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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 1, 247–252

Persistent URL: <http://dml.cz/dmlcz/127881>

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ON SUPER HAMILTONIAN SEMIGROUPS

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(Received July 30, 2001)

Abstract. The concept of super hamiltonian semigroup is introduced. As a result, the structure theorems obtained by A. Cherubini and A. Varisco on quasi commutative semigroups and quasi hamiltonian semigroups respectively are extended to super hamiltonian semigroups.

Keywords: quasi hamiltonian semigroups, super hamiltonian semigroups, quasi commutative semigroups, quasi-groups, strong semilattices of semigroups

MSC 2000: 20M10

1. INTRODUCTION

A semigroup S is called quasi commutative if for all $a, b \in S$, $ab = b^r a$ holds for some positive integer $r \geq 1$. The concept of quasi commutativity was first introduced by N.P. Mukherjee [6] in 1971. Later on, M. Chacron and G. Thierrin [1] called a semigroup S a σ -reflexive semigroup if and only if S satisfies the following condition:

$$\forall a, b \in S, \exists m = m(a, b) \geq 1, ab = (ba)^m.$$

Quasi commutative semigroups, cyclic communicative semigroups and σ -reflexive semigroups were then studied by a number of authors, for example, see [2], [4], [5] and [7].

In generalizing the concept of quasi commutativity, A. Cherubini and A. Varisco [3] in 1983 called a semigroup S a quasi hamiltonian semigroup if for every $a, b \in S$, there exists two positive integers r, s such that $ab = b^r a^s$. Thus, it is clear that the class of quasi hamiltonian semigroups contains the class of quasi commutative

The research of the first author is partially supported by a UGC (HK) grant #2160126 (1999/2001).

semigroups as its special subclass. On the other hand, B. Pondělíček [9] in 1975 called a semigroup S a weakly commutative semigroup if for every $a, b \in S$, $(ab)^n \in bSa$ for some positive integer $n \geq 1$. Thus, quasi commutative semigroups, quasi hamiltonian semigroups and σ -reflexive semigroups can be regarded as special weakly commutative semigroups. It was stated by M. Petrich [8, Corollary II.5.6] that a weakly commutative semigroup is a semilattice of archimedean semigroups. Along with this direction, A. Cherubini and A. Varisco proved in [2] and [3] respectively the following structure theorems:

- (i) A semigroup S is σ -reflexive if and only if S is a semilattice of σ -reflexive archimedean semigroups S_α and for every $a, b \in S$ with $ab \neq ba$, ab belongs to a subgroup of S . (See [3, Theorem 2.8].)
- (ii) A semigroup S is a quasi hamiltonian semigroup if and only if S is a semilattice of archimedean quasi hamiltonian semigroups, and every subsemigroup of S , generated by two elements, is a duo semigroup. (See [3, Theorem 1.9].)

Also, it was shown by N.P. Mukherjee in [6] that:

- (iii) Every quasi commutative semigroup S is uniquely expressible as a semilattice of archimedean semigroups S_α . (See [6, Theorem 4].)

Inspired by the above definitions and results in the literature, we now call a semigroup S a *generalized quasi hamiltonian* semigroup if for every $a, b \in S$, it holds $(ab)^n = b^r a^s$ for some positive integers n, r, s satisfying $2n < r + s$. In view of the results mentioned by M. Chacron and G. Thirerrin (see [1, Theorems 1 and 2]), it is natural to call the generalized quasi hamiltonian semigroups with central idempotents the *super hamiltonian* semigroups. What we are going to show is that for a generalized quasi hamiltonian semigroup S , S is super hamiltonian if and only if S is a strong semilattice of quasi-groups. This theorem describes the structure of super hamiltonian semigroups and as a consequence, we observe that the quasi-groups are the basic building blocks for super hamiltonian semigroups. Thus, the structure theorem for the quasi hamiltonian semigroups in [3] is extended.

The reader is referred to M. Petrich [8] for notations if it is necessary.

2. SUPER HAMILTONIAN SEMIGROUPS

We shall first give some definitions that will be used throughout the paper.

Definition 2.1. An element a in a semigroup S is called a quasi regular element if there exists an integer $n \geq 1$ and $x \in S$ such that $a^n = a^n x a^n$. We shall call a semigroup a quasi regular semigroup if all elements of S are quasi regular.

Quasi regular semigroups have been extensively investigated (see [12]). It is trivial to see that a quasi regular semigroup contains some idempotents.

Definition 2.2 [11]. A quasi regular semigroup S is called a quasi-group if there exists only one idempotent in S , that is, $|E| = 1$.

Definition 2.3 [10]. A semigroup S is called a t -archimedean semigroup if for all $a, b \in S$, there exists a positive integer $n \geq 1$ such that $b^n \in aS \cap Sa$.

The following result is stated in [3] for quasi-hamiltonian semigroups. In fact, it can be easily verified that the result holds for generalized quasi hamiltonian semigroups as well.

Proposition 2.4. *A generalized quasi hamiltonian semigroup S is a semilattice of t -archimedean semigroups S_α .*

Hereafter, we shall write the generalized quasi hamiltonian semigroup S as $S = \bigcup_{\alpha \in Y} S_\alpha$, where S_α is a t -archimedean subsemigroup of S for every $\alpha \in Y$. We also call S_α the t -archimedean component of the semigroup S .

We now prove the following lemmas which are the crucial lemmas in the establishment of our main theorem.

Lemma 2.5. *Let S be a generalized quasi hamiltonian semigroup. Then every t -archimedean component of S contains a unique idempotent of S .*

Proof. Since S is a generalized quasi hamiltonian semigroup, by Proposition 2.4, $S = \bigcup_{\alpha \in Y} S_\alpha$, where each S_α is a t -archimedean subsemigroup of S . Now, let $a \in S$. Then $a \in S_\alpha$ for some $\alpha \in Y$. Since S_α is a subsemigroup of S , we have $\langle a \rangle \subseteq S_\alpha$. Since S is a generalized quasi hamiltonian semigroup, we have $(a^2)^n = a^r a^s$ for some positive integers n, r and s , that is, $a^{2n} = a^{r+s}$ with $2n \neq r + s$. It is now clear that a is periodic and hence there exists a positive integer $m \geq 1$ such that $a^m = e \in S_\alpha$. We claim that the idempotent e in S_α is unique. In fact, if we let e, f be idempotents of S_α , then, since S_α is a t -archimedean semigroup, there exists x, y in S_α such that $e = fx$ and $f = ye$. This leads to $e = fe = ye = f$. Thus, S_α contains exactly one idempotent of S . The proof is completed. \square

Lemma 2.6. *The t -archimedean semigroup S_α of a generalized quasi hamiltonian semigroup S is a quasi-group.*

Proof. In view of Lemma 2.5, we only need to prove that the t -archimedean component S_α of S is quasi regular. To this end, we let $a \in S_\alpha$. Then we consider the elements a, e in S_α , where e is the unique idempotent in S_α , by Lemma 2.5. Since S_α is a t -archimedean semigroup, there exist elements u, v, x in S_α and a positive integer $n \geq 1$ such that $a^n = eu = ve$ and $e = a^n x$. These equalities now lead to $a^n = ea^n = a^n x a^n$. Thereby, S_α is a quasi-group. \square

Summing up Lemma 2.5 and Lemma 2.6, we obtain the following result.

Lemma 2.7. *If S is a generalized quasi hamiltonian semigroup then S is a semilattice of quasi-groups S_α .*

The above lemma improves the result of A. Cherubini and A. Varisco [3] for quasi hamiltonian semigroups. Also, we observe that for every element a in a quasi-group S_α , a^n is an idempotent for some positive integer n . Thus, it is trivial to see that in the quasi-group S_α , the idempotent is in the center of S_α .

By using Lemma 2.7, we now establish the following theorem for super hamiltonian semigroups.

Theorem 2.8 (Main Theorem). *Let S be a generalized quasi hamiltonian semigroup. Then S is a super hamiltonian semigroup if and only if S is a strong semilattice of quasi-groups S_α .*

Proof. (\Rightarrow) Let S be a super hamiltonian semigroup. Then by Lemma 2.7, $S = \bigcup_{\alpha \in Y} S_\alpha$, where Y is a semilattice and each S_α is a quasi-group for every $\alpha \in Y$. Now, let e_α be the identity element of the quasi-group S_α . Define a mapping $\varphi_{\alpha,\beta}: S_\alpha \rightarrow S_\beta$ by $a\varphi_{\alpha,\beta} = ae_\beta$ for any $a \in S_\alpha$ and $\alpha, \beta \in Y$ with $\alpha \geq \beta$. It is trivial to see that $\varphi_{\alpha,\alpha}$ is the identity mapping on S_α . Also, by the multiplication on S , $ae_\beta \in S_\beta$. Suppose that a, b are two arbitrary elements of S_α . Then we have

$$\begin{aligned} (ab)\varphi_{\alpha,\beta} &= (ab)e_\beta \\ &= ae_\beta be_\beta \quad (\text{since } e_\beta \text{ is in the center of } S_\beta) \\ &= a\varphi_{\alpha,\beta} b\varphi_{\alpha,\beta}. \end{aligned}$$

This shows that $\varphi_{\alpha,\beta}$ is a homomorphism. Moreover, since S is a super hamiltonian semigroup, we can easily verify that $e_\alpha e_\beta = e_\gamma$ for $\alpha, \beta, \gamma \in Y$ with $\alpha\beta = \gamma$, where e_γ is the idempotent in S_γ . Thus, for $\alpha \geq \beta \geq \gamma$ with $a \in S_\alpha$, we have

$$a\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = (ae_\beta)e_\gamma = ae_\gamma = a\varphi_{\alpha,\gamma}.$$

Because the element a is arbitrarily chosen in S , we have $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. Hence, the maps $\varphi_{\alpha,\beta}$ are structure homomorphisms of the strong semilattice of quasi-groups S_α , that is, $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ is a strong semilattice of the quasi-groups S_α .

(\Leftarrow) Let S be a strong semilattice of the quasi-groups S_α . Then for any element $a \in S$ and any idempotent $e \in E \subseteq S$, there exist some $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and

$e \in S_\beta \cap E$. Since $S_{\alpha\beta} \in S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ and $\varphi_{\alpha,\beta}$ are the structure homomorphisms of the strong semilattice $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$, we have

$$\begin{aligned} ae &= a\varphi_{\alpha,\alpha\beta}e\varphi_{\beta,\alpha\beta} \\ &= e\varphi_{\beta,\alpha\beta}a\varphi_{\alpha,\alpha\beta} \quad (\text{since } S_{\alpha\beta} \text{ is a quasi-group}) \\ &= ea. \end{aligned}$$

This shows that e lies in the center of S as well. Since S itself has been assumed to be a generalized quasi hamiltonian semigroup, we deduce that S is super hamiltonian. This finishes the proof. \square

Remark. A. Cherubini and A. Varisco have remarked in [2] that the idempotents of a quasi commutative semigroup S are in the center of S . Our Theorem 2.8 extends their remark from quasi commutative semigroups to super hamiltonian semigroups.

Example 2.9. Let $S_\alpha = \{a, b, e\}$, $S_\beta = \{c, f\}$ and $S_{\alpha\beta} = \{u, v, w, x, y, z\}$ be respectively quasi-groups on a semilattice $Y = \{\alpha, \beta, \alpha\beta\}$.

The Cayley tables of S_α , S_β and $S_{\alpha\beta}$ are respectively the following

$$S_\alpha: \begin{array}{c|ccc} * & a & b & e \\ \hline a & b & e & e \\ b & e & e & e \\ e & e & e & e \end{array} \quad S_\beta: \begin{array}{c|cc} * & c & f \\ \hline c & f & f \\ f & f & f \end{array}$$

$$S_{\alpha\beta}: \begin{array}{c|cccccc} * & u & v & w & x & y & z \\ \hline u & u & v & w & x & y & z \\ v & v & u & y & z & w & x \\ w & w & z & u & y & x & v \\ x & x & y & z & u & v & w \\ y & y & x & v & w & z & u \\ z & z & w & x & v & u & y \end{array}$$

Define the mapping $\theta_{\alpha,\alpha\beta}: S_\alpha \rightarrow S_{\alpha\beta}$ by $x \mapsto u$ for any $x \in S_\alpha$; $\theta_{\beta,\alpha\beta}: S_\beta \rightarrow S_{\alpha\beta}$ by $y \mapsto u$ for any $y \in S_\beta$ and let $\theta_{\alpha,\alpha}$ be the identity mapping for any $\alpha \in Y$. Then we can easily check that the mappings $\theta_{\alpha,\beta}$ form a family of structure homomorphisms of the strong semilattice $S = [Y; S_\alpha; \theta_{\alpha,\beta}]$, where $S = \bigcup_{\alpha \in Y} S_\alpha = S_\alpha \cup S_\beta \cup S_{\alpha\beta}$. By using the above structure homomorphisms, we obtain the following Cayley table to the semigroup S :

*	<i>a</i>	<i>b</i>	<i>e</i>	<i>c</i>	<i>f</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>a</i>	<i>b</i>	<i>e</i>	<i>e</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>b</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>e</i>	<i>e</i>	<i>e</i>	<i>e</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>c</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>f</i>	<i>f</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>f</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>f</i>	<i>f</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>u</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>u</i>	<i>v</i>	<i>w</i>	<i>x</i>	<i>y</i>	<i>z</i>
<i>v</i>	<i>v</i>	<i>v</i>	<i>v</i>	<i>v</i>	<i>v</i>	<i>v</i>	<i>u</i>	<i>y</i>	<i>z</i>	<i>w</i>	<i>x</i>
<i>w</i>	<i>w</i>	<i>w</i>	<i>w</i>	<i>w</i>	<i>w</i>	<i>w</i>	<i>z</i>	<i>u</i>	<i>y</i>	<i>x</i>	<i>v</i>
<i>x</i>	<i>x</i>	<i>x</i>	<i>x</i>	<i>x</i>	<i>x</i>	<i>x</i>	<i>y</i>	<i>z</i>	<i>u</i>	<i>v</i>	<i>w</i>
<i>y</i>	<i>y</i>	<i>y</i>	<i>y</i>	<i>y</i>	<i>y</i>	<i>y</i>	<i>x</i>	<i>v</i>	<i>w</i>	<i>z</i>	<i>u</i>
<i>z</i>	<i>z</i>	<i>z</i>	<i>z</i>	<i>z</i>	<i>z</i>	<i>z</i>	<i>w</i>	<i>v</i>	<i>v</i>	<i>u</i>	<i>y</i>

Then, we can check that S is a super hamiltonian semigroup, but S is not commutative and in fact not quasi hamiltonian as well (indeed, it suffices to consider e.g. v, w).

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