists a finite subset K' of K such that $K \subseteq V(K')$, so $K \in V_H(K')$ and hence $K' \in V_H^{-1}(K) \subseteq U_H(K)$. Thus $K \subseteq U^{-1}(K')$. We conclude that $\mathcal{U}^{-1}|K$ is precompact. \square

Theorem 3. *Let (X, \mathcal{U}) be a Hausdorff quasi-uniform space. Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is co-uniformly locally compact if and only if (X, \mathcal{U}) is co-uniformly locally compact and $\mathcal{U}^{-1}|K$ is precompact for each $K \in \mathcal{K}_0(X)$.*

Proof. Suppose that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is co-uniformly locally compact. By Proposition 5, $(\mathcal{K}_0(X), \mathcal{U}_*)$ is cofinally co-complete. Furthermore, it is a Hausdorff quasi-uniform space because Hausdorffness is preserved by the Vietoris topology \mathcal{T}_V of $(X, \mathcal{T}(\mathcal{U}))$ on $\mathcal{K}_0(X)$, and $\mathcal{T}_V \subseteq \mathcal{T}(\mathcal{U}_*)$ on $\mathcal{K}_0(X)$ (see [18]). Therefore \mathcal{U}_* is

point symmetric on $\mathcal{K}_0(X)$ by Lemma 8. It follows from Lemma 9 that $\mathcal{U}^{-1}|K$ is precompact for each $K \in \mathcal{K}_0(X)$.

Let us prove that (X, \mathcal{U}) is co-uniformly locally compact. Let $U \in \mathcal{U}$ be such that $\overline{U_H^{-1}(K)}$ is compact for each $K \in \mathcal{K}_0(X)$. Then $\overline{U^{-1}(x)}$ is compact for each $x \in X$. Indeed, given $x \in X$ let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $\overline{U^{-1}(x)}$. It easily follows that $\{x_\lambda\} \in \overline{U_H^{-1}(\{x\})}$ for each $\lambda \in \Lambda$ and since $\overline{U_H^{-1}(\{x\})}$ is compact, there exists $C \in \overline{U_H^{-1}(\{x\})}$ which is a cluster point of $(\{x_\lambda\})_{\lambda \in \Lambda}$. Therefore y is a cluster point of $(x_\lambda)_{\lambda \in \Lambda}$ for each $y \in C$. We have shown that $\overline{U^{-1}(x)}$ is compact, and hence (X, \mathcal{U}) is co-uniformly locally compact.

Conversely, suppose that (X, \mathcal{U}) is co-uniformly locally compact and $\mathcal{U}^{-1}|K$ is precompact for each $K \in \mathcal{K}_0(X)$. Let $U \in \mathcal{U}$ be such that $\overline{U^{-1}(x)}$ is compact for each $x \in X$ and let $V \in \mathcal{U}$ with $V^3 \subseteq U$.

Let $K \in \mathcal{K}_0(X)$. We shall prove that $\overline{V_H^{-1}(K)}$ is compact. Let $(K_\lambda)_{\lambda \in \Lambda}$ be a net in $\overline{V_H^{-1}(K)}$. Then $K_\lambda \subseteq \overline{V^{-1}(K)}$ for all $\lambda \in \Lambda$, and thus $K_\lambda \in \mathcal{K}_0(\overline{V^{-1}(K)})$.

Next we show that $(\mathcal{K}_0(\overline{V^{-1}(K)}), \mathcal{U}_*)$ is compact. Since $\mathcal{U}^{-1}|K$ is precompact, there exists a finite subset $\{x_1, \dots, x_n\}$ of K such that $K \subseteq \bigcup_{i=1}^n V^{-1}(x_i)$. Hence $\overline{V^{-1}(K)} \subseteq \bigcup_{i=1}^n \overline{U^{-1}(x_i)}$. Therefore $\overline{V^{-1}(K)}$ is compact. Since, by our hypothesis, $\mathcal{U}^{-1}|K'$ is precompact whenever $K' \in \mathcal{K}_0(\overline{V^{-1}(K)})$, it follows from Lemmas 2 and 4 that $(\mathcal{K}_0(\overline{V^{-1}(K)}), \mathcal{U}_*)$ is compact.

Consequently, the net $(K_\lambda)_{\lambda \in \Lambda}$ has a cluster point C in $\mathcal{K}_0(\overline{V^{-1}(K)})$, and hence $\overline{V_H^{-1}(K)}$ is compact. We conclude that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is co-uniformly locally compact. \square

The following result should be compared with Propositions 1 and 3 and Remark 1b).

Corollary 3. *Let X be a Hausdorff space. Then*

- a) $(\mathcal{K}_0(X), \mathcal{CF}_*)$ is co-uniformly locally compact if and only if (X, \mathcal{CF}) is co-uniformly locally compact.
- b) $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is co-uniformly locally compact if and only if (X, \mathcal{LF}) is co-uniformly locally compact.

Proof. a) Apply Theorem 3 and Lemma 5.

b) If $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is co-uniformly locally compact, then (X, \mathcal{LF}) is co-uniformly locally compact by Theorem 3. Conversely, if (X, \mathcal{LF}) is co-uniformly locally compact, then X is a locally compact space by Proposition 7, so $\mathcal{CF} = \mathcal{LF}$. Hence $(\mathcal{K}_0(X), \mathcal{LF}_*)$ is co-uniformly locally compact by part a). \square

Related to the proof of the “only if” part in Theorem 3 above, we give an example of a T_1 quasi-uniform (actually, quasi-metric) space (X, \mathcal{U}) such that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is not T_1 .

Example 3. Let $X = \{1/n : n \in \mathbb{N}\}$ and let d be the quasi-metric on X given by $d(1/n, 1/m) = 1/m$ if $n \neq m$, and $d(1/n, 1/n) = 0$, whenever $n, m \in \mathbb{N}$. Clearly the topology generated by d is the co-finite topology on X .

Let $Y = \{1/2n : n \in \mathbb{N}\}$. Then $X, Y \in \mathcal{K}_0(X)$ and an easy computation shows that $d_*(X, Y) = 0$. Hence the Hausdorff-Bourbaki quasi-pseudo-metric d_* is not a quasi-metric on $\mathcal{K}_0(X)$.

Lemma 10. *Let X be a space and $K \in \mathcal{K}_0(X)$. Then $\mathcal{S}\mathcal{C}^{-1}|K$ is precompact if and only if every lower semicontinuous real-valued function on X is bounded on K .*

Proof. Suppose that $\mathcal{S}\mathcal{C}^{-1}|K$ is precompact and let f be a lower semicontinuous real-valued function on X . Thus $U \in \mathcal{S}\mathcal{C}$ where $U = \{(x, y) : f(x) - f(y) < 1\}$. Let K' be a finite subset of K such that $K \subseteq U^{-1}(K')$. Put $M = \max\{f(z) : z \in K'\}$. Then $f(x) \leq 1 + M$ for all $x \in K$. Since f is also lower semicontinuous on K , it is lower bounded on K ([9], Proposition 3.17). Hence f is bounded on K .

Conversely, let f be a lower semicontinuous real-valued function f on X and let $\varepsilon > 0$. Put $U = \{(x, y) : f(x) - f(y) < \varepsilon\}$. By assumption, $f(K)$ is a bounded subset of \mathbb{R} , so there is a finite subset K' of K such that $f(K) \subseteq \bigcup_{z \in K'}]f(z) - \varepsilon, f(z) + \varepsilon[$. Hence $K \subseteq U^{-1}(K')$. Consequently, $\mathcal{S}\mathcal{C}^{-1}|K$ is precompact. \square

Proposition 8. *Let X be a Hausdorff space. Then $(\mathcal{K}_0(X), \mathcal{S}\mathcal{C}_*)$ is co-uniformly locally compact if and only if X is a σ -compact space such that every lower semicontinuous real-valued function on X is bounded on each compact subset of X .*

Proof. Apply Proposition 6, Theorem 3 and Lemma 10. \square

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