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ON THE EMBEDDING OF ORDERED SEMIGROUPS
INTO ORDERED GROUP

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Abstract. It was shown in [7] that any right reversible, cancellative ordered semigroup can be embedded into an ordered group and as a consequence, it was shown that a commutative ordered semigroup can be embedded into an ordered group if and only if it is cancellative. In this paper we introduce the concept of L -maher and R -maher semigroups and use a technique similar to that used in [7] to show that any left reversible cancellative ordered L or R -maher semigroup can be embedded into an ordered group.

Keywords: semicommutative semigroups, maher semigroups, ordered semigroups

MSC 2000: 06F05

1. INTROUCTION AND PRELIMINARIES

The concept of L -semicommutative (R -semicommutative) semigroups was first introduced in [1]. A semigroup $(S, *)$ is called L -semicommutative if and only if $\forall a, b \in S: a * b * a = a^2 * b$ and R -semicommutative if and only if $\forall a, b \in S: a * b * a = b * a^2$. Clearly any commutative semigroup is both L -semicommutative and R -semicommutative and any cancellative L -semicommutative or R -semicommutative semigroup is commutative.

A semigroup $(S, *)$ is called left (right) reversible if $\forall a, b \in S: a * S \cap b * S \neq \emptyset$ ($S * a \cap S * b \neq \emptyset$). An R -semicommutative (L -semicommutative) semigroup is left (right) reversible. It is well known that any right reversible cancellative semigroup can be embedded in a group [2], Theorem 1.23. Kehayopulu and Tsingelis [7] proved that a commutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative. The following theorem is an immediate consequence of the main theorem in [7].

Theorem 1. An L -semicommutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative.

In order to have the dual result of the above theorem for R -semicommutative semigroups, we introduce the concept of L -maher and R -maher semigroups which are generalizations of L and R -semicommutative semigroups, respectively, and show that both of them are embeddable in ordered groups.

Definition 2. A semigroup $(S, *)$ is called *cancellative* if

$$\forall a, b, c \in S: c * a = c * b \implies a = b \quad \text{and} \quad a * c = b * c \implies a = b.$$

A semigroup $(S, *)$ is called *R-maher* if

$$\forall a, b, x, y \in S: (a * x = b * y \implies (a * b * a) * x = (b * a * b) * y)$$

and *L-maher* if

$$\forall a, b, x, y \in S: (x * a = y * b \implies x * (a * b * a) = y * (b * a * b)).$$

Commutative semigroups are both L and R -maher. L -semicommutative semigroups and R -semicommutative semigroups are L and R -maher, respectively. An ordered semigroup $(S, *, \leq_S)$ is called *cancellative* if

$$\forall a, b, c \in S: (c * a \leq_S c * b \implies a \leq_S b \quad \text{and} \quad a * c \leq_S b * c \implies a \leq_S b).$$

Let $(S, *, \leq_S)$, (T, \diamond, \leq_T) be ordered semigroups. Let $f: S \rightarrow T$ be a mapping. Then f is called isotone if $x, y \in S$, $x \leq_S y$ implies that $f(x) \leq_T f(y)$. f is called reverse isotone if $x, y \in S$, $f(x) \leq_T f(y)$ implies that $x \leq_S y$. Clearly, any reverse isotone mapping is injective. f is called a homomorphism if it is isotone and

$$\forall x, y \in S: f(x * y) = f(x) \diamond f(y).$$

f is called an isomorphism if it is an onto reverse isotone homomorphism. S is embeddable in T if $(S, *, \leq_S)$ is isomorphic to a subsemigroup of (T, \diamond, \leq_T) .

2. CONSTRUCTING AN ORDERED GROUP FROM AN ORDERED SEMIGROUP

Let $(S, *)$ be a left reversible semigroup and $a, b \in S$. We define a relation $E(a, b)$ on S as follows: $(x, y) \in E(a, b) \iff a * x = b * y$.

Clearly, a semigroup $(S, *)$ is R -maher if $(x, y) \in E(a, b) \implies (x, y) \in E(a * b * a, b * a * b)$.

The following lemma will be used repeatedly in this paper.

Lemma 3. *Let $(S, *)$ be a left reversible semigroup. Then*

- 1) $\forall a, b \in S: E(a, b) \neq \emptyset$,
- 2) $\forall a, b \in S: E(a, b) = (E(b, a))^{-1}$,
- 3) if $(S, *)$ is cancellative and $a, b, c, d \in S$, then

$$E(a, b) \cap E(c, d) \neq \emptyset \iff E(a, b) = E(c, d),$$

- 4) if $(S, *)$ is cancellative and $a, b \in S$, then $E(a, a) = E(b, b)$.

Proof. Parts 1 and 2 follow from the definitions of left reversibility and $E(a, b)$.

3) (\Rightarrow) Assume that $E(a, b) \cap E(c, d) \neq \emptyset$. Let $(x, y) \in E(a, b) \cap E(c, d)$, $(u, v) \in E(c, d)$, $(z, t) \in E(u, x)$. Then

$$a * x = b * y, \quad c * x = d * y, \quad c * u = d * v, \quad u * z = x * t.$$

It then follows that

$$\begin{aligned} d * (v * z) &= (d * v) * z = (c * u) * z = c * (u * z) \\ &= c * (x * t) = (c * x) * t = (d * y) * t = d * (y * t). \end{aligned}$$

Since $(S, *)$ is cancellative, $v * z = y * t$. Also,

$$\begin{aligned} (a * u) * z &= a * (u * z) = a * (x * t) = (a * x) * t \\ &= (b * y) * t = b * (y * t) = b * (v * z) = (b * v) * z. \end{aligned}$$

Since $(S, *)$ is cancellative, it follows that $a * u = b * v$ and thus $(u, v) \in E(a, b)$, which concludes that $E(c, d) \subseteq E(a, b)$. Similarly, $E(a, b) \subseteq E(c, d)$ and therefore $E(a, b) = E(c, d)$.

(\Leftarrow) Immediate.

4) follows immediately from part 3, since $(a, a) \in E(a, a) \cap E(b, b)$. □

In what follows $(S, *)$ is a left reversible cancellative semigroup. We define a relation ξ on $S \times S$ as follows:

$$((a, b), (c, d)) \in \xi \iff E(a, c) = E(b, d).$$

The following lemma shows that ξ is an equivalence relation on S .

Lemma 4. *Let $(S, *)$ be a left reversible, cancellative semigroup. Then ξ is an equivalence relation on $S \times S$.*

Proof. From Lemma 3(4) we immediately get that the relation ξ is reflexive.

To show that ξ is symmetric, let $((a, b), (c, d)) \in \xi$. Then $E(a, c) = E(b, d)$ and Lemma 3(2) implies that

$$E(c, a) = (E(a, c))^{-1} = (E(b, d))^{-1} = E(d, b),$$

which shows that $((c, d), (a, b)) \in \xi$.

To show that ξ is transitive, let $((a, b), (c, d)) \in \xi$ and $((c, d), (e, f)) \in \xi$. Then $E(a, c) = E(b, d)$, $E(c, e) = E(d, f)$. Let $(x, y) \in E(a, c)$, $(u, v) \in E(d, f)$ and $(t, z) \in E(y, u)$. Then

$$(1) \quad a * x = c * y, \quad b * x = d * y, \quad d * u = f * v, \quad c * u = e * v, \quad y * t = u * z.$$

By a straightforward calculations we get

$$b * (x * t) = f * (v * z), \quad a * (x * t) = e * (v * z),$$

which means that $(x * t, v * z) \in E(b, f) \cap E(a, e)$. Thus Lemma 3(3) implies that $E(a, e) = E(b, f)$ and therefore $((a, b), (e, f)) \in \xi$. \square

Let $(a, b) \in S \times S$. We denote by $[(a, b)]_\xi$ the equivalence class of (a, b) with respect to the equivalence relation ξ .

Lemma 5. *Let $(S, *)$ be a left reversible cancellative semigroup and let $a, b, c, d \in S$. Then*

- 1) $[(a, a)]_\xi = [(b, b)]_\xi$,
- 2) $((a, b), (c, d)) \in \xi \iff ((b, a), (d, c)) \in \xi$.

Proof. 1) Since $E(a, b) = E(a, b)$, it follows by the definition of ξ that $((a, a), (b, b)) \in \xi$ and thus $[(a, a)]_\xi = [(b, b)]_\xi$.

2) The definition of ξ implies that

$$\begin{aligned} ((a, b), (c, d)) \in \xi &\iff E(a, c) = E(b, d) \iff \\ E(b, d) = E(a, c) &\iff ((b, a), (d, c)) \in \xi. \end{aligned}$$

\square

Let \odot be defined on $(S \times S)/\xi$ as follows:

$$[(a, b)]_\xi \odot [(c, d)]_\xi = [(c * y, b * x)]_\xi \quad \text{for some } (x, y) \in E(a, d).$$

The following lemma shows that \odot is an associative binary operation on $(S \times S)/\xi$.

Lemma 6. Let $(S, *)$ be a left reversible cancellative semigroup. Then \odot is an associative binary operation on $(S \times S)/\xi$.

Proof. We first show that \odot is well defined. So let $[(a, b)]_\xi = [(a', b')]_\xi$ and $[(c, d)]_\xi = [(c', d')]_\xi$ in $(S \times S)/\xi$. Then

- (2) $[(a, b)]_\xi \odot [(c, d)]_\xi = [(c * y, b * x)]_\xi$ for some $(x, y) \in E(a, d)$,
(3) $[(a', b')]_\xi \odot [(c', d')]_\xi = [(c' * y', b' * x')]_\xi$ for some $(x', y') \in E(a', d')$.

Since $((a, b), (a', b')), ((c, d), (c', d')) \in \xi$, we have $E(a, a') = E(b, b')$, $E(c, c') = E(d, d')$. Also, $a * x = d * y$, $a' * x' = d' * y'$. Let $(\alpha, \beta) \in E(b * x, b' * x')$. Then

$$\begin{aligned} b * x * \alpha = b' * x' * \beta &\implies (x * \alpha, x' * \beta) \in E(b, b') = E(a, a') \\ &\implies (a * x) * \alpha = (a' * x') * \beta \implies (d * y) * \alpha = (d' * y') * \beta \implies \\ (y * \alpha, y' * \beta) &\in E(d, d') = E(c, c') \implies (c * y) * \alpha = (c' * y') * \beta. \end{aligned}$$

Thus $(\alpha, \beta) \in E(c * y, c' * y')$ and $E(b * x, b' * x') \cap E(c * y, c' * y') \neq \emptyset$; hence $E(b * x, b' * x') = E(c * y, c' * y')$. This means that $((b * x, c * y), (b' * x', c' * y')) \in \xi$. Then (2) and (3) and Lemma 5(2) imply that $[(a, b)]_\xi \odot [(c, d)]_\xi = [(a', b')]_\xi \odot [(c', d')]_\xi$. Therefore, \odot is well defined.

It remains to show that \odot is associative. So, let $[(a, b)]_\xi, [(c, d)]_\xi, [(e, f)]_\xi \in (S \times S)/\xi$. Then

- (4) $([(a, b)]_\xi \odot [(c, d)]_\xi) \odot [(e, f)]_\xi = [(c * y, b * x)]_\xi \odot [(e, f)]_\xi$
 $= [(e * z, (b * x) * t)]_\xi$ for some $(x, y) \in E(a, d), (t, z) \in E(c * y, f)$,
(5) $[(a, b)]_\xi \odot (([c, d)]_\xi \odot [(e, f)]_\xi) = [(a, b)]_\xi \odot [(e * y', d * x')]_\xi$
 $= [((e * y') * z', (b * t'))]_\xi$ for some $(x', y') \in E(c, f), (t', z') \in E(a, d * x')$.

Let $(\alpha, \beta) \in E(e * z, (e * y') * z')$. Then

$$\begin{aligned} (e * z) * \alpha = ((e * y') * z') * \beta &\implies z * \alpha = (y' * z') * \beta \\ &\implies (f * z) * \alpha = (f * y') * (z' * \beta) \\ &\implies c * y * t * \alpha = c * x' * z' * \beta \\ &\implies y * t * \alpha = x' * z' * \beta \\ &\implies d * y * t * \alpha = d * x' * z' * \beta \\ &\implies a * x * t * \alpha = a * t' * \beta \\ &\implies x * t * \alpha = t' * \beta \\ &\implies (b * x * t) * \alpha = (b * t') * \beta. \end{aligned}$$

This concludes that $(\alpha, \beta) \in E(b * x * t, b * t')$ and $E(e * z, (e * y') * z') \cap E((b * x) * t, b * t') \neq \emptyset$. Hence $E(e * z, (e * y') * z') = E((b * x) * t, b * t')$ and thus $((e * z, (b * x) * t), ((e * y') * z', b * t')) \in \xi$. The combination of the last statement with (4) and (5) implies that \odot is associative. \square

Theorem 7. *Let $(S, *)$ be a left reversible cancellative semigroup. Then $((S \times S)/\xi, \odot)$ is a group.*

Proof. By Lemma 6, \odot is an associative binary operation on $(S \times S)/\xi$. It remains to show that $(S \times S)/\xi$ contains an identity element with respect to the binary operation \odot and that for each element in $(S \times S)/\xi$ there exists an inverse element in $(S \times S)/\xi$ with respect to \odot .

For any $a \in S$, $[(a, a)]_\xi$ is the identity of $(S \times S)/\xi$ with respect to \odot . Indeed, if $[(c, d)]_\xi \in (S \times S)/\xi$ then

$$[(a, a)]_\xi \odot [(c, d)]_\xi = [(c * y, a * x)]_\xi \quad \text{for some } (x, y) \in E(a, d).$$

Take $(\alpha, \beta) \in E(a * x, d)$, then

$$\begin{aligned} (a * x) * \alpha = d * \beta &\Rightarrow (d * y) * \alpha = d * \beta \Rightarrow y * \alpha = \beta \Rightarrow c * (y * \alpha) = c * \beta \\ &\Rightarrow (\alpha, \beta) \in E(c * y, c) \Rightarrow E(a * x, d) \cap E(c * y, c) \neq \emptyset, \end{aligned}$$

hence $E(a * x, d) = E(c * y, c)$. Thus $((c * y, a * x), (c, d)) \in \xi$ and $[(c * y, a * x)]_\xi = [(c, d)]_\xi$. Similarly $[(c, d)]_\xi \odot [(a, a)]_\xi = [(c, d)]_\xi$.

To show that each element of $(S \times S)/\xi$ has an inverse in $(S \times S)/\xi$, let $[(a, b)]_\xi \in (S \times S)/\xi$. Then $[(b, a)]_\xi \in (S \times S)/\xi$ and

$$[(a, b)]_\xi \odot [(b, a)]_\xi = [(b * y, b * x)]_\xi \quad \text{for some } (x, y) \in E(a, a).$$

Since $a * x = a * y$ and $(S, *)$ is cancellative, we have $x = y$ and

$$[(a, b)]_\xi \odot [(b, a)]_\xi = [(b * x, b * x)]_\xi = [(a, a)]_\xi \quad (\text{see Lemma 3(4)}).$$

Similarly, if $(x, y) \in E(b, b)$ then $x = y$ and

$$[(b, a)]_\xi \odot [(a, b)]_\xi = [(a * y, a * x)]_\xi = [(a * x, a * y)]_\xi = [(a, a)]_\xi \quad (\text{see Lemma 3(4)}).$$

This yields $[(a, b)]_\xi^{-1} = [(b, a)]_\xi$. \square

Since every R -semicommutative semigroup is left reversible, we obtain

Corollary 8. *Let $(S, *)$ be an R -semicommutative cancellative semigroup. Then $((S \times S)/\xi, \odot)$ is a group.*

Let $(S, *, \leq_S)$ be a left reversible cancellative ordered semigroup. As in [7], we define a relation “ \preceq ” on $(S \times S)/\xi$ as follows:

$$\begin{aligned} ([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq &\iff \exists(x, y) \in [(a, b)]_\xi, (z, t) \in [(c, d)]_\xi; \\ &(\forall(k, h) \in E(y, t): x * k \leq_S z * h). \end{aligned}$$

The following lemma and its proof is a modification of equivalent statements proved in [7].

Lemma 9. *Let $(S, *, \leq_S)$ be a left reversible cancellative ordered semigroup. Then*

- 1) $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq \iff \forall(x, y) \in [(a, b)]_\xi, \forall(z, t) \in [(c, d)]_\xi,$
 $(\forall(k, h) \in E(y, t): x * k \leq_S z * h);$
- 2) $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq \iff \exists(x, y) \in [(a, b)]_\xi, \exists(z, t) \in [(c, d)]_\xi,$
 $\exists(k, h) \in E(y, t): (x * k \leq_S z * h).$

Proof. (\Leftarrow) immediate.

(\Rightarrow) Let $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq$, $(x, y) \in [(a, b)]_\xi$, $(z, t) \in [(c, d)]_\xi$ and $(k, h) \in E(y, t)$. Since $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq$ there exists, by definition of \preceq , $(x_1, y_1) \in [(a, b)]_\xi$, $(z_1, t_1) \in [(c, d)]_\xi$ such that

$$(6) \quad \forall(u, v) \in E(y_1, t_1): x_1 * u \leq_S z_1 * v.$$

Since $(k, h) \in E(y, t)$, we have $y * k = t * h$. Since $((x, y), (x_1, y_1)) \in \xi$, $E(x, x_1) = E(y, y_1)$. Similarly, $E(z, z_1) = E(t, t_1)$. Let $(s, w) \in E(x_1, x)$, $(e, f) \in E(z_1, z)$. Then $x * w = x_1 * s$, $y * w = y_1 * s$, $z * f = z_1 * e$, $t * f = t_1 * e$. Let $(\lambda, \varrho) \in E(w, k)$, $(\alpha, \beta) \in E(h * \varrho, f)$. Then $w * \lambda = k * \varrho$ and $h * \varrho * \alpha = f * \beta$. Hence,

$$\begin{aligned} (y_1 * s) * \lambda * \alpha &= (y * w) * \lambda * \alpha = y * (k * \varrho) * \alpha = (t * h) * \varrho * \alpha \\ &= t * (h * \varrho * \alpha) = t * (f * \beta) = (t * f) * \beta = (t_1 * e) * \beta. \end{aligned}$$

This shows that $(s * \lambda * \alpha, e * \beta) \in E(y_1, t_1)$ and by (6) we have

$$x_1 * (s * \lambda * \alpha) \leq_S z_1 * (e * \beta) = (z * f) * \beta = z * (h * \varrho * \alpha).$$

On the other hand,

$$(x_1 * s) * \lambda * \alpha = (x * w) * \lambda * \alpha = x * (w * \lambda) * \alpha = x * (k * \varrho) * \alpha.$$

Therefore, $x * k * \varrho * \alpha \leq_S z * h * \varrho * \alpha$. Since $(S, *, \leq_S)$ is cancellative, we have $x * k \leq_S z * h$. Since $(x, y), (z, t)$ and (k, h) were arbitrarily chosen from $[(a, b)]_\xi$, $[(c, d)]_\xi$ and $E(y, t)$, respectively, we have

$$\forall(x, y) \in [(a, b)]_\xi, \forall(z, t) \in [(c, d)]_\xi: (\forall(k, h) \in E(y, t): x * k \leq_S z * h).$$

2) (\implies) immediate.

(\impliedby) Let $(x, y) \in [(a, b)]_\xi, (z, t) \in [(c, d)]_\xi$ and let $(k, h) \in E(y, t)$ be such that $x * k \leq_S z * h$. Let (u, v) be arbitrarily chosen in $E(y, t)$ and $(\lambda, \varrho) \in E(k, u)$. Then

$$\begin{aligned} (t * h) * \lambda &= (y * k) * \lambda = y * (k * \lambda) = y * (u * \varrho) \\ &= (y * u) * \varrho = (t * v) * \varrho \Rightarrow h * \lambda = v * \varrho, \\ x * (u * \varrho) &= x * (k * \lambda) = (x * k) * \lambda \leq_S (z * h) * \lambda \\ &= z * (h * \lambda) = z * (v * \varrho) \Rightarrow x * u \leq_S z * v. \end{aligned}$$

□

Theorem 10. *Let $(S, *, \leq_S)$ be a left reversible cancellative ordered semigroup. Then $((S \times S)/\xi, \odot, \preceq)$ is an ordered group.*

Proof. Theorem 8 shows that $((S \times S)/\xi, \odot)$ is a group. It remains to show that it is ordered by \preceq . So let $[(a, b)]_\xi \in (S \times S)/\xi$. Then $(a, b) \in [(a, b)]_\xi$ and for any $(x, y) \in E(b, b)$ we have $x = y$ and thus $a * x \leq_S a * y$. It then follows, by the definition of \preceq , that $([(a, b)]_\xi, [(a, b)]_\xi) \in \preceq$. Therefore \preceq is reflexive.

Let $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq, ([(c, d)]_\xi, [(a, b)]_\xi) \in \preceq$. Then, by Lemma 9(1),

$$\forall(x, y) \in [(a, b)]_\xi, \forall(z, t) \in [(c, d)]_\xi, \forall(k, h) \in E(y, t): x * k \leq_S z * h.$$

Let $(k, h) \in E(b, d)$. Since $(a, b) \in [(a, b)]_\xi$ and $(c, d) \in [(c, d)]_\xi$, we have $a * k \leq_S c * h$. Also, $(h, k) \in E(d, b)$ and thus $c * h \leq_S a * k$, which implies that $a * k = c * h$. Hence, we have $(k, h) \in E(b, d) \cap E(a, c)$ and $E(a, c) \cap E(b, d) \neq \emptyset$. It means that $E(a, c) = E(b, d)$ and $[(a, b)]_\xi = [(c, d)]_\xi$. Therefore \preceq is anti-symmetric.

Let $([(a, b)]_\xi, [(c, d)]_\xi) \in \preceq, ([(c, d)]_\xi, [(e, f)]_\xi) \in \preceq, (k, h) \in E(b, d)$ and $(u, v) \in E(d, f)$. Since $(a, b) \in [(a, b)]_\xi, (c, d) \in [(c, d)]_\xi$, it follows by Lemma 9(2) that $a * k \leq_S c * h$. Similarly, since $(c, d) \in [(c, d)]_\xi, (e, f) \in [(e, f)]_\xi$, it follows by Lemma 9(2) that $c * u \leq_S e * v$. Let $(\lambda, \varrho) \in E(h, u)$. Then

$$\begin{aligned} (b * k) * \lambda &= (d * h) * \lambda = d * (h * \lambda) = d * (u * \varrho) = (d * u) * \varrho = (f * v) * \varrho \\ &\Rightarrow (k * \lambda, v * \varrho) \in E(b, f). \end{aligned}$$

Also, since $a * k \leq_S c * h$ and $c * u \leq_S e * v$, we have

$$(a * k) * \lambda \leq_S (c * h) * \lambda = c * (h * \lambda) = c * (u * \varrho) = (c * u) * \varrho \leq_S (e * v) * \varrho.$$

Thus, we have $(a, b) \in [(a, b)]_\xi$, $(e, f) \in [(e, f)]_\xi$ and $(k * \lambda, v * \varrho) \in E(b, f)$ such that $a * (k * \lambda) \leq_S e * (v * \varrho)$. Hence by Lemma 9(2) $[(a, b)]_\xi, [(e, f)]_\xi \in \preceq$. Therefore \preceq is transitive.

We have shown that the relation \preceq is a partial order on the set $(S \times S)/\xi$.

Let $[(a, b)]_\xi \preceq [(c, d)]_\xi, [(e, f)]_\xi \in (S \times S)/\xi$ and $(k, h) \in E(b, d)$. Then $a * k \leq_S c * h$ and

$$[(a, b)]_\xi \odot [(e, f)]_\xi = [(e * v, b * u)]_\xi \quad \text{for some } (u, v) \in E(a, f);$$

$$[(c, d)]_\xi \odot [(e, f)]_\xi = [(e * v', d * u')]_\xi \quad \text{for some } (u', v') \in E(c, f).$$

Remember that $[(e * v, b * u)]_\xi \preceq [(e * v', d * u')]_\xi$ if and only if there exist $(l, m) \in [(e * v, b * u)]_\xi, (n, r) \in [(e * v', d * u')]_\xi$ and $(p, q) \in E(m, r)$ such that $l * p \leq_S n * q$.

This can be achieved as follows:

Let $(x, y) \in E(k, u), (z, t) \in E(h * x, u')$. Then

$$\begin{aligned} b * (u * y) * z &= b * (k * x) * z = (b * k) * (x * z) = (d * h) * (x * z) \\ &= d * (h * x * z) = d * (u' * t) \Rightarrow (y * z, t) \in E(b * u, d * u'). \end{aligned}$$

Also,

$$\begin{aligned} (f * v) * (y * z) &= (a * u) * (y * z) = a * (u * y) * z = a * (k * x) * z \\ &= (a * k) * (x * z) \leq_S (c * h) * (x * z) = c * (h * x * z) \\ &= c * (u' * t) = (c * u') * t = (f * v') * t \Rightarrow v * y * z \leq_S v' * t \\ &\Rightarrow e * v * y * z \leq_S e * v' * t. \end{aligned}$$

Take $(l, m) = (e * v, b * u) \in [(e * v, b * u)]_\xi, (n, r) = (e * v', d * u') \in [(e * v', d * u')]_\xi$ and $(p, q) = (y * z, t) \in E(b * u, d * u')$. Then $l * p \leq_S n * q$ and, by Lemma 9(2), we have

$$[(e * v, b * u)]_\xi \preceq [(e * v', d * u')]_\xi.$$

Hence, $[(a, b)]_\xi \odot [(e, f)]_\xi \preceq [(c, d)]_\xi \odot [(e, f)]_\xi$.

Similarly, $[(e, f)]_\xi \odot [(a, b)]_\xi \preceq [(e, f)]_\xi \odot [(c, d)]_\xi$. □

3. EMBEDDING THEOREMS

The following theorem is an immediate consequence of the main theorem of [7].

Theorem 11. Any right reversible ordered L -maher semigroup is embeddable in an ordered group if and only if it is cancellative.

The dual of the above theorem, for the left reversible semigroups, is still true as stated and proved in the next theorem.

Theorem 12. Let $(S, *, \leq_S)$ be a left reversible ordered R -maher semigroup. Then $(S, *, \leq_S)$ is embeddable in an ordered group if and only if it is cancellative.

Proof. (\Leftarrow) Assume that $(S, *, \leq_S)$ is cancellative. Define $f: S \rightarrow (S \times S)/\rho$ as follows: $\forall x \in S: f(x) = [(x^2, x)]_\xi$. To show that f is a homomorphism, let $x, y \in S$. Then

$$f(x) \odot f(y) = [(x^2, x)]_\xi \odot [(y^2, y)]_\xi = [(y^2 * t, x * z)]_\xi \quad \text{for some } (z, t) \in E(x^2, y).$$

Let $u \in S$. Then $(u, y * u) \in E(x * y, x)$ and $(u, y * u) \in E((x * y)^2, (x * y * x))$.

Hence, $E(x * y, x) \cap E((x * y)^2, x * y * x) \neq \emptyset$ and we have $((x * y * x, x), ((x * y)^2, x * y)) \in \xi$. So

$$f(x * y) = [(x * y)^2, (x * y)]_\xi = [(x * y * x, x)]_\xi.$$

Also, since $x^2 * z = y * t$, we have $y * x^2 * z = y^2 * t$. Hence,

$$f(x) \odot f(y) = [(y^2 * t, x * z)]_\xi = [(y * x^2 * z, x * z)]_\xi.$$

Similarly, if $v \in S$ then $(v, z * v) \in E(x * z, x) \cap E(y * x^2 * z, y * x^2)$. Hence, we have $((y * x^2 * z, x * z), (y * x^2, x)) \in \xi$ and

$$f(x) \odot f(y) = [(y * x^2 * z, x * z)]_\xi = [(y * x^2, x)]_\xi.$$

Since $x^2 * z = y * t$, we have $(x * z, t) \in E(x, y)$ and since $(y * x^2) * (x * z) = (y * x) * (x^2 * z) = (y * x) * (y * t)$, we have $(x * z, t) \in E(y * x^2, y * x * y)$. Thus, we have $(x * z, t) \in E(x, y) \cap E(y * x^2, y * x * y) \neq \emptyset$. Therefore, $((y * x^2, x), (y * x * y, y)) \in \xi$ and

$$f(x) \odot f(y) = [(y * x^2, x)]_\xi = [(y * x * y, y)]_\xi.$$

Since $(S, *)$ is R -maher and $(x * z, t) \in E(x, y)$, we have $(x * z, t) \in E(x * y * x, y * x * y)$. This implies that $((x * y * x, x), (y * x * y, y)) \in \xi$ and

$$f(x) \odot f(y) = [(y^2 * t, x * z)]_\xi = [(y * x * y, y)]_\xi = [(x * y * x, x)]_\xi = f(x * y).$$

It remains to show that f is isotone and reverse isotone. So, let $a, b, x, y \in S$ be such that $a \leq_S b$ and $f(x) \preceq f(y)$. We will show that $f(a) \preceq f(b)$ and $x \leq_S y$. We have

$$f(a) = [(a^2, a)]_\xi, \quad f(b) = [(b^2, b)]_\xi.$$

Let $(k, h) \in E(a, b)$. Then

$$a * k = b * h \implies a^2 * k = a * b * h \leq_S b^2 * h.$$

Thus, $(a^2, a) \in [(a^2, a)]_\xi$, $(b^2, b) \in [(b^2, b)]_\xi$, $(k, h) \in E(a, b)$ and $a^2 * k \leq_S b^2 * h$, which implies that $f(a) \preceq f(b)$. Therefore, f is isotone.

Since $f(x) \preceq f(y)$, $[(x^2, x)]_\xi \preceq [(y^2, y)]_\xi$ and since $(x^2, x) \in [(x^2, x)]_\xi$, $(y^2, y) \in [(y^2, y)]_\xi$, there exists $(k, h) \in E(x, y)$ such that $x^2 * k \leq_S y^2 * h$. Hence, $x^2 * k = x * (x * k) = x * (y * h)$. Thus, $x * y * h \leq_S y^2 * h$. Since $(S, *, \leq_S)$ is cancellative, we have $x \leq_S y$ and f is reverse isotone.

(\implies) Assume that $(S, *, \leq_S)$ is embeddable in an ordered group (G, \otimes, \leq_G) . Then there exists a reverse isotone (thus isotone) homomorphism $f: (S, *, \leq_S) \longrightarrow (G, \otimes, \leq_G)$. Now assume that $a, b, c \in S$ are such that $a * c \leq_S b * c$. Then

$$\begin{aligned} f(a * c) \leq_G f(b * c) &\implies f(a) \otimes f(c) \leq_G f(b) \otimes f(c) \\ \xrightarrow{\text{“(}G, \otimes\text{) is a group}} f(a) \leq_G f(b) &\xrightarrow{\text{“}f \text{ reverse isotone}} a \leq_S b. \end{aligned}$$

Similarly, $c * a \leq_S c * b$ implies that $a \leq_S b$. □

Since every ordered R -semicommutative semigroup is a left reversible R -maher ordered semigroup, we get

Corollary 13. *An R -semicommutative ordered semigroup is embeddable in an ordered group if and only if it is cancellative.*

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