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SOME CHARACTERIZATIONS OF COMPLETENESS FOR TRELLISES IN TERMS OF JOINS OF CYCLES

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Abstract. This paper gives some new characterizations of completeness for trellises by introducing the notion of a cycle-complete trellis. One of our results yields, in particular, a characterization of completeness for trellises of finite length due to K. Gladstien (see K. Gladstien: Characterization of completeness for trellises of finite length, Algebra Universalis *3* (1973), 341–344).

Keywords: pseudo-ordered set, trellis, p-chain, ascending well-ordered p-chain, cycle-complete trellis, complete trellis

MSC 2000: 06B05

1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A. A *pseudo-ordered set* or a *psoset* $\langle A; \trianglelefteq\rangle$ consists of a nonempty set A and a pseudo-order $\trianglelefteq$ on A. For $a, b \in A$, if $a \trianglelefteq b$ and $a \neq b$, then we write $a \vartriangleleft b$. For a subset B of A, the notions of a lower bound, an upper bound, the greatest lower bound (GLB or meet, denoted by $\bigwedge B$), the least upper bound (LUB or join, denoted by $\bigvee B$), a minimal element, a maximal element, the minimum (or the least) element and the maximum (or the greatest) element are defined analogously to the corresponding notions in a poset. As in the case of posets (see [1]), for the empty set Φ, $\bigvee \Phi$ exists in A if and only if $\bigwedge A$ exists or equivalently A has the minimum element 0 and $\bigvee \Phi = \bigwedge A = 0$. By a *trellis* (also called a T-*lattice* in [2] and a *weakly associative lattice* in [3]) we mean a poset any two of whose elements have a GLB and a LUB. A trellis in which every subset has a GLB and a LUB is called a *complete trellis*. The notion of a trellis as a nonassociative generalization of a lattice is due to E. Fried [2] and H. L. Skala [6].

Define a relation $\sqsubset_B$ on a subset B of a psoset $\langle A; \trianglelefteq\rangle$ by setting $b \sqsubset_B b'$ for two elements b and b' of B if there exists a finite sequence $(b_1, \ldots, b_n)$ of elements of B such that $b \lhd b_1 \lhd \ldots \lhd b_n \lhd b'$. If $b \trianglelefteq b_1 \trianglelefteq \ldots \trianglelefteq b_n \trianglelefteq b'$ then we write $b \sqsubseteq_B b'$. If for each pair of elements b and b' of B at least one of the relations $b \sqsubseteq_B b'$ or $b' \sqsubseteq_B b$ holds, then B will be called a *pseudo-chain* or a *p-chain*. If both these relations hold for each pair of elements, B is said to be a cycle. A one-element cycle is called a *trivial cycle*. It is known that a cycle having a maximum element is a trivial cycle (see [4]). The empty set Φ is also regarded as a cycle. A p-chain $C = \{a_i \mid i = 1, 2, \ldots\}$ of elements of a psoset $\langle A; \trianglelefteq\rangle$ is said to be a *descending p-chain* in A if $a_1 \rhd a_2 \rhd \ldots$ A psoset $\langle A; \trianglelefteq\rangle$ is said to satisfy the *descending p-chain condition* if there is no infinite descending p-chain of elements of A. A p-chain satisfying the descending p-chain condition is called an *ascending well-ordered p-chain*. An ascending p-chain, ascending p-chain condition and descending well-ordered p-chain are defined similarly.

It is proved in our paper [5] that a trellis A is complete if and only if every ascending well-ordered p-chain in A has a join. In this paper, using the notion of a cycle-complete trellis, we obtain some new characterizations of completeness for trellises, one of which yields, in particular, a result of K. Gladstien [4] for trellises of finite length.

2. Definitions and results

Let $\langle A; \trianglelefteq\rangle$ be a psoset and H a nonempty subset of A. Define an equivalence relation $\sim$ on H by, for $a, b \in H$, $a \sim b$ if there exists a cycle C of elements of H such that $a, b \in C$. For a $\in H$, let $[a]_H$ denote the equivalence class in H containing a with respect to the equivalence relation $\sim$, i.e. $[a]_H = \{x \in H \mid x \sim a\}$. Clearly $[a]_H$ is a maximal cycle (with respect to set inclusion) in H containing a. Let $H^* = \{[a]_H \mid a \in H\}$. Then the binary relation $\trianglelefteq^*$ on H^* defined for $[a]_H, [b]_H \in H^*$ by $[a]_H \trianglelefteq^* [b]_H$ if a $\sqsubseteq_H b$, is clearly a partial order on H^*.

Let $\langle A; \trianglelefteq\rangle$ be a psoset. We call a subset S of A *join-closed* if, whenever T is a subset of S such that $\bigvee T$ exists in A, then $\bigvee T \in S$. We call a subset S of A *up-directed* if every pair of elements of S has an upper bound in S. If any two-elements of A have a LUB, then it is clear that any join-closed subset of A is up-derected.

Remark 1. We make the following observations.

(i) If H is a nonempty up-directed subset of a psoset $\langle A; \trianglelefteq\rangle$, then $\langle H^*; \trianglelefteq^*\rangle$ is an up-directed poset.

(ii) An up-directed psoset $\langle A; \trianglelefteq\rangle$ has the maximum element a if and only if the poset $\langle A^*; \trianglelefteq^*\rangle$ has the maximum element $[a]_A$ where $[a]_A = \{a\}$.

For brevity, a trellis $\langle A; \trianglelefteq \rangle$ is said to be *cycle-complete* if every cycle in A has a join. It is clear that any lattice with a minimum element is a cycle-complete trellis. The following theorem gives some characterizations of completeness for trellises in terms of cycle-completeness.

Theorem 1. *For a trellis $\langle A; \trianglelefteq \rangle$, the following statements are equivalent.*

(1) *A is complete.*

(2) *A is cycle-complete and for every join-closed subset S of A, the poset S^* has a maximum element.*

(3) *A is cycle-complete and for every subset H of A, the poset $(H^{\nabla})^*$ has a maximum element, where H^{∇} denotes the set of all lower bounds of H in A.*

Proof. (1) $\Rightarrow$ (2): Clearly A is cycle-complete by (1). Also, for any join-closed subset S of A, $\bigvee S = a$ exists in A and $a \in S$. Hence a is the maximum element of S. This implies S^* has the maximum element $[a]_S = \{a\}$ by (ii) of Remark 1.

(2) $\Rightarrow$ (3): Follows by noting that H^{∇} in join-closed.

(3) $\Rightarrow$ (1): To show that A is complete it is enough to show that for any subset H of A, $\bigwedge H$ exists in A (see [6]). Let H be a subset of A. Then $H^{\nabla} \neq \Phi$ as $0 = \bigvee \Phi$ exists in A and therefore $0 \in H^{\nabla}$ since H^{∇} is join-closed. By (3), $(H^{\nabla})^*$ has the maximum, say $[a]_{H^{\nabla}}$. Then $[a]_{H^{\nabla}}$, being a cycle in H^{∇}, is also a cycle in A. Therefore $\bigvee [a]_{H^{\nabla}} = x$ exists in A and $x \in H^{\nabla}$. Now $[x]_{H^{\nabla}} \in (H^{\nabla})^*$ and $[a]_{H^{\nabla}} \trianglelefteq^* [x]_{H^{\nabla}}$ as $a \trianglelefteq x$. But $[a]_{H^{\nabla}}$ is the maximum of $(H^{\nabla})^*$. Thus $[a]_{H^{\nabla}} = [x]_{H^{\nabla}}$, consequently x is the maximum of the cycle $[a]_{H^{\nabla}}$. Hence $[a]_{H^{\nabla}} = \{x\}$ so that $a = x$. Therefore by (ii) of Remark 1, H^{∇} has the maximum element a and hence $a = \bigwedge H$. Thus A is complete. □

Let $\langle P; \leqslant \rangle$ be a poset and $\mathbf{S}$ the set of all ascending well-ordered chains in P. Define a binary relation $\leqslant$ on $\mathbf{S}$ for $C, D \in \mathbf{S}$ by $C \leqslant D$ if $C = D$ or $C = \{x \in D \mid x < d\}$ for some $d \in D$. Then $\langle \mathbf{S}; \leqslant \rangle$ is a poset and, by using Zorn's lemma, it follows that $\langle \mathbf{S}; \leqslant \rangle$ has a maximal element (see [1]). Any maximal element of the poset $\langle \mathbf{S}; \leqslant \rangle$ is called a *maximal ascending well-ordered chain* in P.

Remark 2. Let P be an up-directed poset. Then it is clear that the following statements are equivalent.

(i) P has the maximum element.

(ii) Every subchain of P has an upper bound.

(iii) Every ascending well-ordered chain in P has an upper bound.

(iv) Every maximal ascending well-ordered chain in P has an upper bound (or equivalently has the maximum).

(v) P has a maximal element.

In (2) of Theorem 1, we note that S^* is an up-directed poset by (i) of Remark 1. Therefore replacing P by S^* in the above remark, some equivalent formulations of (2) can be obtained. We make similar observations for (3) of Theorem 1 since H^∇ is join-closed.

Lemma 1. *A psoset $\langle A; \trianglelefteq\rangle$ satisfies the ascending p-chain condition if and only if it satisfies the following conditions.*

(1) *All cycles of A are finite.*

(2) *The poset $\langle A^*; \trianglelefteq^*\rangle$ satisfies the ascending chain condition.*

Proof. ($\Rightarrow$): (1) If C is an infinite cycle in A, then we can find infinitely many elements a_0, a_1, $a_2, \ldots$ in C. Then $a_0 \sqsubset_c a_1 \sqsubset_c a_2 \sqsubset_c \ldots$. This implies, for each $i \geqslant 0$, that there exists an integer $n_i \geqslant 0$ and $a_{ij} \in C$ for $0 \leqslant j \leqslant n_i$ such that $a_i = a_{i0} \trianglelefteq a_{i1} \trianglelefteq \ldots \trianglelefteq a_{in_i} = a_{i+1}$. These elements a_{ij} of C form an infinite ascending p-chain in A, which is a contradiction to the hypothesis.

(2) If $\langle A^*; \trianglelefteq^*\rangle$ does not satisfy the ascending chain condition, then in A^* there exists an infinite chain of the form $[a_0]_A \triangleleft^* [a_1]_A \triangleleft^* \ldots$. This implies $a_i \sqsubset_A a_{i+1}$ for $i \geqslant 0$. Now, arguing as in (1), we obtain an infinite ascending p-chain, which is a contradiction to the hypothesis.

($\Leftarrow$): Assume that (1) and (2) hold for $\langle A; \trianglelefteq\rangle$. If there exists an infinite ascending p-chain in $\langle A; \trianglelefteq\rangle$, say $a_0 \triangleleft a_1 \triangleleft \ldots$, then $[a_0]_A \trianglelefteq^* [a_1]_A \trianglelefteq^* \ldots$ in the poset $\langle A^*; \trianglelefteq^*\rangle$. By (2), this implies that there exists $n \geqslant 0$ such that $[a_n]_A = [a_{n+i}]_A$ for every $i \geqslant 1$. This implies $a_{n+i} \in [a_n]_A$ for every $i \geqslant 1$. Thus $[a_n]_A$ is an infinite cycle in A, a contradiction to (1). Therefore $\langle A; \trianglelefteq\rangle$ satisfies the ascending p-chain condition. □

We now obtain a useful corollary of Theorem 1.

Corollary 1. *A trellis $\langle A; \trianglelefteq\rangle$ satisfying the ascending p-chain condition is complete if and only if it is cycle-complete.*

Proof. ($\Rightarrow$): Obvious.

($\Leftarrow$): We verify the second part of the condition (2) of Theorem 1. Let S be a join-closed subset of A. Then $S \neq \Phi$ since $\bigvee \Phi = 0$ exists in A so that $0 \in S$. Also, S satisfies the ascending p-chain condition since A satisfies the same condition. Then S^* is nonempty and S^* satisfies the ascending chain condition by Lemma 1. Therefore S^* has a maximal element. But then S^* has the maximum by Remark 2. Hence $\langle A; \trianglelefteq\rangle$ is complete by Theorem 1. □

According to K. Gladstien [4], a psoset A is of *finite length* if there exists a finite p-chain in A such that the number of its elements is the maximum possible.

Corollary 2 (Theorem 2 in [4]). *A trellis $\langle A; \trianglelefteq\rangle$ of finite length is complete if and only if every cycle has a GLB and a LUB.*

Proof. Follows from Corollary 1, by noting that any trellis of finite length satisfies the ascending p-chain condition. $\square$

It is proved in [5] that a trellis A is complete if and only if every ascending well-ordered p-chain in A has a join. However, if A is cycle-complete this statement can be simplified as in Theorem 2 below. First we state a lemma, the proof of which is similar to that of Lemma 2.1 of [5].

Lemma 2. *Let $\langle A; \trianglelefteq\rangle$ be a psoset and let $A^{\square}$ denote the set of all acyclic ascending well-ordered p-chains in A. Define a relation $\leqslant$ on $A^{\square}$ by setting $C \leqslant D$ for $C, D \in A^{\square}$. If $C = D$ or $C = \{x \in D \setminus x \sqsubset_D d\}$ for some $d \in D$. Then $\langle A^{\square}; \leqslant\rangle$ is a poset and has a maximal element.*

Theorem 2. *A trellis $\langle A; \trianglelefteq\rangle$ is complete if and only if it is cycle-complete and every acyclic ascending well-ordered p-chain in A has a join.*

Proof. ($\Rightarrow$): Obvious.

($\Leftarrow$): Let H be any subset of A. It is enough to show that $\bigwedge H$ exists in A. Let H^{∇} be the set of all lower bounds of H and P the set of all acyclic ascending well-ordered p-chains in H^{∇}. An application of Lemma 2 yields that the poset $\langle P; \leqslant\rangle$ has a maximal element M. By hypothesis $\bigvee M = a$ exists in A. Since H^{∇} is join-closed, $a \in H^{\nabla}$. Clearly $M \cup \{a\} \in P$. If $a \notin M$, then $M < M \cup \{a\}$ as $M = \{x \in M \cup \{a\} \mid x \sqsubset_{M\cup\{a\}} a\}$, a contradiction to the maximality of M. Thus a is the maximum of M. Now $[a]_{H^{\nabla}}$, being a cycle in A, $\bigvee[a]_{H^{\nabla}} = t$ exists in A and $t \in H^{\nabla}$.

Claim. $t = a$.

If $t \neq a$, then $t \vartriangleright a$. But then $M \cup \{t\}$ is clearly an ascending well-ordered p-chain in H^{∇}. Further, $M \cup \{t\}$ is acyclic. For otherwise, it would contain a nontrivial cycle C containing t. This implies $C \cup \{a\}$ is a nontrivial cycle in $M \cup \{t\}$ containing a. But then $C \cup \{a\} \subseteq [a]_{H^{\nabla}}$ since $C \cup \{a\} \subseteq H^{\nabla}$. Hence $t \in [a]_{H^{\nabla}}$ so that t is the maximum of $[a]_{H^{\nabla}}$ and $[a]_{H^{\nabla}} = \{t\}$. Thus $a = t$, a contradiction. Therefore $M \cup \{t\} \in P$. Now $M < M \cup \{t\}$, a contradiction to the maximality of M. Therefore $t = a$.

We claim that $a = \bigwedge H$. For otherwise, there would exist an element $b \in H^{\nabla}$ such that $b \ntrianglelefteq a$. Then $a \vee b \in H^{\nabla}$ and $a \vee b \vartriangleright a$. Now it follows that $M \cup \{a \vee b\} \in P$ and $M < M \cup \{a \vee b\}$, a contradiction to the maximality of M. Thus $a = \bigwedge H$. Hence A is complete. $\square$

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