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GRACEFUL SIGNED GRAPHS

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Abstract. A (p,q)-sigraph S is an ordered pair (G,s) where $G=(V,E)$ is a (p,q)-graph and s is a function which assigns to each edge of G a positive or a negative sign. Let the sets E^+ and E^- consist of m positive and n negative edges of G, respectively, where $m+n=q$. Given positive integers k and d, S is said to be (k,d)-graceful if the vertices of G can be labeled with distinct integers from the set $\{0,1,\ldots,k+(q-1)d\}$ such that when each edge uv of G is assigned the product of its sign and the absolute difference of the integers assigned to u and v the edges in E^+ and E^- are labeled $k,k+d,k+2d,\ldots,k+(m-1)d$ and $-k,-(k+d),-(k+2d),\ldots,-(k+(n-1)d)$, respectively.

In this paper, we report results of our preliminary investigation on the above new notion, which indeed generalises the well-known concept of (k,d)-graceful graphs due to B. D. Acharya and S. M. Hegde.

Keywords: signed graphs, (k,d)-graceful signed graphs

MSC 2000: 05C78

1. Introduction

Throughout this paper, unless mentioned otherwise, by a “graph” we shall mean a finite simple graph without loops as treated in F. Harary [15]; in particular, by a (p,q)-graph we mean a graph with p vertices and q edges.

Abundant literature exists as of today concerned with the structure of graphs admitting a variety of functions assigning real numbers to their elements so that certain given conditions are satisfied; hence, a graph G together with such a function f is called a *real weight network* and if, in particular, f is *injective* then it is called a *(vertex, edge or mixed) valuation of* G (e.g., see A. Rosa [18], S. W. Golomb [13], G. S. Bloom [8], [9], J. C. Bermond et al [7], B. D. Acharya [1]–[3], [6], P. J. Slater [19]–[22], B. D. Acharya and S. M. Hedge [4], [5], T. Grace [14], G. J. Chang et al [11], I. Cahit [10], M. Maheo and H. Thuillier [17], A. Kotzig [16]; see J. A. Gallian [12] for a recent review). Perhaps the most famous notion amongst them all is that of a *graceful graph*, viz., a (p,q)-graph G that admits a vertex valuation $f\colon V(G)\to\{0,1,\dots,q\}$ of G such that the induced edge-valuation g_f defined by

$$g_f(uv)=|f(u)-f(v)|,\ \ \forall uv\in E(G), \tag{1.1}$$

called the *bandwidth function*, has the property that $\{g_f(e)\colon e\in E(G)\}:=g_f(E(G))=\{1,2,\dots,q\}$; f is then called a *graceful numbering* of G. A few graphs which are graceful in this sense are shown in Figure 1. The following are some significant general facts known about graceful graphs.

Theorem 1 ([13], [18]). *The complete graph K_n of order n is graceful if and only if $n\leqslant 4$.*

Theorem 2 ([13], [18]). *If G is an eulerian (p,q)-graph with $q\equiv 0,2 \pmod 4$ then G is not graceful.*

Theorem 3 ([1]). *Every graph can be embedded as an induced subgraph in a graceful graph.*

In the following theorems, $\lceil.\rceil$ (respectively, $\lfloor.\rfloor$) means the *ceiling* (*floor*) function which assigns to each real number r the least (greatest) integer not less (greater) than r.

Theorem 4 ([13]). *A necessary condition for a (p,q)-graph $G=(V,E)$ to be graceful is that it be possible to partition its vertex set $V:=V(G)$ into two subsets V_0 and V_{e} such that there are exactly $\lceil q/2\rceil$ edges each of which joins a vertex of V_0 with one of V_{e}.*

In a Group Discussion on Graph Labeling Problems held in Karnataka Regional Engineering College (KREC), Surathkal, during August 16–25, 1999, B. D. Acharya [6] raised the following problem:

Problem 1(m). Let $\mathbb{N}$ denote the set of natural numbers, $k\geqslant 0$ and $d\geqslant 1$ be arbitrarily given integers, $G=(V,E)$ be a connected (p,q)-graph, and $\mathcal{G}=\{G_i\}$ be any collection of its edge-disjoint (p_i,q_i)-subgraphs G_i, $1\leqslant i\leqslant m$, whose union is G. Is it then possible to find an injective function $f\colon V\to\{0,1,2,\dots,k+(q-1)d\}$ such that the band-width function $g_f\colon E\to\mathbb{N}$, defined by (1.1), has the property that $g_f(E(G_i))=\{k_i,k_i+d_i,k_i+2d_i,\dots,k_i+(q_i-1)d_i\}$ for some integers $k_i\geqslant 0$ and $d_i\geqslant 1$, $1\leqslant i\leqslant m$? If so, what should be the relationships amongst the parameters p_i, q_i, k_i, d_i, k and d? If so, G is said to be *(k,d)-gracefully packable*, $\mathcal{G}$ is called a *(k,d)-graceful packing* of G and f is called a *(k,d)-graceful packing code* for G.

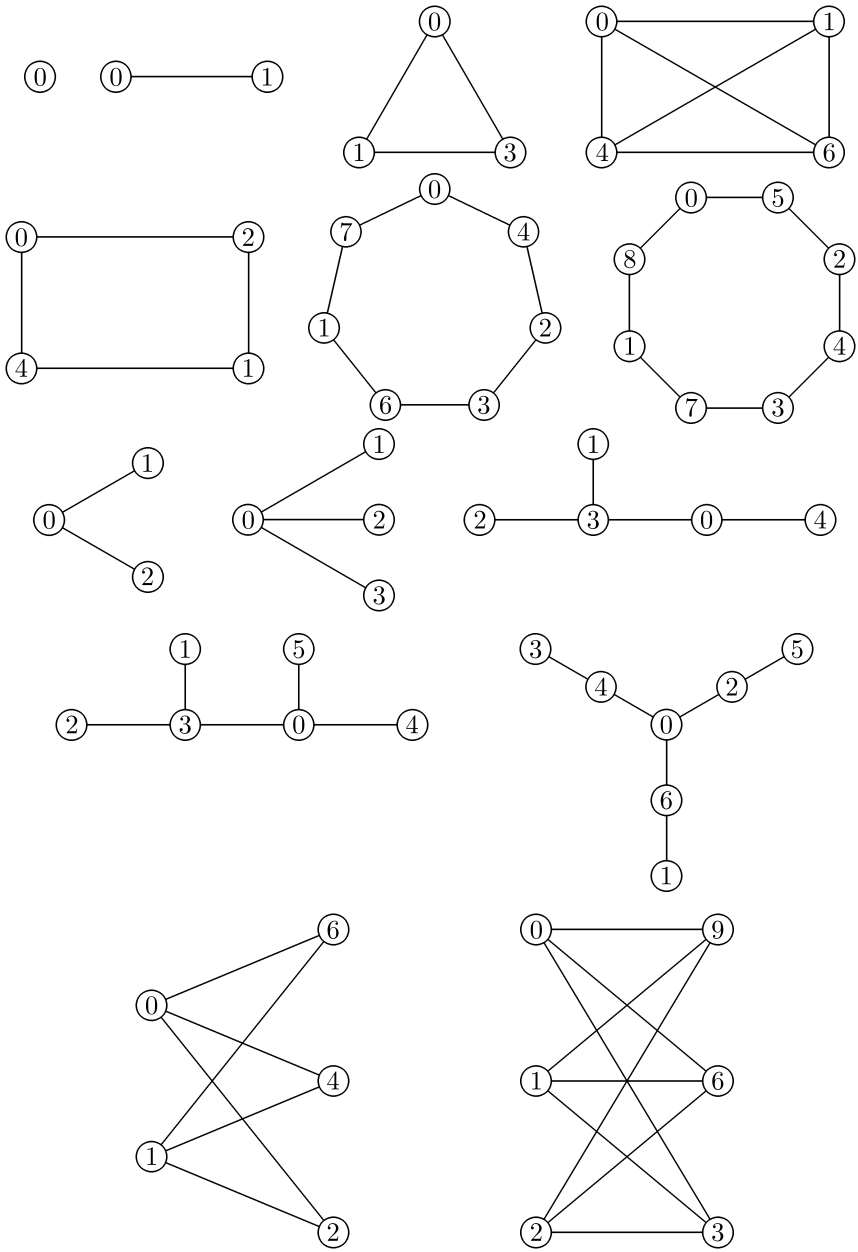

Fig. 1

In this paper, we attempt to solve Problem 1(2) using the notion of *signed graphs* (or, briefly, *sigraphs*) and taking $k_1 = k_2 = k$, $d_1 = d_2 = d$.

2. How to gracefully label a sigraph?

A (p, q)-*sigraph* S is an ordered pair (G, s) where $G = (V, E)$ is a (p, q)-graph and s is a function which assigns to each edge of G a *positive* or a *negative* sign. Let the sets E^+ and E^- consist of m positive and n negative edges of G, respectively, where $m + n = q$. Given positive integers k and d, B. D. Acharya [6] defines S to be (k, d)-*graceful* if the vertices of S can be labeled with distinct integers from the set $\{0, 1, \ldots, k + (q-1)d\}$ such that when each edge uv of S is assigned the product of its sign and the absolute difference of the integers assigned to u and v the edges in E^+ and E^- are labeled $k, k+d, k+2d, \ldots, k+(m-1)d$ and $-k, -(k+d), -(k+2d), \ldots, -(k+(n-1)d)$, respectively. In particular, a $(1,1)$-graceful labeling is called a *sigraceful labeling* and S is called *graceful* if it admits a sigraceful labeling.

For example, consider the *sistar* $K_{1,(m,n)}$ which is defined as a sigraph on the star $K_{1,r}$, $r = m + n$, having m positive edges and n negative edges. Let the central vertex of $K_{1,r}$ be labeled $k + (n-1)d$ (respectively, $k + (m-1)d$), the $n(m)$ pendant vertices of the negative (respectively, positive) edges be labeled $0, d, 2d, \ldots, (n-1)d$ (respectively, $0, d, 2d, \ldots, (m-1)d$) and the $m(n)$ pendant vertices of the positive (respectively, negative) edges be labeled $2k+(n-1)d, 2k+nd, 2k+(n+1)d, \ldots, 2k+(n+m-2)d$ (respectively, $2k+(m-1)d, 2k+md, 2k+(m+1)d, \ldots, 2k+(m+n-2)d$). Then, it may be easily seen that $K_{1,(m,n)}$ is (k, d)-gracefully numbered. Some more (k, d)-gracefully labeled small sigraphs taking specific values of k and d are displayed in Figure 2.

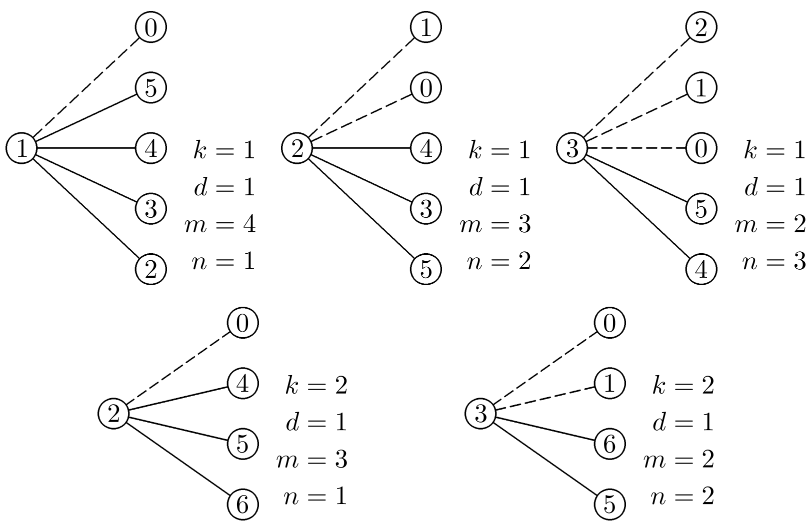

Fig. 2

In general, we shall call a sigraph S *arbitrarily graceful* if it is (k,d)-graceful for all values of k and d. Thus, we have seen above that the *star sigraph* $K_{1,(m,n)}$ is arbitrarily graceful. Are there any other such sigraphs? It may be interesting to describe some specific classes of such sigraphs, since many infinite classes of graphs are known to be arbitrarily graceful (e.g., see [3], [4]). In general, we pose

Problem 2. Determine (arbitrarily) *graceful sigraphs.*

In particular, one may be interested to solve Problem 2 for specific classes of sigraphs on certain standard classes of graphs, say *acyclic sigraphs*. For example, *are all sitrees (arbitrarily) graceful*? That all trees are graceful is the well-known and long-standing Ringel-Kotzig conjecture (e.g., see [8]). We give below a necessary condition for a sigraph to admit a graceful labeling.

Theorem 5. *Let $S=(G,s)$ be any (p,q)-sigraph with $G=(V,E)$ as its underlying graph and let the sets E^+ and E^- consist of m positive and n negative edges of G, respectively, where $m+n=q$. A necessary condition for S to be (k,d)-graceful for some positive integers k and d, which are not simultaneously even, is that it be possible to partition $V(G):=V(S)$ into two subsets V_{o} and V_{e} such that the numbers $m^+(V_{\mathrm{o}},V_{\mathrm{e}})$, $m^-(V_{\mathrm{o}},V_{\mathrm{e}})$ of positive and negative edges of S respectively each of which joins a vertex of V_{o} with one of V_{e} are given as described below:*

(i) *when k and d are both odd,*

$$m^+(V_{\mathrm{o}},V_{\mathrm{e}})=\lfloor\tfrac{1}{2}(m+1)\rfloor \quad \textit{and} \quad m^-(V_{\mathrm{o}},V_{\mathrm{e}})=\lfloor\tfrac{1}{2}(n+1)\rfloor;$$

(ii) *when k is even and d is odd,*

$$m^+(V_{\mathrm{o}},V_{\mathrm{e}})=\lfloor\tfrac{1}{2}m\rfloor \quad \textit{and} \quad m^-(V_{\mathrm{o}},V_{\mathrm{e}})=\lfloor\tfrac{1}{2}n\rfloor; \quad \textit{and}$$

(iii) *when k is odd and d is even,*

$$m^+(V_{\mathrm{o}},V_{\mathrm{e}})=m \quad \textit{and} \quad m^-(V_{\mathrm{o}},V_{\mathrm{e}})=n.$$

Proof. Since S is (k,d)-graceful there must exist a (k,d)-graceful numbering f of S. Let $V_{\mathrm{o}}=\{u\in V(S)\colon f(u) \text{ is odd}\}$ and $V_{\mathrm{e}}=V(S)-V_{\mathrm{o}}$. Now, the conclusions (i), (ii) and (iii) are obvious to see due to the fact that in each of these cases every edge receiving an odd number in g_f must join a vertex of V_{o} with one of V_{e} and $m^+(V_{\mathrm{o}},V_{\mathrm{e}})$ and $m^-(V_{\mathrm{o}},V_{\mathrm{e}})$ precisely count these edges since f is given to be (k,d)-graceful. $\square$

Corollary 5.1. *If $S = (G, s)$ is a (k, d)-graceful sigraph with k odd and d even, then G is bipartite.*

Problem 3. What can we say about (k, d)-graceful sigraphs when both k and d are required to be even?

In the next section, we shall report the results of our preliminary study of sigraphs that do, or do not, admit a (k, d)-graceful numbering for certain given values of k and d.

3. Some specific classes of graceful sigraphs

A finite *caterpillar* is a tree such that the removal of its *pendant vertices* (i.e., vertices of degree 1) results in a simple path. If we want to extend this definition to include infinite trees, we need to adopt the following approach: A tree T is called a *caterpillar* if its vertex set $V(T)$ can be arranged in two columns (or rows) L (for *left*) and R (for *right*) on the euclidean plane such that (i) $\{L, R\}$ is a *bipartition* of T (i.e., $L \cap R = \emptyset$ and each of L and R is a nonempty independent set of vertices of T such that $L \cup R = V(T)$), (ii) every edge of T is drawn as a "Jordan curve" (i.e., a simple curve which does not intersect itself and is *topologically homeomorphic* to the closed unit interval) with its vertices as the two ends of the curve, and (iii) no two of these curves (representing edges of T) intersect at a point other than possibly at their ends. *Unless mentioned otherwise, by a tree we shall mean one which may possibly be infinite.*

Let $S = (G, s)$ be any sigraph. We shall say that it is *bifurcated* if the *positive subgraph* $S^+ = (V^+, E^+(S))$ and the *negative subgraph* $S^- = (V^-, E^-(S))$ of S are both connected subgraphs of S, where V^+ (V^-, respectively) denotes the subset of $V(G)$ consisting of the ends of the positive (negative) edges in S. Next, for any vertex v of S, let E_v^+ (E_v^-, respectively) denote the set of positive (negative) edges of S that are incident at v. Then, $d^+(v)$ ($d^-(v)$, respectively) denotes the cardinality of E_v^+ (E_v^-, respectively) and is called the *positive (negative) degree* of v in S. Then the *degree* $d(v)$ of v in S is defined as the cardinality of $E_v^+ \cup E_v^-$. If S is finite, then clearly $d(v) = d^+(v) + d^-(v)$.

Theorem 6. *Every finite bifurcated signed caterpillar is graceful.*

P r o o f. Let (T, s) be any finite bifurcated signed caterpillar of order p. Since T is a bipartite graph, it has a bipartition $\{L, R\}$ and since T is a (finite) caterpillar we may assume that L and R are the left and the right columns of vertices, respectively, in a *plane vertical imbedding* of T described above. Now, if either $E^+(T)$ or $E^-(T)$

is empty, then T may be treated as the usual graph-theoretical caterpillar which is well known to be graceful (e.g., see [18]). Hence, we shall assume that (T, s) is a *heterogeneous* sigraph (i.e., a sigraph S in which both $E^+(S)$ and $E^-(S)$ are nonempty). Then, clearly, since T is basically a tree there must exist a unique vertex u such that both E_u^+ and E_u^- are nonempty as also the positive and the negative subgraphs (each of which is connected, by hypothesis) of (T, s) must be positioned one *above* (with respect to the plane vertical embedding of the signed caterpillar) the other as seen from u. Without loss of generality, we consider a plane vertical imbedding of (T, s) in which the negative subgraph T^- appears above the positive subgraph T^+ at the vertex u. For convenience, we shall refer to such an imbedding of (T, s) a *negative-up plane imbedding*. Hence, in such an imbedding of (T, s) any vertex v for which $E_v^+ = \emptyset$ (called a *negative vertex*) appears above u and any vertex w for which $E_w^- = \emptyset$ (called a *positive vertex*) appears below u. Without loss of generality, we may assume that $u \in L$; *otherwise*, we may permute the columns L and R to interchange their labels L' and R'. Hence, choose the lowest negative vertex adjacent to u, say v_1. Then, $v_1 \in R$ we label the negative vertices in R lying above v_1 successively $v_2, v_3, \ldots, v_r$ as we go up, with v_r as the "highest" negative vertex in R. Next, we choose the "highest" negative vertex adjacent to v_r in L, if any, and label it as u_{r+1}. We then label all the vertices below u_{r+1} in L successively $u_{r+2}, u_{r+3}, \ldots, u_{r+s}$, with u_{r+s} as the "lowest" vertex in L. Let u_{r+m} be the negative vertex immediately above u in L so that in the above procedure u is labeled as u_{r+m+1}. Since $E_u^+ \neq \emptyset$, there must exist a "lowest" vertex in R, which we label as v_{r+s+1}. Clearly, due to the plane imbedding of T, it follows that u_{r+s} must be adjacent to v_{r+s+1}. Hence, we label the other positive vertices above v_{r+s+1} in R successively $v_{r+s+2}, v_{r+s+3}, \ldots, v_p$ as we go up in the right column R. Now, let $f\colon V(T) \to \{0, 1, \ldots, p-1\}$ be a function defined by saying $f(u_i) = i-1$ and $f(v_j) = j-1$ for each $i \in \{r+1, r+2, \ldots, r+s\}$ and $j \in \{1, 2, \ldots, r\} \cup \{r+s+1, r+s+2, \ldots, p\}$. It is not hard to verify that f so defined is indeed a sigraceful numbering of T. $\square$

Some gracefully labeled bifurcated signed caterpillars are shown in Figure 3.

In general, B. D. Acharya [6] has offered the following generalisation of the famous Ringel-Kotzig conjecture.

Conjecture 1. All bifurcated signed trees are graceful.

Next a (p, q)-sigraph with m positive edges and n negative edges will be called a (p, m, n)-sigraph.

Theorem 7. *Let $S = (G, s)$ be any (p, m, n)-sigraph such that G is eulerian. If S is graceful, then $m^2 + n^2 + m + n \equiv 0 \pmod 4$.*

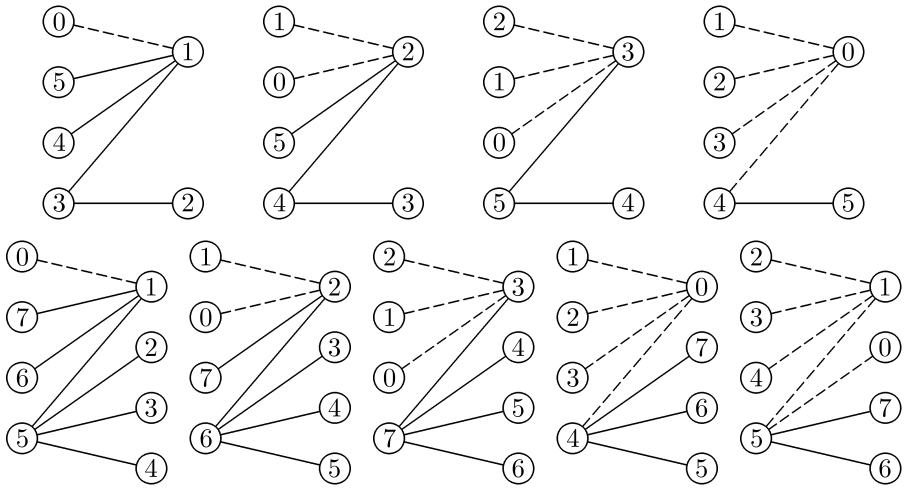

Fig. 3

P r o o f. Let f be any sigraceful numbering of S and Z be any eulerian circuit in S. Then, since the signs of the edges in S are disregarded while computing $\sum\limits_{e\in Z} g_f(e) \equiv \sum\limits_{e_i\in Z} (a_i - a_{i+1})$, where $g_f(e_i) = |a_i - a_{i+1}|$ for each i, the result follows from a result in [13] (p. 26, Theorem 2), that for any integer-valued function f defined on the vertex set of G, if each edge uv of G is given an edge number equal to its band-width $|f(u) - f(v)|$ then the sum of the band-widths of the edges forming any circuit of G is even as also that the edge set of any eulerian graph can be decomposed into disjoint subsets each of which spans a circuit. □

Corollary 7.1. *If a* (k, m, n) *signed cycle* Z_k, $k = m + n \geqslant 3$, *is graceful then* $m^2 + n^2 + m + n \equiv 0 \pmod 4$.

Corollary 7.2 ([13])**.** *If* G *is a graceful eulerian* (p, q)*-graph then* $q \equiv 0$, *or* 3 (mod 4).

So, the signed cycles of lengths $\equiv 1 \pmod 4$ are not graceful. We will now begin examining the sufficiency part of Corollary 7.1. Since this is known to be true in the case of all-positive cycles (see [18]), we shall deal only with the heterogeneous case. Towards this end, in a heterogeneously signed cycle Z_n, $n \geqslant 3$, by a *negative (positive) section* we mean a maximal subgraph consisting of only the negative (positive) edges of Z_n. The following result demonstrates that there are further necessary conditions for Z_n to be graceful!

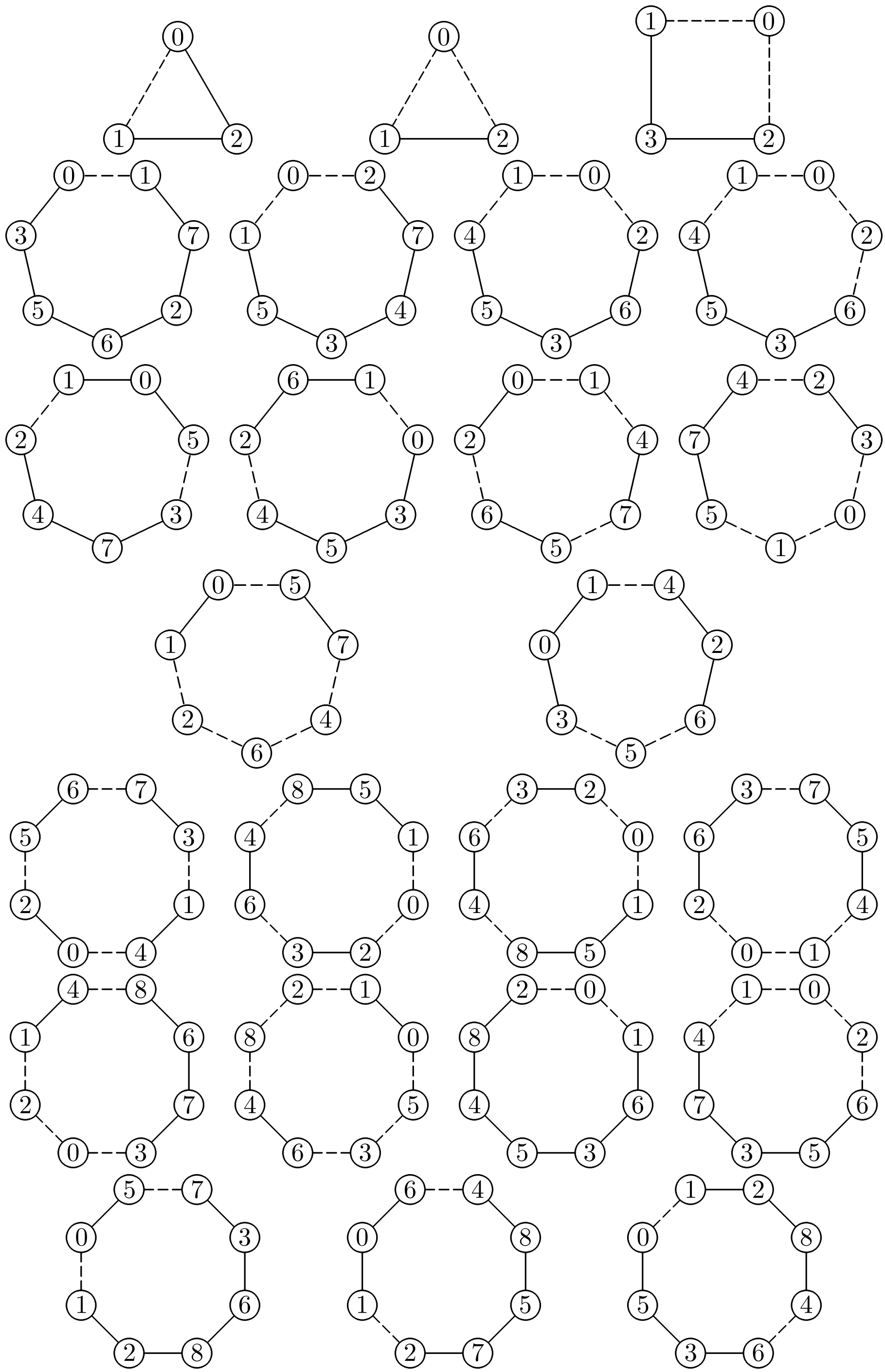

Fig. 4

Theorem 8. *Let Z_n be a heterogeneous signed cycle of length $n \equiv 0 \pmod 4$. If Z_n is graceful then the number of negative sections of odd lengths is even.*

Proof. Let Z_n be any heterogeneous signed cycle of length n, $n \equiv 0 \pmod 4$, possessing a sigraceful numbering f and let $l_1, l_2, \ldots, l_k$ be the lengths of the negative sections, $k \geqslant 1$. Suppose the number of negative sections of odd lengths in Z_n is odd, say $2r+1$ for some positive integer r. Without loss of generality, we may assume $l_1, l_2, \ldots, l_{2r+1}$ are the odd ones. Let $l_i = 2a_i + 1$ for $i \in \{1, 2, \ldots, 2r+1\}$ and $l_i = 2b_i$ for $i \in \{2r+2, 2r+3, \ldots, k\}$. Since the sum of the band-widths of the edges in a cycle is even, for some positive integer m we must have

$$2m = \sum_{e \in Z_n} g_f(e) = \sum_{e \in E^-(Z_n)} g_f(e) + \sum_{e \in E^+(Z_n)} g_f(e) = \binom{q^- + 1}{2} + \binom{q^+ + 1}{2}, \tag{3.1}$$

which implies

$$4m = q^-(q^- + 1) + q^+(q^+ + 1). \tag{3.2}$$

Now,

$$q^- = \sum_{i=1}^{2r+1} l_i + \sum_{i=2r+2}^{k} l_i = 2\left(r + \sum_{i=1}^{2r+1} a_i + \sum_{i=2r+2}^{k} b_i\right) + 1. \tag{3.3}$$

Thus, q^- is of the form

$$q^- = 2R + 1, \qquad R = \left(r + \sum_{i=1}^{2r+1} a_i + \sum_{i=2r+2}^{k} b_i\right). \tag{3.4}$$

Now, since $n \equiv 0 \pmod 4$, there exists a positive integer x such that $n = 4x$. Further, since n is the number of edges in Z_n, we also have $4x = q^- + q^+$. Therefore, $q^+ = n - q^- = (4x - 2R - 1)$. Substituting this in (3.2), we get $4m = (2R+1) \times (2R+2) + (4x - 2R - 1)(4x - 2R)$ which implies

$$2m = (R+1)(2R+1) + (2x - R)(4x - 2R - 1). \tag{3.5}$$

If $R = 2t + 1$ for some positive integer t, then (3.5) gives $2m = 2(t+1)(4t+3) + (2x - 2t - 1)(4x - 4t - 3) \equiv 1 \pmod 2$, a preposterous statement. Therefore, $R = 2t$ for some positive integer t. Again, substituting this in (3.5) gives $2m = (2t+1)(4t+1) + 2(x-t)(4x - 4t - 1) \equiv 1 \pmod 2$, a preposterous statement again. Thus, our assumption that the number of negative sections of odd lengths in Z_n is odd cannot be sustained, and hence it must be even as claimed. $\square$

Sufficiency of the condition in Theorem 8 is not guaranteed in general, since it may be verified that Z_4 having two negative sections of length 1 each is not graceful. However, *it appears that excepting this case, the sufficiency of the condition must also hold.* Before embarking on examining this contention, all the structural possibilities amenable for gracefulness of Z_4 are exhaustively illustrated in Figure 4.

An exhaustive treatment of the sufficiency part of Theorem 8 for values of $n \equiv 0$, 3 (mod 4), $n \geqslant 7$, being quite involved and tedious, will be dealt with separately elsewhere.

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