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THE GENERALIZED HOLDITCH THEOREM FOR THE  
HOMOTHETIC MOTIONS ON THE PLANAR KINEMATICS

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*Abstract.* W. Blaschke and H. R. Müller [4, p. 142] have given the following theorem as a generalization of the classic Holditch Theorem: Let  $E/E'$  be a 1-parameter closed planar Euclidean motion with the rotation number  $\nu$  and the period  $T$ . Under the motion  $E/E'$ , let two points  $A = (0, 0)$ ,  $B = (a + b, 0) \in E$  trace the curves  $k_A, k_B \subset E'$  and let  $F_A, F_B$  be their orbit areas, respectively. If  $F_X$  is the orbit area of the orbit curve  $k$  of the point  $X = (a, 0)$  which is collinear with points  $A$  and  $B$  then

$$F_X = \frac{[aF_B + bF_A]}{a + b} - \pi\nu ab.$$

In this paper, under the 1-parameter closed planar homothetic motion with the homothetic scale  $h = h(t)$ , the generalization given above by W. Blaschke and H. R. Müller is expressed and

$$F_X = \frac{[aF_B + bF_A]}{a + b} - h^2(t_0)\pi\nu ab,$$

is obtained, where  $\exists t_0 \in [0, T]$ .

*Keywords:* Holditch Theorem, homothetic motion, Steiner formula

*MSC 2000:* 53A17

## 1. INTRODUCTION

Let  $E$  and  $E'$  be moving and fixed Euclidean planes and  $\{O; \mathbf{e}_1, \mathbf{e}_2\}$  and  $\{O'; \mathbf{e}'_1, \mathbf{e}'_2\}$  be their coordinate systems, respectively. By taking  $\mathbf{OO}' = \mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$  for  $u_1, u_2 \in \mathbb{R}$ , the motion defined by the transformation

$$(1.1) \quad \mathbf{x}' = h\mathbf{x} - \mathbf{u}$$

is called 1-parameter planar homothetic motion and denoted by  $E/E'$ , where  $h$  is a homothetic scale of the motion  $E/E'$  and  $\mathbf{x}$  and  $\mathbf{x}'$  are the position vectors with

respect to the moving and fixed rectangular coordinate systems of a point  $X \in E$ , respectively. The homothetic scale  $h$  and the vectors  $\mathbf{x}$ ,  $\mathbf{x}'$  and  $\mathbf{u}$  are continuously differentiable functions of a real parameter  $t$ . Furthermore, at the initial time  $t = 0$  the coordinate systems  $\{O; \mathbf{e}_1, \mathbf{e}_2\}$  and  $\{O'; \mathbf{e}'_1, \mathbf{e}'_2\}$  are coincident. Taking  $\varphi = \varphi(t)$  as the rotation angle between  $\mathbf{e}_1$  and  $\mathbf{e}'_1$ , the equation

$$(1.2) \quad \begin{aligned} \mathbf{e}_1 &= \cos \varphi \mathbf{e}'_1 + \sin \varphi \mathbf{e}'_2 \\ \mathbf{e}_2 &= -\sin \varphi \mathbf{e}'_1 + \cos \varphi \mathbf{e}'_2 \end{aligned}$$

can be written. If

$$(1.3) \quad \begin{aligned} u_j(t+T) &= u_j(t), \quad j = 1, 2 \\ \varphi(t+T) &= \varphi(t) + 2\pi\nu, \quad \forall t \in [0, T] \end{aligned}$$

then the motion  $E/E'$  is called *1-parameter closed planar homothetic motion* with the period  $T > 0$  and the rotation number  $\nu \in \mathbb{Z}$ . To avoid the cases of the pure translation and the pure rotation we assume that

$$\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0.$$

Under the 1-parameter closed planar homothetic motions, if  $P = (p_1, p_2)$  is the pole point of the motion at the time  $t$  then the sliding velocity of a fixed point  $X = (x_1, x_2) \in E$  with respect to  $E'$  is

$$(1.4) \quad d\mathbf{x}' = \{(x_1 - p_1) dh - (x_2 - p_2) h d\varphi\} \mathbf{e}_1 + \{(x_1 - p_1) h d\varphi + (x_2 - p_2) dh\} \mathbf{e}_2.$$

Furthermore, the orbit area  $F_X$  of the point  $X$ , given by Gauss area formula [3], is

$$(1.5) \quad F_X = \frac{1}{2} \oint (x'_1 dx'_2 - x'_2 dx'_1),$$

where the integration is taken along the closed orbit curve of  $X$ . Then, we obtain

$$(1.6) \quad \begin{aligned} 2F_X &= (x_1^2 + x_2^2) \int_0^T h^2(t) d\varphi(t) - 2x_1 \int_0^T p_1(t) h^2(t) d\varphi(t) \\ &\quad - 2x_2 \int_0^T p_2(t) h^2(t) d\varphi(t) + \int_0^T \{u_1(t) p_1(t) h(t) d\varphi(t) \\ &\quad + u_2(t) p_2(t) h(t) d\varphi(t) + u_1(t) p_2(t) dh(t) - u_2(t) p_1(t) dh(t)\} \\ &\quad + x_1 \int_0^T \{u_2(t) dh(t) - 2p_2(t) h(t) dh(t) + h(t) du_2(t)\} \\ &\quad + x_2 \int_0^T \{-u_1(t) dh(t) + 2p_1(t) h(t) dh(t) - h(t) du_1(t)\}, \quad [1]. \end{aligned}$$

Moreover, using the mean value theorem of integral-calculus for the closed interval  $0 \leq t \leq T$ , there exists at least a point  $t_0 \in [0, T]$  such that

$$(1.7) \quad \int_0^T h^2(t) d\varphi(t) = \int_0^T h^2(t)\dot{\varphi}(t) dt = 2h^2(t_0)\pi\nu.$$

By taking  $\nu \neq 0$ , the Steiner point  $S = (s_1, s_2)$  for the closed planar homothetic motion can be written as

$$(1.8) \quad s_j = \frac{\int_0^T h^2(t)p_j(t) d\varphi(t)}{\int_0^T h^2(t) d\varphi(t)}, \quad j = 1, 2.$$

Thus, from Eqs. (1.6), (1.7) and (1.8) we get

$$(1.9) \quad F_X = F_0 + h^2(t_0)\pi\nu(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2) + \mu_1x_1 + \mu_2x_2, \quad [1],$$

where  $F_0$  is the orbit area of the origin of the moving coordinate system and

$$(1.10) \quad \begin{aligned} \mu_1 &= \frac{1}{2} \int_0^T \{-2h(t)p_2(t) dh(t) + h(t) du_2(t) + u_2(t) dh(t)\}, \\ \mu_2 &= \frac{1}{2} \int_0^T \{2h(t)p_1(t) dh(t) - h(t) du_1(t) - u_1(t) dh(t)\}. \end{aligned}$$

Eq. (1.9) is called *the Steiner area formula* for the 1-parameter closed planar homothetic motion.

## 2. THE GENERALIZED HOLDITCH THEOREM FOR THE CLOSED PLANAR HOMOTHETIC MOTIONS

**Theorem 1.** *Let  $E/E'$  be 1-parameter planar homothetic motion with the rotation number  $\nu$ . Let  $F_A, F_B$  denote the orbit areas of the orbit curves  $k_A, k_B \subset E'$  of the points  $A = (0, 0), B = (a + b, 0) \in E$ , respectively. If  $F_X$  is the orbit area of the orbit curve  $k$  of the point  $X = (a, 0)$ , which is collinear with points  $A$  and  $B$ , then*

$$(2.1) \quad F_X = \frac{[aF_B + bF_A]}{a + b} - h^2(t_0)\pi\nu ab.$$

**Proof.** From Eq. (1.9), for the orbit areas  $F_A, F_B$  and  $F_X$ , we obtain

$$(2.2) \quad F_A = F_0,$$

$$(2.3) \quad F_B = F_0 + h^2(t_0)\pi\nu[(a + b)^2 - 2s_1(a + b)] + \mu_1(a + b),$$

and

$$(2.4) \quad F_X = F_0 + h^2(t_0)\pi\nu(a^2 - 2s_1a) + \mu_1a.$$

From Eqs. (2.2) and (2.3), we have

$$(2.5) \quad \frac{aF_B + bF_A}{a + b} - h^2(t_0)\pi\nu ab = F_0 + h^2(t_0)\pi\nu(a^2 - 2s_1a) + \mu_1a.$$

Then, from Eqs. (2.4) and (2.5), we get Eq. (2.1). □

**Special case 1.** In the case of the homothetic scale  $h \equiv 1$ , we get

$$(2.6) \quad F_X = \frac{[aF_B + bF_A]}{a + b} - \pi\nu ab,$$

which was given by [4].

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