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μ-STATISTICALLY CONVERGENT FUNCTION SEQUENCES

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Abstract. In the present paper we are concerned with convergence in μ-density and μ-statistical convergence of sequences of functions defined on a subset D of real numbers, where μ is a finitely additive measure. Particularly, we introduce the concepts of μ-statistical uniform convergence and μ-statistical pointwise convergence, and observe that μ-statistical uniform convergence inherits the basic properties of uniform convergence.

Keywords: pointwise and uniform convergence, μ-statistical convergence, convergence in μ-density, finitely additive measure, additive property for null sets

MSC 2000: 40A30

1. Introduction

Steinhaus [19] introduced the idea of statistical convergence (see also Fast [10]). If K is a subset of $\mathbb{N}$, the set of natural numbers, then the asymptotic density of K, denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leqslant n \colon k \in K\}|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set B. A sequence $x = (x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k \colon |x_k - L| \geqslant \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write st-$\lim x = L$ or $x_k \to L$ (stat). Note that convergent sequences are statistically convergent but not conversely ([2], [11]).

Statistical convergence has been investigated in a number of recent papers [2], [6], [11], [12], [13], [14], [18]. Some generalizations of statistical convergence have appeared in the study of locally convex spaces [16], strong integral summability [5], finitely additive set functions [6]. It is also connected with subsets of the Stone-Čech compactification of the set of natural numbers [7], [9]. Some results on characterizing Banach spaces with separable duals via statistical convergence may be found in [8]. This notion of convergence is also considered in measure theory [17] and trigonometric series [21].

Connor [3] gave an extension of the notion of statistical convergence where the asymptotic density is replaced by a finitely additive set function. Through the present paper, let μ be a finitely additive set function taking values in $[0,1]$ defined on a field Γ of subsets of $\mathbb{N}$ such that if $|A|<\infty$, then $\mu(A)=0$; if $A\subset B$ and $\mu(B)=0$, then $\mu(A)=0$; $\mu(\mathbb{N})=1$. Such a set function satisfying the above criteria will be called a measure. Following Connor [3], [4] we say that:

(i) x is μ-density convergent to L if there is an $A\in\Gamma$ such that $(x-L)\chi_A$ is a null sequence and $\mu(A)=1$, where χ_A is the characteristic function of A.

(ii) x is μ-statistically convergent to L, and write $\operatorname{st}_\mu\text{-}\lim x=L$, provided $\mu(\{k\colon |x_k-L|\geqslant\varepsilon\})=0$ for every $\varepsilon>0$.

If $T=(t_{nk})$ is a nonnegative regular summability method, then T can be used to generate a measure as follows: for each $n\in\mathbb{N}$, set $\mu_n(A)=\sum\limits_{k=1}^{\infty}t_{nk}\chi_A(k)$ for each $A\subseteq\mathbb{N}$. Let $\Gamma\colon=\{A\subseteq\mathbb{N}\colon \lim\limits_n\mu_n(A)=0 \text{ or } \lim\limits_n\mu_n(A)=1\}$. Define $\mu_T\colon \Gamma\to[0,1]$ by

$$\mu_T(A)=\lim_{n\to\infty}\mu_n(A)=\lim_{n\to\infty}\sum_{k=1}^{\infty}t_{nk}\chi_A(k).$$

Then μ_T and Γ satisfy the requirements of the preceding definitions. If T is the Cesàro matrix of order one, then μ_T-statistical convergence is equivalent to statistical convergence.

It is known (Connor [3]) that (i) implies (ii), but not conversely. These two definitions are equivalent ([3], [4]) if μ has the so-called additive property for null sets: if, given a collections of null sets $\{A_j\}_{j\in\mathbb{N}}\subseteq\Gamma$, there exists a collection $\{B_i\}_{i\in\mathbb{N}}\subseteq\Gamma$ with the properties $|A_i\bigtriangleup B_i|<\infty$ for each $i\in\mathbb{N}$, $B=\bigcup\limits_{i=1}^{\infty}B_i\in\Gamma$, and $\mu(B)=0$.

In the present paper we are concerned with convergence in μ-density and μ-statistical convergence of sequences of functions defined on a subset D of $\mathbb{R}$, the set of real numbers. Particularly, we introduce the concepts of μ-statistical uniform convergence and μ-statistical pointwise convergence, and observe that μ-statistical uniform convergence inherits the basic properties of uniform convergence.

2. μ-STATISTICALLY AND μ-DENSITY CONVERGENT FUNCTION SEQUENCES

Let $D \subset \mathbb{R}$ and let (f_n) be a sequence of real functions on D.

Definition 2.1. (f_n) converges μ-density pointwise to $f \Leftrightarrow \forall \varepsilon > 0$ and $\forall x \in D$, $\exists K_x \in \Gamma$, $\mu(K_x) = 1$ and $\exists n_0 = n_0(\varepsilon, x) \in K_x \ni \forall n \geqslant n_0$ and $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.

In this case we will write $f_n \to f$ (μ-density) on D.

Definition 2.2. (f_n) converges μ-density uniform to $f \Leftrightarrow \forall \varepsilon > 0$, $\exists K \in \Gamma$, $\mu(K) = 1$ and $\exists n_0 = n_0(\varepsilon) \in K \ni \forall n \geqslant n_0$ and $n \in K$ and $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$.

In this case we will write $f_n \rightrightarrows f$ (μ-density) on D.

Definition 2.3. (f_n) converges μ-statistically pointwise to $f \Leftrightarrow \forall \varepsilon > 0$ and $\forall x \in D$, $\mu(\{n\colon |f_n(x) - f(x)| \geqslant \varepsilon\}) = 0$.

In this case we will write $f_n \to f$ (μ-stat) on D. We note that this definition includes the definition given in [20].

Definition 2.4. The sequence (f_n) of bounded functions on D converges μ-statistically uniformly to $f \Leftrightarrow \text{st}_\mu\text{-lim}\,\|f_n - f\|_B = 0$, where the norm $\|\cdot\|_B$ is the usual supremum norm on $B(D)$, the space of bounded functions on D.

In this case we will write $f_n \rightrightarrows f$ (μ-stat) on D. Observe that $f_n \rightrightarrows f$ (μ-stat) on D if and only if $\text{st}_\mu\text{-lim}\Big(\sup_{x\in D} |f_n(x) - f(x)|\Big) = 0$.

As in the ordinary case the property of Definition 2.1 implies that of Definition 2.3; and, of course for bounded functions, the property of Definition 2.2 implies that of Definition 2.4. If μ has the additive property for null sets, then Definitions 2.1 and 2.3 are equivalent, and Definitions 2.2 and 2.4 are equivalent.

The next result is a μ-statistical analogue of a well-known result.

Theorem 2.1. *Let all functions f_n be continuous on D. If $f_n \rightrightarrows f$ (μ-density) on D, then f is continuous on D.*

P r o o f. Assume $f_n \rightrightarrows f$ (μ-density) on D. Then, for every $\varepsilon > 0$, there exists a set $K \in \Gamma$ of measure 1 and $n_0 = n_0(\varepsilon) \in K$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for each $x \in D$ and for all $n \geqslant n_0$ and $n \in K$. Let $x_0 \in D$. Since f_{n_0} is continuous at $x_0 \in D$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f_{n_0}(x) - f_{n_0}(x_0)| < \varepsilon/3$ for each $x \in D$. Now for all $x \in D$ for which $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \leqslant |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \varepsilon.$$

Since $x_0 \in D$ is arbitrary, f is continuous on D. □

Now Theorem 2.1 yields immediately the following

Corollary 2.2. *Let all functions f_n be continuous on a compact subset D of $\mathbb{R}$, and let μ be a measure with the additive property for null sets. If $f_n \rightrightarrows f$ (μ-stat) on D, then f is continuous on D.*

The next example shows that neither of the converses of Theorem 2.1 and Corollary 2.2 are true.

Example 2.1. Let $\mu(K) = 1$. Define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1, & n \notin K, \\ \dfrac{2nx}{1+n^2x^2}, & n \in K. \end{cases}$$

Then we have $f_n \to f = 0$ (μ-density) on $[0,1]$. Hence we get $f_n \to f = 0$ (μ-stat) on $[0,1]$. Though all f_n and f are continuous on $[0,1]$, it follows from Definition 2.4 that the μ-statistical convergence of (f_n) is not uniform for

$$c_n := \max_{0 \leqslant x \leqslant 1} |f_n(x) - f(x)| = 1 \quad \text{and} \quad \text{st}_\mu\text{-lim}\, c_n = 1 \neq 0.$$

The following result is an analogue of Dini's theorem.

Theorem 2.3. *Let μ be a measure with the additive property for null sets. Let D be a compact subset of $\mathbb{R}$ and let (f_n) be a sequence of continuous functions on D. Assume that f is continuous and $f_n \to f$ (μ-stat) on D. Also, let (f_n) be monotonic decreasing on D; i.e. $f_n(x) \geqslant f_{n+1}(x)$ $(n = 1, 2, \ldots)$ for every $x \in D$. Then $f_n \rightrightarrows f$ (μ-stat) on D.*

P r o o f. Write $g_n(x) := f_n(x) - f(x)$. By hypothesis, each g_n is continuous and $g_n \to 0$ (μ-stat) on D, also (g_n) is a monotonic decreasing sequence on D. Now, since $g_n \to 0$ (μ-stat) on D and μ has the additive property for null sets, $g_n \to 0$ (μ-density) on D. Hence for every $\varepsilon > 0$ and each $x \in D$ there exists $K_x \in \Gamma$ of measure 1 and a number $n(x) := n(\varepsilon, x) \in K_x$ such that $0 \leqslant g_n(x) < \varepsilon/2$ for all $n \geqslant n(x)$ and $n \in K_x$. Since $g_{n(x)}$ is continuous at $x \in D$, for every $\varepsilon > 0$ there is an open set $J(x)$ which contains x such that $|g_{n(x)}(t) - g_{n(x)}(x)| < \varepsilon/2$ for all $t \in J(x)$. Hence given $\varepsilon > 0$, by monotonicity we have

$$\begin{aligned} 0 \leqslant g_n(t) \leqslant g_{n(x)}(t) &= g_{n(x)}(t) - g_{n(x)}(x) + g_{n(x)}(x) \\ &\leqslant |g_{n(x)}(t) - g_{n(x)}(x)| + g_{n(x)}(x) < \varepsilon \end{aligned}$$

for every $t \in J(x)$ and for all $n \geqslant n(x)$ and $n \in K_x$. Since $D \subset \bigcup_{x \in D} J(x)$ and D is a compact set, by the Heine-Borel theorem D has a finite open covering such that $D \subset J(x_1) \cup J(x_2) \cup \ldots \cup J(x_m)$. Now, let $K := K_{x_1} \cap K_{x_2} \cap \ldots \cap K_{x_m}$ and $N = \max\{n(x_1), n(x_2), \ldots, n(x_m)\}$. Observe that $\mu(K) = 1$. Then $0 \leqslant g_n(t) < \varepsilon$ for every $t \in D$ and for all $n \geqslant N$ and $n \in K$. So $g_n \rightrightarrows 0$ (μ-density) on D. Consequently, $g_n \rightrightarrows 0$ (μ-stat) on D, which completes the proof. □

The following theorem is the Cauchy criterion for μ-statistical uniform convergence.

Theorem 2.4. *Let μ be a measure with the additive property for null sets, and let (f_n) be a sequence of bounded functions on D. Then (f_n) is μ-statistically uniformly convergent on D if and only if for every $\varepsilon > 0$ there is an $n(\varepsilon) \in \mathbb{N}$ such that*

$$\mu(\{n : \|f_n - f_{n(\varepsilon)}\|_B < \varepsilon\}) = 1. \tag{2.1}$$

Note. The sequence (f_n) satisfying the property (2.1) is said to be μ-statistically uniformly Cauchy on D.

P r o o f. Assume that (f_n) converges μ-statistically uniformly to a function f defined on D. Let $\varepsilon > 0$. Then we have $\mu(\{n\colon \|f_n - f\|_B < \varepsilon/2\}) = 1$. We can select an $n(\varepsilon) \in \mathbb{N}$ such that $\|f_{n(\varepsilon)} - f\|_B < \varepsilon/2$. The triangle inequality yields that $\mu(\{n\colon \|f_n - f_{n(\varepsilon)}\|_B < \varepsilon\}) = 1$. Since ε was arbitrary, (f_n) is μ-statistically uniformly Cauchy on D.

Conversely, assume that (f_n) is μ-statistically uniformly Cauchy on D. Let $x \in D$ be fixed. By (2.1), for every $\varepsilon > 0$ there is an $n(\varepsilon) \in \mathbb{N}$ such that $\mu(\{n\colon |f_n(x) - f_{n(\varepsilon)}(x)| < \varepsilon\}) = 1$. Hence $\{f_n(x)\}$ is μ-Cauchy, so by Proposition 3 of Connor [4] we have that $\{f_n(x)\}$ converges μ-statistically to $f(x)$. Then $f_n \to f$ (μ-stat) on D. Now we shall show that this convergence must be uniform. Note that since μ has the additive property for null sets, by (2.1) there is a $K \in \Gamma$ of measure 1 such that $\|f_n - f_{n(\varepsilon)}\|_B < \varepsilon/2$ for all $n \geqslant n(\varepsilon)$ and $n \in K$. So for every $\varepsilon > 0$ there is a $K \in \Gamma$ of measure 1 and $n(\varepsilon) \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon \tag{2.2}$$

for all $n, m \geqslant n(\varepsilon)$ and $n, m \in K$ and for each $x \in D$. Fixing n and applying the limit operator on $m \in K$ in (2.2), we conclude that for every $\varepsilon > 0$ there is a $K \in \Gamma$ of measure 1 and an $n(\varepsilon) \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geqslant n_0$ and for each $x \in D$. Hence $f_n \rightrightarrows f$ (μ-density) on D, consequently $f_n \rightrightarrows f$ (μ-stat) on D. □

3. Applications

Using μ-statistical uniform convergence, we can also get some applications. We merely state the following theorems and omit the proofs.

Theorem 3.1. *Let μ be a measure with the additive property for null sets. If a function sequence (f_n) converges μ-statistically uniformly on $[a,b]$ to a function f and each f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$. Moreover,*

$$\mathrm{st}_\mu\text{-lim} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \mathrm{st}_\mu\text{-lim}\, f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 3.2. *Let μ be a measure with the additive property for null sets. Suppose that (f_n) is a function sequence such that each (f_n) has a continuous derivative on $[a,b]$. If $f_n \to f$ (μ-stat) on $[a,b]$ and $f_n' \rightrightarrows g$ (μ-stat) on $[a,b]$, then $f_n \rightrightarrows f$ (μ-stat) on $[a,b]$, where f is differentiable, and $f' = g$.*

4. Function sequences that preserve μ-statistical convergence

This section is motivated by a paper of Kolk [15]. Recall that a function sequence (f_n) is called convergence-preserving (or conservative) on $D \subset \mathbb{R}$ if the transformed sequence $\{f_n(x_n)\}$ converges for each convergent sequence $x = (x_n)$ from D [15]. In this section, analogously, we describe the function sequences which preserve the μ-statistical convergence of sequences. Our arguments also give a sequential characterization of the continuity of μ-statistical limit functions of μ-statistically uniformly convergent function sequences. This result is complementary to Theorem 2.1.

First we introduce the following definition.

Definition 4.1. Let $D \subset \mathbb{R}$ and let (f_n) be a sequence of real functions on D. Then (f_n) is called a *function sequence preserving μ-statistical convergence* (or μ-statistically conservative) on D if the transformed sequence $\{f_n(x_n)\}$ converges μ-statistically for each μ-statistically convergent sequence $x = (x_n)$ from D. If (f_n) is μ-statistically conservative and preserves the limits of all μ-statistically convergent sequences from D, then (f_n) is called *μ-statistically regular* on D.

Hence, if (f_n) is conservative on D, then (f_n) is μ-statistically conservative on D. But the following example shows that the converse of this result is not true.

Example 4.1. Let $K \in \Gamma$ be a set such that $\mathbb{N} \setminus K$ is infinite and $\mu(K) = 1$. Define $f_n\colon [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & n \in K, \\ 1, & n \notin K. \end{cases}$$

Suppose that (x_n) from $[0,1]$ is an arbitrary sequence such that $\mathrm{st}_\mu\text{-}\lim x_n = L$. Then, for every $\varepsilon > 0$, $\mu(\{n\colon |f_n(x) - 0| \geqslant \varepsilon\}) = \mu(\mathbb{N} \setminus K) = 0$. Hence $\mathrm{st}_\mu\text{-}\lim f_n(x_n) = 0$, so (f_n) is μ-statistically conservative on $[0,1]$. But observe that (f_n) is not conservative on $[0,1]$.

Now we have

Theorem 4.1. *Let μ be a measure with the additive property for null sets and let (f_k) be a sequence of functions defined on a closed interval $[a,b] \subset \mathbb{R}$. Then (f_k) is μ-statistically conservative on $[a,b]$ if and only if (f_k) converges μ-statistically uniformly on $[a,b]$ to a continuous function.*

Proof. *Necessity.* Assume that (f_k) is μ-statistically conservative on $[a,b]$. Choose the sequence $(v_k) = (t,t,\ldots)$ for each $t \in [a,b]$. Since $\mathrm{st}_\mu\text{-}\lim v_k = t$, $\mathrm{st}_\mu\text{-}\lim f_k(v_k)$ exists, hence $\mathrm{st}_\mu\text{-}\lim f_k(t) = f(t)$ for all $t \in [a,b]$. We claim that f is continuous on $[a,b]$. To prove this we suppose that f is not continuous at a point $t_0 \in [a,b]$. Then there exists a sequence (u_k) in $[a,b]$ such that $\lim u_k = t_0$, but $\lim f(u_k)$ exists and $\lim f(u_k) \neq f(t_0)$. Since (f_k) is μ-statistically pointwise convergent to f on $[a,b]$ and μ has the additive property for null sets, we obtain $f_k \to f$ (μ-density) on $[a,b]$. Hence, for each j, $\{f_k(u_j) - f(u_j)\} \to 0$ (μ-density). It follows from Corollary 9 of Connor [4] that there exists $\lambda\colon \mathbb{N} \to \mathbb{N}$ such that $\mu(\{\lambda(k)\colon k \in \mathbb{N}\}) = 1$ and

$$\lim_k [f_{\lambda(k)}(u_j) - f(u_j)] = 0$$

for each j. Now, by the "diagonal process" [1, p. 192] we can choose an increasing index sequence (n_k) in such a way that $\mu(\{n_k\colon k \in \mathbb{N}\}) = 1$ and $\lim_k [f_{n_k}(u_k) - f(u_k)] = 0$. Now define a sequence $x = (t_i)$ by

$$t_i = \begin{cases} t_0, & i = n_k \text{ and } i \text{ is odd}, \\ u_k, & i = n_k \text{ and } i \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence $t_i \to t_0$ (μ-density), which implies $\mathrm{st}_\mu\text{-}\lim t_i = t_0$. But if $i = n_k$ and i is odd, then $\lim f_{n_k}(t_0) = f(t_0)$, and if $i = n_k$ and i is even, then $\lim f_{n_k}(u_k) = \lim[f_{n_k}(u_k) - f(u_k)] + \lim f(u_k) \neq f(t_0)$. Hence $\{f_i(t_i)\}$ is not μ-density convergent since the sequence $\{f_i(t_i)\}$ has two disjoint subsequences of positive measure that converge to two different limit values. So, the sequence $\{f_i(t_i)\}$ is not μ-statistically convergent, which contradicts the hypothesis. Thus f must be continuous on $[a,b]$. It remains to prove that (f_k) converges μ-statistically uniformly on $[a,b]$ to f. Assume that (f_k) is not μ-statistically uniformly convergent to f on $[a,b]$, then (f_k) is not μ-density uniformly convergent to f on $[a,b]$. Hence, for an arbitrary index sequence (n_k) with $\mu(\{n_k\colon\ k\in\mathbb{N}\})=1$, there exists a number $\varepsilon_0>0$ and numbers $t_k\in[a,b]$ such that $|f_{n_k}(t_k)-f(t_k)|\geqslant 2\varepsilon_0$ $(k\in\mathbb{N})$. The bounded sequence $x=(t_k)$ contains a convergent subsequence (t_{k_i}), $\mathrm{st}_\mu\text{-}\lim t_{k_i}=\alpha$, say. By the continuity of f, $\lim f(t_{k_i})=f(\alpha)$. So there is an index i_0 such that $|f(t_{k_i})-f(\alpha)|<\varepsilon_0$ $(i\geqslant i_0)$. For the same i's, we have

$$(4.1)\qquad |f_{n_{k_i}}(t_{k_i})-f(\alpha)|\geqslant |f_{n_{k_i}}(t_{k_i})-f(t_{k_i})|-|f(t_{k_i})-f(\alpha)|\geqslant\varepsilon_0.$$

Now, defining

$$u_j=\begin{cases}\alpha, & j=n_{k_i}\text{ and } j \text{ is odd},\\ t_{k_i}, & j=n_{k_i}\text{ and } j\text{ is even},\\ 0, & \text{otherwise},\end{cases}$$

we get $u_j\to\alpha$ (μ-density). Hence $\mathrm{st}_\mu\text{-}\lim u_j=\alpha$. But if $j=n_{k_i}$ and j is odd, then $\lim f_{n_{k_i}}(\alpha)=f(\alpha)$, and if $j=n_{k_i}$ and j is even, then, by (4.1), $\lim f_{n_{k_i}}(t_{k_i})\neq f(\alpha)$. Hence $\{f_j(t_j)\}$ is not μ-density convergent since the sequence $\{f_j(t_j)\}$ has two disjoint subsequences of positive measure that converge to two different limit values. So, the sequence $\{f_j(t_j)\}$ is not μ-statistically convergent, which contradicts the hypothesis. Thus (f_k) must be μ-statistically uniformly convergent to f on $[a,b]$.

Sufficiency. Assume that $f_n\rightrightarrows f$ (μ-stat) on $[a,b]$ and f is continuous. Let $x=(x_n)$ be a μ-statistically convergent sequence in $[a,b]$ with $\mathrm{st}_\mu\text{-}\lim x_n=x_0$. Since μ has the additive property for null sets, $x_n\to x_0$ (μ-density), so there is an index sequence $\{n_k\}$ such that $\lim\limits_k x_{n_k}=x_0$ and $\mu(\{n_k\colon\ k\in\mathbb{N}\})=1$. By the continuity of f at x_0, $\lim\limits_k f(x_{n_k})=f(x_0)$. Hence $f(x_n)\to f(x_0)$ (μ-density). Let $\varepsilon>0$ be given. Then there exists $K_1\in\Gamma$ of measure 1 and a number $n_1\in K_1$ such that $|f(x_n)-f(x_0)|<\varepsilon/2$ for all $n\geqslant n_1$ and $n\in K_1$. By assumption μ has the additive property for null sets. Hence the μ-statistical uniform convergence is equivalent to the μ-density uniform convergence, so there exists a $K_2\in\Gamma$ of measure 1 and a number $n_2\in K_2$ such that $|f_n(t)-f(t)|<\varepsilon/2$ for every $t\in[a,b]$ for all $n\geqslant n_2$ and $n\in K_2$. Let $N:=\max\{n_1,n_2\}$ and $K:=K_1\cap K_2$. Observe that $\mu(K)=1$. Hence taking $t=x_n$ we have

$$|f_n(x_n)-f(x_0)|\leqslant|f_n(x_n)-f(x_n)|+|f(x_n)-f(x_0)|<\varepsilon$$

for all $n\geqslant N$ and $n\in K$. This shows that $f_n(x_n)\to f(x_0)$ (μ-density) which necessarily implies that $\mathrm{st}_\mu\text{-}\lim f_n(x_n)=f(x_0)$, whence the proof follows. □

Theorem 4.1 contains the following necessary and sufficient condition for the continuity of μ-statistical limit functions of function sequences that converge μ-statistically uniformly on a closed interval.

Theorem 4.2. *Let μ be a measure with the additive property for null sets and let (f_k) be a sequence of functions that converges μ-statistically uniformly on a closed interval $[a,b]$ to a function f. The* st_μ-lim *function f is continuous on $[a,b]$ if and only if (f_k) is μ-statistically conservative on $[a,b]$.*

Now, we study the μ-statistical regularity of function sequences. If (f_k) is μ-statistically regular on $[a,b]$, then obviously $\mathrm{st}_\mu\text{-lim}\, f_k(t) = t$ for all $t \in [a,b]$. So, taking $f(t) = t$ in Theorem 4.1, we immediately get the following

Theorem 4.3. *Let μ be a measure with the additive property for null sets and let (f_k) be a sequence of functions on $[a,b]$. Then (f_k) is μ-statistically regular on $[a,b]$ if and only if (f_k) is μ-statistically uniformly convergent on $[a,b]$ to the function f defined by $f(t) = t$.*

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