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# AN EXAMPLE OF A POSITIVE SEMIDEFINITE DOUBLE SEQUENCE WHICH IS NOT A MOMENT SEQUENCE

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*Abstract.* The first explicit example of a positive semidefinite double sequence which is not a moment sequence was given by Friedrich. We present an example with a simpler definition and more moderate growth as $(m, n) \to \infty$.

*Keywords*: double sequence, positive definite, moment sequence

*MSC 2000*: 43A35, 44A60

## 1. Introduction

Suppose $(S, +)$ is an abelian semigroup with zero. A function $\varphi\colon S \to \mathbb{R}$ is *positive semidefinite* if

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) \geqslant 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{R}$, and *positive definite* if the same sum is positive whenever the $s_j$ are pairwise distinct and the $c_j$ are not all 0. Denote by $\mathscr{P}(S)$ the set of all positive semidefinite functions on $S$. A *character* on $S$ is a function $\sigma\colon S \to \mathbb{R}$ satisfying $\sigma(0) = 1$ and $\sigma(s+t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by $S^*$ the set of all characters. A function $\varphi\colon S \to \mathbb{R}$ is a *moment function* if there is a measure $\mu$ on $S^*$ such that

$$\varphi(s) = \int_{S^*} \sigma(s)\,\mathrm{d}\mu(\sigma), \quad s \in S. \tag{1}$$

Denote by $\mathscr{H}(S)$ the set of all moment functions on $S$. We have $\mathscr{H}(S) \subset \mathscr{P}(S)$ since if (1) holds then

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) = \int_{S^*} \left( \sum_{j=1}^{n} c_j \sigma(s_j) \right)^2 \mathrm{d}\mu(\sigma) \geqslant 0.$$

The semigroup $S$ is *semiperfect* if $\mathscr{H}(S) = \mathscr{P}(S)$. For these topics, see the monograph by Berg, Christensen, and Ressel [2], especially Chapter 6.

For $k \in \mathbb{N}$ consider the semigroup $S = \mathbb{N}_0^k$. The moment functions on $S$ are the *moment sequences* (more precisely, moment multisequences if $k > 1$), that is, functions $\varphi\colon S \to \mathbb{R}$ such that

$$\varphi(n) = \int_{\mathbb{R}^k} x^n \,\mathrm{d}\mu(x), \quad n \in S$$

for some measure $\mu$ on $\mathbb{R}^k$, with the notation $x^n = x_1^{n_1} \dots x_k^{n_k}$ for $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $n = (n_1, \dots, n_k) \in \mathbb{N}_0^k$. Hamburger's Theorem [6] asserts that $S$ is semiperfect if $k = 1$. On the other hand, if $k \geqslant 2$ then $S$ is non-semiperfect as shown by Berg, Christensen, and Jensen [1] and independently by Schmüdgen [8]. Each set of authors appealed to the Hahn-Banach Theorem and so produced no explicit example of a function $\varphi \in \mathscr{P}(\mathbb{N}_0^2) \setminus \mathscr{H}(\mathbb{N}_0^2)$. The first such example was given by Friedrich [5]. In his example,

$$\varphi(0, n) = \exp\left\{ \left[ \binom{n/2+2}{2} + 1 \right]! \, \log \binom{n/2+2}{2}! \right\}$$

for even $n \geqslant 8$. This raised the question: How fast must $\varphi(m, n)$ grow as $m+n \to \infty$ if $\varphi \in \mathscr{P}(\mathbb{N}_0^2) \setminus \mathscr{H}(\mathbb{N}_0^2)$? It was shown in [3] that there is a function $\varphi \in \mathscr{P}(\mathbb{N}_0^2) \setminus \mathscr{H}(\mathbb{N}_0^2)$ such that $\varphi(m, n) = O((m+n)^{a(m+n)})$ as $n \to \infty$ for each $a > 1$, and the constant 1 is the best possible.

The example in [3] involves the integral

$$\int_0^\infty x^n \mathrm{e}^{-x/(1+(\log x)^2)} \,\mathrm{d}x,$$

which we have not been able to evaluate. The purpose of the present note is to exhibit a funciton $\varphi \in \mathscr{P}(\mathbb{N}_0^2) \setminus \mathscr{H}(\mathbb{N}_0^2)$, of growth intermediate between the example of Friedrich and the example from [3], which has the merit of being of an extremely simple form.

Let $S$ be the semigroup $\mathbb{N}_0 \setminus \{1\}$. The non-semiperfectness of $S$ was shown by Nakamura and Sakakibara [7]. We shall show that if $\gamma$ is the positive solution to the equation $\sum_{n=1}^{\infty} \gamma^{n^2} = \frac{1}{2}$ and $a = \gamma^{-1/4}$ then the function $f\colon S \to \mathbb{R}$ defined by

$$f(n) = \begin{cases} a^{n^2} & \text{if } n \text{ is even and } \quad n \neq 2, \\ 0 & \text{if } n \text{ is odd or } \quad n = 2 \end{cases}$$

is in $\mathscr{P}(S) \setminus \mathscr{H}(S)$. Any larger value of $a$ can be used instead. (For example, take $a = 2$.) Now define $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ by $\varphi(m,n) = f(2m+3n)$ for $(m,n) \in \mathbb{N}_0^2$. Then $\varphi \in \mathscr{P}(\mathbb{N}_0^2) \setminus \mathscr{H}(\mathbb{N}_0^2)$.

## 2. The example

Suppose $S$ is a set. A kernel (that is, a function) $\Phi\colon S \times S \to \mathbb{C}$ is *positive semidefinite* if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \Phi(s_j, s_k) \geqslant 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{C}$, and *positive definite* if the same sum is positive whenever the $s_j$ are pairwise distinct and the $c_j$ are not all 0. Every positive semidefinite kernel $\Phi$ is *hermitian* in the sense that $\Phi(t,s) = \overline{\Phi(s,t)}$ for all $s, t \in S$.

**Theorem 1.** *If $\Phi\colon S \times S \to \mathbb{C}$ is hermitian and such that $\Phi(s,s) = 1$ and*

$$\sum_{t\colon t \neq s} |\Phi(s,t)| \leqslant 1 \tag{2}$$

*for all $s \in S$ then $\Phi$ is positive semidefinite (and positive definite if strict inequality holds in (2)).*

Proof. For $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ pairwise distinct, and $c_1, \ldots, c_n \in \mathbb{C}$ we have

$$\begin{aligned} \sum_{j,k=1}^{n} c_j \overline{c_k} \Phi(s_j, s_k) &= \sum_{j=1}^{n} |c_j|^2 + \sum_{j,k\colon j \neq k} c_j \overline{c_k} \Phi(s_j, s_k) \\ &\geqslant \sum_{j=1}^{n} |c_j|^2 - \sum_{j,k\colon j \neq k} |c_j||c_k||\Phi(s_j, s_k)| \\ &\geqslant \sum_{j=1}^{n} |c_j|^2 - \frac{1}{2} \sum_{j,k\colon j \neq k} (|c_j|^2 + |c_k|^2)|\Phi(s_j, s_k)| \\ &= \sum_{j=1}^{n} |c_j|^2 \Big(1 - \sum_{k\colon k \neq j} |\Phi(s_j, s_k)|\Big) \geqslant \sum_{j=1}^{n} |c_j|^2 \Big(1 - \sum_{t\colon t \neq s_j} |\Phi(s_j, t)|\Big) \geqslant 0, \end{aligned}$$

with strict inequality if we have strict inequality in (2) and if the $c_j$ are not all 0. □

**Corollary 1.** *If $S$ is an abelian semigroup with zero and if $f\colon S \to \mathbb{R}$ satisfies $f(2s) > 0$ for all $s \in S$ and*

$$\sum_{t\colon t\neq s} \frac{|f(s+t)|}{\sqrt{f(2s)f(2t)}} \leqslant 1$$

*for all $s \in S$ then $f$ is positive semidefinite.*

P r o o f. For any function $\lambda\colon S \to \mathbb{R}\setminus\{0\}$, the function $f$ is positive semidefinite if and only if the kernel $(s,t) \mapsto \lambda(s)\lambda(t)f(s+t)\colon S \to \mathbb{R}$ so is. Now apply this to $\lambda(s) = f(2s)^{-1/2}$, and apply the Theorem. □

**Theorem 2.** *With $S$ and $f$ as at the end of the Introduction, the function $f$ is positive definite but not a moment function.*

P r o o f. Apply the Corollary. Denoting by $2\mathbb{Z}$ the set of all even integers, for $j \in S$ we have

$$\begin{aligned}\sum_{k\colon k\neq j} \frac{f(j+k)}{\sqrt{f(2j)f(2k)}} &\leqslant \sum_{k\colon k\neq j, k-j\in 2\mathbb{Z}} \frac{a^{(j+k)^2}}{\sqrt{a^{(2j)^2}a^{(2k)^2}}} \\ &= \sum_{k\colon k\neq j, k-j\in 2\mathbb{Z}} a^{-(k-j)^2} < 2\sum_{n=1}^{\infty} a^{-4n^2} = 1.\end{aligned}$$

This proves that $f$ is positive definite. To see that $f$ is not a moment function, suppose it is. Choose a measure $\mu$ on $S^*$ such that $f(s) = \int_{S^*} \sigma(s)\,\mathrm{d}\mu(\sigma)$ for $s \in S$. Then $0 < a^{16} = f(4) = \int_{S^*} \sigma(4)\,\mathrm{d}\mu(\sigma) = \int_{S^*} \sigma(2)^2\,\mathrm{d}\mu(\sigma)$, so with $A = \{\sigma \in S^* \mid \sigma(2) \neq 0\}$ we have $\mu(A) > 0$. Now for $\sigma \in A$ we actually have $\sigma(2) > 0$. Indeed, $\sigma(2)^3 = \sigma(6) = \sigma(3)^2 \geqslant 0$, and taking third roots we obtain $\sigma(2) \geqslant 0$. Since $\sigma \in A$, it follows that $\sigma(2) > 0$. Now $0 < \int_A \sigma(2)\,\mathrm{d}\mu(\sigma) = \int_{S^*} \sigma(2)\,\mathrm{d}\mu(\sigma) = f(2) = 0$, a contradiction. □

**Corollary 2.** *The function $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ given by $\varphi(m,n) = f(2m+3n)$ is positive semidefinite but not a moment sequence.*

P r o o f. Define a homomorphism $h$ of $\mathbb{N}_0^2$ onto $S$ by $h(m,n) = 2m+3n$, so $\varphi = f \circ h$. Since $f$ is positive semidefinite, so is $\varphi$. If $\varphi$ is a moment function then it follows from [4], Proposition 1, that so is $f$, a contradiction. □

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