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ON THE WEAK-OPEN IMAGES OF METRIC SPACES

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Abstract. In this paper, we give characterizations of certain weak-open images of metric spaces.

Keywords: $g$-metrizable spaces, weak-bases, weak-open mappings, $\sigma$-mappings, $\pi$-mappings, $cs$-mappings

MSC 2000: 54E99, 54C10

1. Introduction

To find internal characterizations of certain images of metric spaces is one of the central problems in General Topology. Recently, S. Xia [4] introduced the concept of weak-open mappings. By using it, certain $g$-first countable spaces are characterized as images of metric spaces under various weak-open mappings. Papers [6], [8], [9], [10], [11], [20] have done some wonderful work on $g$-metrizable spaces, but have only investigated internal characterizations of $g$-metrizable spaces. The present paper establishes the relationships between $g$-metrizable spaces (spaces with compact-countable weak-bases) and metric spaces by means of weak-open mappings, $\pi$-mappings and $\sigma$-mappings (weak-open mappings and $cs$-mappings, respectively).

In this paper, all spaces are regular and $T_1$, all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers, $\omega$ denotes $\mathbb{N} \cup \{0\}$. For a collection $\mathcal{P}$ of subsets of a space $X$ and a mapping $f: X \to Y$, denote $f(\mathcal{P}) = \{f(P): P \in \mathcal{P}\}$.

Definition 1.1. Let $\mathcal{P}$ be a cover of a space $X$. $\mathcal{P}$ is called compact-countable if for each compact subset $K$ of $Y$, only countably many members of $\mathcal{P}$ intersect $K$.

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Definition 1.2. Let \( P = \bigcup \{ P_x : x \in X \} \) be a collection of subsets of a space \( X \) satisfying that for each \( x \in X \),
(1) \( P_x \) is a network of \( x \) in \( X \),
(2) if \( U, V \in P_x \), then \( W \subset U \cap V \) for some \( W \in P_x \).

\( P \) is called a weak-base for \( X \) if a subset \( G \) of \( X \) is open in \( X \) if and only if for each \( x \in G \), there exists \( P \in P_x \) such that \( P \subset G \).

A space \( X \) is called a \( g \)-metrizable space if \( X \) has a \( \sigma \)-locally finite weak-base.

Definition 1.3. Let \( f : X \to Y \) be a mapping.
(1) \( f \) is a weak-open mapping if there exists a weak-base \( B = \bigcup \{ B_y : y \in Y \} \) for \( Y \), and for \( y \in Y \) there exists \( x(y) \in f^{-1}(y) \) satisfying condition (\( \ast \)): for each open neighbourhood \( U \) of \( x(y) \), \( B_y \subset f(U) \) for some \( B_y \in B_y \).
(2) \( f \) is a cs-mapping if for each compact subset \( K \) of \( Y \), \( f^{-1}(K) \) is separable in \( X \).
(3) \( f \) is a \( \sigma \)-mapping if there exists a base \( \mathcal{B} \) for \( X \) such that \( f(\mathcal{B}) \) is a \( \sigma \)-locally finite collection of subsets of \( Y \).
(4) \( f \) is a \( \pi \)-mapping if \( (X, d) \) is a metric space, and for each \( y \in Y \) and its open neighbourhood \( V \) in \( Y \), \( d(f^{-1}(y), X \setminus f^{-1}(V)) > 0 \).

It is easy to check that a weak-open mapping is a quotient mapping.

2. The weak-open \( \sigma \)-image of a metric space

Lemma 2.1. Suppose \( (X, d) \) is a metric space and \( f : X \to Y \) is a quotient mapping. Then \( Y \) is a symmetric space if and only if \( f \) is a \( \pi \)-mapping.

Theorem 2.2. The following are equivalent for a space \( X \):
(1) \( Y \) is a \( g \)-metrizable space.
(2) \( Y \) is a weak-open, \( \pi \), \( \sigma \)-image of a metric space.
(3) \( Y \) is a weak-open \( \sigma \)-image of a metric space.

Proof. (1) \( \Rightarrow \) (2) Suppose \( Y \) is a \( g \)-metrizable space, then \( Y \) has a \( \sigma \)-locally finite weak-base. Let \( \mathcal{P} = \bigcup \{ \mathcal{P}_i : i \in \mathbb{N} \} \) be a \( \sigma \)-locally finite weak-base for \( Y \), where each \( \mathcal{P}_i = \{ P_\alpha : \alpha \in A_i \} \) is locally finite in \( Y \) which is closed under finite intersections and \( Y \in \mathcal{P}_i \subset \mathcal{P}_{i+1} \). For each \( i \in \mathbb{N} \), endow \( A_i \) with discrete topology. Then \( A_i \) is a metric space. Put

\[ X = \left\{ \alpha = (\alpha_i) \in \prod_{i \in \mathbb{N}} A_i : \{ P_{\alpha_i} : i \in \mathbb{N} \} \subset \mathcal{P} \text{ forms a network} \right\}, \]

at some point \( x(\alpha) \in Y \).
and endow $X$ with the subspace topology induced from the usual product topology of the collection \( \{ A_i : i \in \mathbb{N} \} \) of metric spaces. Then $X$ is a metric space. Since $Y$ is Hausdorff, $x(\alpha)$ is unique in $Y$ for each $\alpha \in X$. We define $f : X \to Y$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in X$. Because $\mathcal{P}$ is a $\sigma$-locally finite weak-base for $Y$, we conclude that $f$ is surjective. For each $\alpha = (\alpha_i) \in X$, $f(\alpha) = x(\alpha)$. Suppose $V$ is an open neighbourhood of $x(\alpha)$ in $Y$. Then there exists $n \in \mathbb{N}$ such that $x(\alpha) \in P_{\alpha_n} \subset V$. If we set $W = \{ c \in X : \text{the } n\text{-th coordinate of } c \text{ is } \alpha_n \}$, then $W$ is an open neighbourhood of $\alpha$ in $X$ and $f(W) \subset P_{\alpha_n} \subset V$. Hence $f$ is continuous. We will show that $f$ is a weak-open $\sigma$-mapping.

(i) $f$ is a $\sigma$-mapping.

For each $n \in \mathbb{N}$ and $\alpha_n \in A_n$, put

$$V(\alpha_1, \ldots, \alpha_n) = \{ \beta \in X : \text{for each } i \leq n, \text{ the } i\text{-th coordinate of } \beta \text{ is } \alpha_i \}.$$  

It is easy to check that $\{ V(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N} \}$ is a locally neighbourhood base of $\alpha$ in $X$.

Let $\mathcal{B} = \{ V(\alpha_1, \ldots, \alpha_n) : \alpha_i \in A_i \ (i \leq n) \text{ and } n \in \mathbb{N} \}$; then $\mathcal{B}$ is a base for $X$. To prove that $f$ is a $\sigma$-mapping, we only need to check that $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap_{i \leq n} P_{\alpha_i}$ for each $n \in \mathbb{N}$ and $\alpha_n \in A_n$ because $f(\mathcal{B})$ is $\sigma$-locally finite in $Y$ by this result.

For each $n \in \mathbb{N}$, $\alpha_n \in A_n$ and $i \leq n$ we have $f(V(\alpha_1, \ldots, \alpha_n)) \subset P_{\alpha_i}$, hence $f(V(\alpha_1, \ldots, \alpha_n)) \subset \bigcap_{i \leq n} P_{\alpha_i}$. On the other hand, for each $x \in \bigcap_{i \leq n} P_{\alpha_i}$ there is $\beta = (\beta_j) \in X$ such that $f(\beta) = x$. For each $j \in \mathbb{N}$, $P_{\beta_j} \in \mathcal{P}_j \subset \mathcal{P}_{j+n}$, hence there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$, then $\alpha \in V(\alpha_1, \ldots, \alpha_n)$ and $f(\alpha) = x$. Thus $\bigcap_{i \leq n} P_{\alpha_i} \subset f(V(\alpha_1, \ldots, \alpha_n))$, hence $f(V(\alpha_1, \ldots, \alpha_n)) = \bigcap_{i \leq n} P_{\alpha_i}$.

Therefore, $f$ is a $\sigma$-mapping.

(ii) $f$ is a weak-open mapping.

Denote $\mathcal{P}_y = \{ P \in \mathcal{P} : y \in P \}$; then $\mathcal{P} = \bigcup \{ \mathcal{P}_y : y \in Y \}$.

For each $y \in Y$, by is the idea $\mathcal{P}$, there exists $(\alpha_i) \in \bigcap_{i \in \mathbb{N}} A_i$ such that $\{ P_{\alpha} : i \in \mathbb{N} \} \subset \mathcal{P}$ is a network of $y$ in $Y$, hence $\alpha = (\alpha_i) \in f^{-1}(y)$.

Suppose $G$ is an open neighbourhood of $\alpha$ in $X$. Then there exists $j \in \mathbb{N}$ such that $V(\alpha_1, \ldots, \alpha_j) \subset G$. Thus $f(V(\alpha_1, \ldots, \alpha_j)) \subset f(G)$. By (i), $f(V(\alpha_1, \ldots, \alpha_j)) = \bigcap_{i \leq j} P_{\alpha_i}$. So $P_y \subset \bigcap_{i \leq j} P_{\alpha_i}$ for some $P_y \in \mathcal{P}_y$. Hence $P_y \subset f(G)$.

Hence there exists a weak-base $\mathcal{P}$ for $Y$ and $\alpha \in f^{-1}(y)$ satisfying the condition $(\ast)$ from Definition 1.3(1). Therefore $f$ is a weak-open mapping.

(iii) $f$ is a $\pi$-mapping.

By (ii), $f$ is a quotient mapping. Since a $g$-metrizable space is symmetric, $f$ is a $\pi$-mapping by Lemma 2.1.
(2) ⇒ (3) is clear.

(3) ⇒ (1). Suppose $Y$ is the image of a metric space $X$ under a weak-open $\sigma$-mapping $f$. Since $f$ is a $\sigma$-mapping, there exists a base $\mathcal{B}$ for $X$ such that $f(\mathcal{B})$ is $\sigma$-locally finite in $Y$. And since $f$ is a weak-open mapping, there exists a weak-base $\mathcal{P} = \bigcup \{ \mathcal{P}_y : y \in Y \}$ for $Y$ such that for each $y \in Y$ there exists $x(y) \in f^{-1}(y)$ satisfying the condition (*) from Definition 1.3. For each $y \in Y$, put

$$\mathcal{F}_y = \{ f(B) : x(y) \in B \in \mathcal{B} \}.$$ 

Obviously, $\mathcal{F} \subset f(\mathcal{B})$, hence $\mathcal{F}$ is $\sigma$-locally finite in $Y$. We will prove that $\mathcal{F}$ is a weak-base for $Y$.

It is obvious that $\mathcal{F}$ satisfies the condition (1) from Definition 1.2. For each $y \in Y$, suppose $U, V \in \mathcal{F}_y$; then there exist $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ such that $x(y) \in B_1 \cap B_2$ and $f(B_1) = U$, $f(B_2) = V$. Since $\mathcal{B}$ is a base for $X$, there exists $B \in \mathcal{B}$ such that $x(y) \in B \subset B_1 \cap B_2$. Thus $f(B) \in \mathcal{F}_y$ and $f(B) \subset f(B_1 \cap B_2) \subset U \cap V$. Hence $\mathcal{F}$ satisfies the condition (2) from Definition 1.2.

Suppose $G \subset Y$ is open in $Y$, then $x(y) \in f^{-1}(G)$ for each $y \in G$. Since $\mathcal{B}$ is a base for $X$, we have $x(y) \in B \subset f^{-1}(G)$ for some $B \in \mathcal{B}$. Thus $f(B) \in \mathcal{F}_y$ and $f(B) \subset G$. On the other hand, suppose that $G \subset Y$ and for $y \in G$ there exists $F \in \mathcal{F}_y$ such that $F \subset G$. Then there exists $B \in \mathcal{B}$ such that $x(y) \in B$ and $F = f(B)$. Since $B$ is an open neighbourhood of $x(y)$, there exists $P_y \in \mathcal{P}_y$ such that $P_y \subset f(B)$. Thus for each $y \in G$ there exists $P_y \in \mathcal{P}_y$ such that $P_y \subset G$. Hence $G$ is open in $Y$ because $\mathcal{P}$ is a weak-base for $Y$. So $\mathcal{F}$ is a weak-base for $Y$.

Therefore $Y$ is a $g$-metrizable space.

\[ \square \]

3. The weak-open $cs$-image of a metric space

**Theorem 3.1.** A space $Y$ has a compact-countable weak-base if and only if $Y$ is a weak-open $cs$-image of a metric space.

**Proof.** Sufficiency. Suppose $Y$ is the image of a metric space $X$ under a weak-open $cs$-mapping $f$. Since $f$ is a weak-open mapping, there exists a weak-base $\mathcal{B} = \bigcup \{ \mathcal{B}_y : y \in Y \}$ for $Y$ such that for each $y \in Y$ there exists $x(y) \in f^{-1}(y)$ satisfying the condition (*) from Definition 1.3. Because $X$ is a metric space, $X$ has a $\sigma$-locally finite base. Let $\mathcal{P}$ be a $\sigma$-locally finite base for $X$. For each $P \in \mathcal{P}$, put

$$\mathcal{B}_P = \{ B \in \mathcal{B} : B \subset f(P) \},$$

$$B_P = \bigcup \mathcal{B}_P,$$
then \( B_P \subset f(P) \). For each compact subset \( K \) of \( Y \), since \( f \) is a \( cs \)-mapping, \( f^{-1}(K) \) is separable in \( X \). So \( f^{-1}(K) \) is a Lindelöf subspace of \( X \). Because a locally finite collection of a Lindelöf space is countable, \( \{ P \in \mathcal{P} : P \cap f^{-1}(K) \neq \emptyset \} \) is countable. Thus \( f(\mathcal{P}) \) is compact-countable. Hence \( \mathcal{B}^* = \{ B_P : P \in \mathcal{P} \} \) is compact-countable. For each \( y \in Y \), put

\[
\mathcal{B}'_y = \{ B_P \in \mathcal{B}^* : B_y \in \mathcal{B}_P \text{ for some } B_y \in \mathcal{B}_y \},
\]

\[
\mathcal{B}''_y = \left\{ \bigcap \mathcal{U} : \mathcal{U} \text{ is a finite subcollection of } \mathcal{B}'_y \right\},
\]

\[
\mathcal{B}'' = \bigcup \{ \mathcal{B}''_y : y \in Y \},
\]

then \( \mathcal{B}'' \) is compact-countable. We will prove that \( \mathcal{B}'' \) is a weak-base for \( Y \). It is easy to check that \( \mathcal{B}'' \) satisfies the condition (1), (2) from Definition 1.2.

Suppose \( V \) is open in \( Y \) for each \( y \in Y \), since \( \mathcal{P} \) is a base for \( X \), then \( x(y) \in P \subset f^{-1}(V) \) for some \( P \in \mathcal{P} \). Thus there exists \( B_y \in \mathcal{B}_y \) such that \( B_y \subset f(P) \), and so \( B_y \subset \mathcal{B}_P \). Hence \( B_P \in \mathcal{B}'_y \subset \mathcal{B}''_y \) and \( B_P \subset f(P) \subset V \). On the other hand, suppose \( V \subset Y \) is such that for each \( y \in V \), \( B \subset V \) for some \( B \in \mathcal{B}'' \). By the properties of \( \mathcal{B}' \) and \( \mathcal{B}'' \) and the condition (2) from Definition 1.2, there exists \( B_y \in \mathcal{B}_y \) such that \( y \in B_y \subset B \subset V \). Because \( \mathcal{B} = \bigcup \{ \mathcal{B}_y : y \in Y \} \) is a weak-base for \( Y \), \( V \) is open in \( Y \). Therefore \( \mathcal{B}'' \) is a weak-base for \( Y \).

**Necessity.** Suppose \( \mathcal{P} \) is a compact-countable weak-base for \( Y \). Endow \( \mathcal{P} \) with discrete topology, then \( \mathcal{P} \) is a metric space. Put \( X = \{ (P_n) \in \mathcal{P}^\omega : \{ P_n : n \in \mathbb{N} \} \text{ is a network of some point } y \in Y \} \), and endow \( X \) with the subspace topology induced by the product topology of the usual product space \( \mathcal{P}^\omega \). Then \( X \) is a metric space. Since \( Y \) is Hausdorff, \( y \) is unique in \( Y \) (in fact, it is easy to check that \( \{ y \} = \bigcap_{n \in \mathbb{N}} P_n \)).

We define \( f : X \to Y \) by \( f((P_n)) = y \) for each \( (P_n) \in X \). For each \( y \in Y \), since \( \mathcal{P} \) is point-countable in \( Y \), denoting \( \{ P \in \mathcal{P} : y \in P \} \) by \( (P_n) \), we have \( (P_n) \in X \) and \( f((P_n)) = y \). Thus \( f \) is a surjection. It is obvious that \( f \) is continuous. We will prove that \( f \) is a weak-open cs-mapping.

(i) \( f \) is a weak-open mapping.

For each \( y \in Y \), denote a collection of weak neighbourhoods of \( y \) in \( Y \) by \( \mathcal{P}_y \); then \( \mathcal{P}_y \) is countable. Set \( \mathcal{P}_y = \{ P_n : n \in \mathbb{N} \} \), then \( f((P_n)) = y \) and \( (P_n) \in f^{-1}(y) \). For each \( n \in \mathbb{N} \), put

\[
B(P_1, \ldots, P_n) = \{ (P'_n) \in X : P'_i = P_i \text{ for each } i \leq n \}.
\]

It is easy to check that \( \{ B(P_1, \ldots, P_n) : n \in \mathbb{N} \} \) is a locally neighbourhood base of the point \( (P_n) \) in \( X \). \( \square \)
Claim. \( f(B(P_1, \ldots, P_n)) = \bigcap_{i \leq n} P_i \) for each \( n \in \mathbb{N} \).

Suppose \( (P'_i) \in B(P_1, \ldots, P_n) \), then \( f((P'_i)) = \bigcap_{i \leq n} P'_i \subset \bigcap_{i \leq n} P_i \). Thus \( f(B(P_1, \ldots, P_n)) \subset \bigcap_{i \leq n} P_i \). On the other hand, suppose \( z \in \bigcap_{i \leq n} P_i \) and set \( \mathcal{P}_z = \{ P_{n+j}^n : j \in \mathbb{N} \} \).

Put
\[
P_r^* = \begin{cases}
P_r, & r \leq n, \\
P_r^n, & r > n,
\end{cases}
\]
then \( (P_r^*) \in B(P_1, \ldots, P_n) \) and \( f((P_r^*)) = z \). Thus \( \bigcap_{i \leq n} P_i \subset f(B(P_1, \ldots, P_n)) \). Hence
\[
f(B(P_1, \ldots, P_n)) = \bigcap_{i \leq n} P_i.
\]

Because \( \mathcal{P} \) is a weak-base for \( Y \) and \( \{ P_n : n \in \mathbb{N} \} = \mathcal{P}_y \), we obtain \( f(B(P_1, \ldots, P_n)) = \bigcap_{i \leq n} P_i \in \mathcal{P}_y \) for each \( n \in \mathbb{N} \). Suppose \( G \) is a open neighbourhood of the point \( (P_n) \) in \( X \); then there exists \( j \in \mathbb{N} \) such that \( B(P_1, \ldots, P_j) \subset G \). So \( f(B(P_1, \ldots, P_j)) \subset f(G) \). By the Claim, \( f(B(P_1, \ldots, P_j)) = \bigcap_{i \leq j} P_i \in \mathcal{P}_y \). Hence there exists a weak-base \( \mathcal{P} \) for \( Y \) and \( (P_n) \in f^{-1}(y) \) satisfying the condition (*) from Definition 1.3(1). Therefore \( f \) is a weak-open mapping.

(ii) \( f \) is a cs-mapping.

For each compact subset \( K \) of \( Y \), since \( \mathcal{P} \) is compact-countable, hence \( \{ P \in \mathcal{P} : P \cap K \neq \emptyset \} \) is countable. Thus \( \{ P \in \mathcal{P} : P \cap K \neq \emptyset \}^{\omega} \cap X \) is a hereditarily separable subspace of \( X \). Because \( f^{-1}(K) \subset \{ P \in \mathcal{P} : P \cap K \neq \emptyset \}^{\omega} \cap X \), thus \( f^{-1}(K) \) is separable in \( X \). Hence \( f \) is a cs-mapping.

Remark 3.2. A mapping \( f : X \to Y \) is an \( s \)-mapping (\( ss \)-mapping [16]) if for each \( y \in Y \), \( f^{-1}(y) \) is separable in \( X \) (for each \( y \in Y \), there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( f^{-1}(V) \) is separable in \( X \)). A mapping \( f : X \to Y \) is a 1-sequence-covering mapping [14] if for each \( y \in Y \) there exists \( x \in f^{-1}(y) \) satisfying the following condition: whenever \( \{ y_n \} \) is a sequence in \( Y \) converging to a point \( y \) in \( Y \), there exists a sequence \( \{ x_n \} \) of \( X \) converging to a point \( x \) in \( X \) such that each \( x_n \in f^{-1}(y_n) \). Obviously, if \( X \) is a metric space, then an \( ss \)-mapping \( \Rightarrow \) a cs-mapping \( \Rightarrow \) an \( s \)-mapping. However, we have the following facts.

Example 1. A weak-open \( s \)-image of a metric space is not a weak-open cs-image of a metric space.

Let
\[
S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{ 0 \}, \quad X = [0, 1] \times S,
\]
and let
\[
Y = [0, 1] \times \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}
\]
have the usual Euclidean topology as a subspace of $[0,1] \times S$. Define a typical
neighbourhood of $(t,0)$ in $X$ to be of the form

$$\{(t,0)\} \cup \left( \bigcup_{k \geq n} V(t, 1/k) \right), \quad n \in \mathbb{N},$$

where $V(t, 1/k)$ is a neighbourhood of $(t,1/k)$ in $[0,1] \times \{1/k\}$. Put

$$M = \left( \bigoplus_{n \in \mathbb{N}} [0,1] \times \{1/n\} \right) \oplus \left( \bigoplus_{t \in [0,1]} \{t\} \times S \right)$$

and define $f$ from $M$ onto $X$ such that $f$ is an obvious mapping.

Then $f$ is a compact-covering, quotient, two-to-one mapping from the locally
compact metric space $M$ onto the separable, regular, non-Lindelöf, $k$-space $X$ (see
Example 2.8.16 of [13] or Example 9.3 of [18]). It is easy to check that $f$ is a
1-sequence-covering mapping. By Theorem 2.5 of [14], $X$ has a point-countable
weak-base. Thus $X$ is a weak-open $s$-image of a metric space by Theorem 2.5 of [4].

$X$ has no compact-countable $k$-network. Indeed, suppose $\mathcal{P}$ is a compact-
countable $k$-network for $X$. Put

$$\mathcal{F} = \{\{(t,0)\}: t \in [0,1]\} \cup \{P \cap Y: P \in \mathcal{P}\}.$$  

Since $[0,1] \times \{0\}$ is a closed discrete subspace of $X$, $\mathcal{F}$ is a $k$-network for $X$. But
$Y$ is a $\sigma$-compact subspace of $X$. Thus $\{P \cap Y: P \in \mathcal{P}\}$ is countable, and so
$\mathcal{F}$ is star-countable. Since a regular $k$-space with a star-countable $k$-network is an
$\aleph_0$-space (see [17]), hence $X$ is a Lindelöf space, a contradiction. Thus $X$ has no
compact-countable $k$-network. By Lemma 7 of [15], $X$ has no compact-countable
weak-base. Hence $X$ is not a weak-open $c$-image of a metric space by Theorem 3.1.

**Example 2.** A weak-open $c$-image of a metric space is not a weak-open $s$-image
of a metric space.

Let $X$ be a paracompact space with a point-countable base and not metrizable.
Then $X$ has a compact-countable base, and so $X$ has a compact-countable weak-base. By Theorem 3.1, $X$ is a weak-open $c$-image of a metric space. But $X$ is not
a 1-sequence-covering $s$-image of a metric space because $X$ is not a metric space.
Thus $X$ is not a weak-open $s$-image of a metric space by Proposition 3.3 of [5].
References


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