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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 2, 431–443

Persistent URL: <http://dml.cz/dmlcz/127901>

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DETERMINANTS OF MATRICES ASSOCIATED WITH
INCIDENCE FUNCTIONS ON POSETS

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(Received September 19, 2001)

Abstract. Let $S = \{x_1, \dots, x_n\}$ be a finite subset of a partially ordered set P . Let f be an incidence function of P . Let $[f(x_i \wedge x_j)]$ denote the $n \times n$ matrix having f evaluated at the meet $x_i \wedge x_j$ of x_i and x_j as its i, j -entry and $[f(x_i \vee x_j)]$ denote the $n \times n$ matrix having f evaluated at the join $x_i \vee x_j$ of x_i and x_j as its i, j -entry. The set S is said to be meet-closed if $x_i \wedge x_j \in S$ for all $1 \leq i, j \leq n$. In this paper we get explicit combinatorial formulas for the determinants of matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S . We also obtain necessary and sufficient conditions for the matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S to be nonsingular. Finally, we give some number-theoretic applications.

Keywords: meet-closed set, greatest-type lower, incidence function, determinant, nonsingularity

MSC 2000: 11C20, 15A57

1. INTRODUCTION

Let $S = \{x_1, \dots, x_n\}$ be a set of n distinct positive integers. The matrix having the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j -entry is called the *greatest common divisor* (GCD) *matrix*, denoted by $[(x_i, x_j)]$. The matrix having the least common multiple $[x_i, x_j]$ of x_i and x_j as its i, j -entry is called the *least common multiple* (LCM) *matrix*, denoted by $[[x_i, x_j]]$. The set S is said to be *factor-closed* if it contains every divisor of x for any $x \in S$. H. J. S. Smith [10] showed that the determinant of the GCD matrix $[(x_i, x_j)]$ on a factor-closed set S is the product $\prod_{i=1}^n \varphi(x_i)$, where φ is Euler's totient function. The set S is said to be *gcd-closed* if

Supported partially by the Lady Davis Fellowship at the Technion, an NNSF of China (10101015 and 10128103) and YSRF of Sichuan University.

$(x_i, x_j) \in S$ for all $1 \leq i, j \leq n$. It is clear that a factor-closed set is a gcd-closed set but not conversely.

Let f be an arithmetical function. Let $[f(x_i, x_j)]$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its i, j -entry. In [10], Smith also considered the determinant of the matrix $[f(x_i, x_j)]$ on a factor-closed set S . It was shown to be the product $\prod_{k=1}^n (f * \mu)(x_k)$, where $f * \mu$ is the Dirichlet product of f and μ . In [4], Bourque and Ligh obtained a generalization of Smith's result. Haukkanen [5] gave an abstract generalization of Bourque and Ligh's result.

Now let f be an incidence function and $S = \{x_1, \dots, x_n\}$ a meet-closed set of a finite partially ordered set (poset) P (for related definitions, see the next section). Let $[f(x_i \wedge x_j)]$ denote the $n \times n$ matrix having f evaluated at the meet $x_i \wedge x_j$ of x_i and x_j as its i, j -entry, and let $[f(x_i \vee x_j)]$ denote the $n \times n$ matrix having f evaluated at the join $x_i \vee x_j$ of x_i and x_j as its i, j -entry. In this paper we will obtain explicit combinatorial formulas for the determinants of the matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S . We will also get necessary and sufficient conditions for the matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ on any meet-closed set S to be nonsingular. In the last section we give some number-theoretic applications.

2. PRELIMINARIES AND DEFINITIONS

Let (P, \leq) be a poset. We say that P is a *meet semilattice* if for any $x, y \in P$ there exists a unique $z \in P$ such that

- (i) $z \leq x$ and $z \leq y$, and
- (ii) if $w \leq x$ and $w \leq y$ for some $w \in P$, then $w \leq z$.

In such a case z is called the *meet* of x and y and is denoted by $x \wedge y$. Let S be a subset of P . We call S *lower-closed* if for every $x, y \in P$ with $x \in S$ and $y \leq x$ we have $y \in S$. We call S *meet-closed* if for every $x, y \in S$ we have $x \wedge y \in S$. It is clear that a lower-closed set is always meet-closed but not conversely. The concepts of "lower-closed" and "meet closed" are generalizations of "factor-closed" and "gcd-closed" [2], [3], respectively.

Let f be a complex-valued function on $P \times P$ such that $f(x, y) = 0$ whenever $x \not\leq y$. Then we say that f is an *incidence function* of P . If f and g are incidence functions of P , their *sum* $f + g$ is defined by $(f + g)(x, y) = f(x, y) + g(x, y)$ and their *convolution* $f * g$ is defined by $(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the *unity* δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The

incidence function ζ is defined by $\zeta(x, y) = 1$ if $x \leq y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ .

In what follows, let $(P, \leq) = (P, \wedge, \vee)$ be a finite meet semilattice. Let S be a subset of P and denote $S = \{x_1, x_2, \dots, x_n\}$ with $x_i < x_j \Rightarrow i < j$. For any incidence function f of P we denote $f(0, x) = f(x)$, where $0 = \min P$. For example, let $(P, \leq) = (\mathbf{Z}^+, |)$. Then $\mu(1, n)$ is the usual number-theoretic function $\mu(n)$.

Proposition 2.1 ([5]). *Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set. Then the determinant of the matrix $[f(x_i \wedge x_j)]$ defined on $S = \{x_1, \dots, x_n\}$ is equal to the product $\prod_{k=1}^n \psi_f(x_k)$, where*

$$(1) \quad \psi_f(x_k) = \sum_{\substack{d \leq x_k \\ d \not\leq x_t, t < k}} (f * \mu)(d).$$

Note that Haukkanen [5] writes this formula without using convolution of incidence functions.

Definition 2.2. Let T be a given subset of P . For any $a, b \in T$ and $a < b$, we say that a is a *greatest-type lower* of b in T , if $a \leq c$, $c < b$ and $c \in T$ implies $c = a$.

If $(P, \leq) = (\mathbf{Z}^+, |)$, then the concept of greatest-type lower reduces to that of *greatest-type divisor* introduced in [7].

Definition 2.3. Let f be a complex-valued function on P . Then f is said to be *semi-multiplicative* if for any $x, y \in P$, one has $f(x)f(y) = f(x \wedge y)f(x \vee y)$.

The above concept of a semi-multiplicative function on P is a generalization of the known concept of a *semi-multiplicative arithmetical function* [9, p. 49].

Definition 2.4. For any incidence function f , we define for any $x \in P$ the function $1/f$ to be 0 if $f(x) = 0$; $1/f(x)$ if $f(x) \neq 0$.

It is easy to check that the following is true.

Proposition 2.5. *Let f be an incidence function. Then f is semi-multiplicative if and only if $1/f$ is semi multiplicative.*

3. COMBINATORIAL FORMULAS FOR $\det[f(x_i \wedge x_j)]$ AND $\det(f[x_i \vee x_j])$

Throughout this paper, denote by $|A|$ the cardinality of any finite set A . In the present section we give reductions for $\psi_f(x_k)$ using the ideas in [6], [7]. First one needs a generalization of the principle of cross-classification in [6] to give a preliminary reduction for the formula of $\psi_f(x_k)$. For an alternative proof using induction, see [8].

Lemma 3.1 ([6, Lemma 1]). *Let R be a given finite set and f any complex-valued function defined on R . For a subset T of R , we denote by \bar{T} the set of those elements of R which are not in T , i.e., $\bar{T} = R \setminus T$. If R_1, \dots, R_m are given m distinct subsets of R , then*

$$\sum_{x \in \bigcap_{i=1}^m \bar{R}_i} f(x) = \sum_{x \in R} f(x) + \sum_{t=1}^m (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq m} \sum_{x \in \bigcap_{j=1}^t R_{i_j}} f(x).$$

Lemma 3.2. *Let f be an incidence function of P . Then*

$$\sum_{x \leq z \leq y} (f * \mu)(x, z) = f(x, y)$$

for all $x, y \in P$. In particular, one has

$$\sum_{z \leq y} (f * \mu)(z) = f(y)$$

for all $y \in P$.

Proof. Let $x, y \in P$ be given. Note that $f * \delta = f$ and $\mu * \zeta = \delta$. Then

$$\begin{aligned} f(x, y) &= (f * \delta)(x, y) = (f * (\mu * \zeta))(x, y) = ((f * \mu) * \zeta)(x, y) \\ &= \sum_{x \leq z \leq y} (f * \mu)(x, z) \zeta(z, y) = \sum_{x \leq z \leq y} (f * \mu)(x, z). \end{aligned}$$

The first assertion is proved. For the other assertion, one needs only to pick $x = \min P$. The proof is complete. □

Lemma 3.3. Let n be an integer. Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set with $x_i < x_j \Rightarrow i < j$. If $\psi_f(x_k)$ is defined as in (1), then

$$(2) \quad \psi_f(x_k) = f(x_k) + \sum_{t=1}^{k-1} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq k-1} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t}),$$

where $f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t})$ denotes f evaluated at the meet of $x_k, x_{i_1}, \dots, x_{i_t}$.

Proof. In Lemma 3.1, let $m = k - 1$ and $R = \{d: d \leq x_k, x_k \in S\}$. For $1 \leq i \leq k - 1$, let $R_i = \{d \in R: d \leq x_i, x_i \in S\}$. Then one has $R_i = \{d: d \leq x_k \wedge x_i\}$. By Lemma 3.1, one has

$$(3) \quad \psi_f(x_k) = \sum_{d \leq x_k} (f * \mu)(d) + \sum_{t=1}^{k-1} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq k-1} \sum_{d \leq x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t}} (f * \mu)(d).$$

By Lemma 3.2, one has $\sum_{d \leq x_k} (f * \mu)(d) = f(x_k)$ and for $1 \leq i_1 < \dots < i_t \leq k - 1$ ($1 \leq t \leq k - 1$), one has

$$(4) \quad \sum_{d \leq x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t}} (f * \mu)(d) = f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_t}).$$

It then follows from Equations (3) and (4) that (2) holds. This completes the proof of Lemma 3.3. \square

Now, we give further reduction for the formula of $\psi_f(x_k)$. The ideas of the proofs of the following two lemmas are due to our article [7].

Lemma 3.4. Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set with $x_i < x_j \Rightarrow i < j$. For $1 \leq k \leq n$, let $I_k = \{i: 1 \leq i \leq k - 1 \text{ and } x_i \not\leq x_k\}$ and $J_k = \{1, 2, \dots, k - 1\} \setminus I_k$. Then

$$(5) \quad \psi_f(x_k) = f(x_k) + \sum_{r=1}^{|J_k|} (-1)^r \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r}).$$

Proof. If $|I_k| = 0$, then it follows from Lemma 3.3 that Lemma 3.4 holds. In what follows let $|I_k| \geq 1$. Note that for $i \in J_k$ one has $x_i \leq x_k$. Since S is meet-closed, $x_1 \leq x_k$. Thus one has $|J_k| \geq 1$. Note also that $|I_k| + |J_k| = k - 1$. By Lemma 3.3, one has

$$(6) \quad \psi_f(x_k) = f(x_k) + \Delta' + \Delta,$$

where

$$\Delta' = \sum_{r=1}^{|J_k|} (-1)^r \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r})$$

and

$$(7) \quad \Delta = \sum_{r=1}^{|J_k|} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k}} \sum_{s=1}^{|I_k|} (-1)^{r+s} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in I_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}).$$

For any given $t_1 < \dots < t_s$, $t_u \in I_k$ ($1 \leq u \leq s$), it follows from the fact that S is meet-closed that $x_k \wedge x_{t_1} \wedge \dots \wedge x_{t_s} \in S$. Let $x_l = x_k \wedge x_{t_1} \wedge \dots \wedge x_{t_s}$. Then $x_l \leq x_k$ and $x_l \leq x_{t_u}$ for $1 \leq u \leq s$. So one has $l \in J_k$. Then by (7), one has

$$\begin{aligned} (8) \quad \Delta &= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in I_k}} \sum_{r=1}^{|J_k|} (-1)^{r+s} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\ &= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in I_k}} \sum_{r=0}^{|J_k|-1} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k, i_j \neq l}} ((-1)^{r+s} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s})) \\ &\quad + (-1)^{r+s+1} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_l \wedge x_{t_1} \wedge \dots \wedge x_{t_s})) \\ &= \sum_{s=1}^{|I_k|} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in I_k}} \sum_{r=0}^{|J_k|-1} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in J_k, i_j \neq l}} ((-1)^{r+s} \cdot f(x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_l)) \\ &\quad + (-1)^{r+s+1} \cdot f(x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_l)) = 0. \end{aligned}$$

Therefore it follows from Equations (6) and (8) that (5) holds. The proof of Lemma 3.4 is complete. \square

Now we can use the concept of greatest-type lower to give a further reduction for $\psi_f(x_k)$.

Lemma 3.5. *Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set. For $1 \leq k \leq n$, let $R_k = \{i: 1 \leq i \leq k-1, x_i \text{ is the greatest-type lower of } x_k \text{ in } S\}$. Then*

$$\psi_f(x_k) = f(x_k) + \sum_{r=1}^{|R_k|} (1)^r \sum_{\substack{i_1 < \dots < i_r \\ i_j \in R_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r}).$$

Proof. For the case $k \leq 2$, the lemma is clearly true. In what follows let $k \geq 3$. Let $J_k = \{i: 1 \leq i \leq k-1 \text{ and } x_i \leq x_k\}$. Then $|J_k| \geq 1$. It is clear that $R_k \subseteq J_k$.

If $|J_k| = 1$, then $J_k = \{1\}$. Note that $|R_k| \geq 1$. So one has $R_k = \{1\} = J_k$. Thus by Lemma 3.4, the result is true. In the following let $|J_k| \geq 2$. Let $L_k = J_k \setminus R_k$. We claim that $L_k \neq \emptyset$. Assuming otherwise implies that $R_k = J_k$. But $1 \in J_k$, hence $1 \in R_k$. From $|J_k| \geq 2$ one deduces that there is an $i \in J_k$, $i \neq 1$, such that $i \in J_k = R_k$. Since S is meet-closed, one has $x_1 < x_i$. This is impossible since x_1 and x_i cannot both be greatest-type lowers of x_k in S . Therefore the claim is true. In a similar way to that in (6), one has by Lemma 3.4 that

$$\psi_f(x_k) = f(x_k) + \overline{\Delta}' + \overline{\Delta},$$

where

$$\overline{\Delta}' = \sum_{r=1}^{|R_k|} (-1)^r \sum_{\substack{i_1 < \dots < i_r \\ i_j \in R_k}} f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r})$$

and

$$\begin{aligned} (9) \quad \Delta &= \sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in R_k}} \sum_{s=1}^{|L_k|} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in L_k}} (-1)^{r+s} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\ &= \sum_{s=1}^{|L_k|} \sum_{\substack{t_1 < \dots < t_s \\ t_u \in L_k}} (-1)^s \sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in R_k}} (-1)^r \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}). \end{aligned}$$

To prove the lemma, one needs only to show that $\overline{\Delta} = 0$, which we will do in the following.

For any given $t_1 < \dots < t_s$ ($1 \leq s \leq |L_k|$), $t_u \in L_k$, $1 \leq u \leq s$, let $T = \{i: i \in R_k, \text{ and } x_{t_u} \leq x_i \text{ for some } t_u, 1 \leq u \leq s\}$ and $Q = R_k \setminus T$. Let $|T| = h$ and $|Q| = h'$. Clearly one has that $1 \leq h \leq |R_k|$ and $0 \leq h' \leq |R_k| - 1$. Then one has

$$\begin{aligned} (10) \quad &\sum_{r=0}^{|R_k|} \sum_{\substack{i_1 < \dots < i_r \\ i_j \in R_k}} (-1)^r \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\ &= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \dots < i_{r'} \\ i_u \in Q}} \sum_{r=0}^h \sum_{\substack{j_1 < \dots < j_r \\ j_v \in T}} (-1)^{r+r'} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_{r'}} \wedge x_{j_1} \wedge \dots \\ &\quad \wedge x_{j_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\ &= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \dots < i_{r'} \\ i_u \in Q}} \sum_{r=0}^h \sum_{\substack{j_1 < \dots < j_r \\ j_v \in T}} (-1)^{r+r'} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \end{aligned}$$

(since by the definition of T one has $x_{j_1} \wedge \dots \wedge x_{j_r} \wedge x_{t_1} \wedge \dots \wedge x_{t_s} = x_{t_1} \wedge \dots \wedge x_{t_s}$ for any $j_1 < \dots < j_r$, $j_v \in T$)

$$\begin{aligned}
&= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \dots < i_{r'} \\ i_u \in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\
&\qquad \qquad \qquad \cdot \left(1 + \sum_{r=1}^h (-1)^r \sum_{\substack{j_1 < \dots < j_r \\ j_v \in T}} 1 \right) \\
&= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \dots < i_{r'} \\ i_u \in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \\
&\qquad \qquad \qquad \cdot \left(1 + \sum_{r=1}^h (-1)^r \cdot \binom{h}{r} \right) \\
&= \sum_{r'=0}^{h'} \sum_{\substack{i_1 < \dots < i_{r'} \\ i_u \in Q}} (-1)^{r'} \cdot f(x_k \wedge x_{i_1} \wedge \dots \wedge x_{i_{r'}} \wedge x_{t_1} \wedge \dots \wedge x_{t_s}) \cdot (1-1)^h = 0.
\end{aligned}$$

It now follows from Equations (9) and (10) that $\bar{\Delta} = 0$. This completes the proof of Lemma 3.5. \square

Theorem 3.6. *Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set and f an incidence function. Then*

$$\det[f(x_i \wedge x_j)] = \prod_{i=1}^n \left(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t}) \right),$$

where $f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S .

Proof. This theorem follows from Proposition 2.1 and Lemma 3.5. \square

Lemma 3.7. *Let f be a semi-multiplicative function and $S = \{x_1, \dots, x_n\}$ a meet-closed set. If $f(x_i) \neq 0$ for all $1 \leq i \leq n$, then*

$$[f(x_i \vee x_j)] = \text{diag}\{f(x_1), \dots, f(x_n)\} \cdot \left[\frac{1}{f}(x_i \wedge x_j) \right] \cdot \text{diag}\{f(x_1), \dots, f(x_n)\}.$$

Proof. It follows from definition of a semi-multiplicative function that this lemma is true. \square

Theorem 3.8. Let $S = \{x_1, \dots, x_n\}$ be a meet-closed set. If f is a semi-multiplicative function satisfying $f(x_i) \neq 0$ for all $1 \leq i \leq n$, then

$$\det [f(x_i \vee x_j)] = \prod_{i=1}^n [f(x_i)]^2 \left(\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} \frac{1}{f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})} \right),$$

where $f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S .

Proof. This theorem follows from Lemma 3.7 and Theorem 3.6 applied to the function $1/f$. The proof is complete. \square

It follows from Theorems 3.6 and 3.8 that the following two corollaries are true.

Corollary 3.9. Let $S = \{x_1, x_2, \dots, x_n\}$ be lower-closed and let f be an incidence function. Then each of the following is true:

- (i) One has $\det[f(x_i \wedge x_j)] = \prod_{i=1}^n (f * \mu)(x_i)$;
- (ii) If f is semi-multiplicative and $f(x_i) \neq 0$ for all $1 \leq i \leq n$, then $\det[f(x_i \vee x_j)] = \prod_{i=1}^n [f(x_i)]^2 ((1/f) * \mu)(x)$.

Corollary 3.10. Let $S = \{x_1, x_2, \dots, x_n\}$ be a chain with $x_1 < x_2 < \dots < x_{n-1} < x_n$ and f an incidence function. Then each of the following is true:

- (i) One has $\det[f(x_i \wedge x_j)] = f(x_1) \prod_{i=2}^n [f(x_i) - f(x_{i-1})]$;
- (ii) If f is semi-multiplicative and $f(x_i) \neq 0$ for all $1 \leq i \leq n$, then $\det[f(x_i \vee x_j)] = f(x_n) \prod_{i=2}^n [f(x_{i-1}) - f(x_i)]$.

Proof. For k , $1 \leq k \leq n$, since $x_1 < x_2 < \dots < x_n$, one has that x_{k-1} is the only greatest-type lower of x_k in S . It then follows from Theorems 3.6 and 3.8 that this corollary is true. \square

4. NONSINGULARITY OF MATRICES $[f(x_i \wedge x_j)]$ AND $[f(x_i \vee x_j)]$

We can now use the results of the preceding section to give a characterization for nonsingularity of matrices $[f(x_i \wedge x_j)]$ and $[f(x_i \vee x_j)]$ as follows.

Theorem 4.1. Let $S = \{x_1, x_2, \dots, x_n\}$ be a meet-closed set and let f be an incidence function. Then the matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if for all $1 \leq i \leq n$, one has

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t}) \neq 0,$$

where $f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S .

Proof. First, one has that the matrix $[f(x_i \wedge x_j)]$ on S is nonsingular if and only if $\det([f(x_i \wedge x_j)]) \neq 0$. From Theorem 3.6 one knows that

$$\det[f(x_i \wedge x_j)] = \prod_{i=1}^n \left(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t}) \right),$$

where $f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})$ denotes f evaluated at the meet of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type lowers of x_i in S . So $[f(x_i \wedge x_j)]$ is nonsingular if and only if for all $1 \leq i \leq n$, one has

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t}) \neq 0,$$

as desired. □

Theorem 4.2. Let $S = \{x_1, x_2, \dots, x_n\}$ be a meet-closed set and let f be a semi-multiplicative function. Then the matrix $[f(x_i \vee x_j)]$ defined on S is nonsingular if and only if for all $1 \leq i \leq n$ one has $f(x_i) \neq 0$ and

$$\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} \frac{1}{f(x \wedge y_{i_1} \wedge \dots \wedge y_{i_t})} \neq 0,$$

where $f(x_i \wedge y_{i_1} \wedge \dots \wedge y_{i_t})$ denotes f evaluated at the meet of $x, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type lowers of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S .

Proof. This theorem follows immediately from Theorem 3.8. □

Corollary 4.3. Let $S = \{x_1, x_2, \dots, x_n\}$ be lower-closed. Then each of the following is true:

- (i) The matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if $(f * \mu)(x_i) \neq 0$ for all $1 \leq i \leq n$;
- (ii) The matrix $(f(x_i \vee x_j))$ defined on S is nonsingular if and only if $f(x_i) \neq 0$ and $((1/f) * \mu)(x_i) \neq 0$ for all $1 \leq i \leq n$.

Corollary 4.4. Let $S = \{x_1, x_2, \dots, x_n\}$ be a chain with $x_1 < x_2 < \dots < x_{n-1} < x_n$. Then each of the following is true:

- (i) The matrix $[f(x_i \wedge x_j)]$ defined on S is nonsingular if and only if $f(x_1) \neq 0$ and for all $k, 2 \leq k \leq n$, one has $f(x_{k-1}) \neq f(x_k)$;
- (ii) The matrix $[f(x_i \vee x_j)]$ defined on S is nonsingular if and only if $f(x_1) \neq 0$, and for all $k, 2 \leq k \leq n$, one has $f(x_k) \neq 0$ and $f(x_{k-1}) \neq f(x_k)$.

5. APPLICATIONS TO MATRICES $[f(x_i, x_j)]$ AND $(f[x_i, x_j])$

In the present section, we give number-theoretic applications of the results presented in Sections 3 and 4.

Theorem 5.1. Let $S = \{x_1, \dots, x_n\}$ be a gcd-closed set and let f be an arithmetical function. Then

$$\det[f(x_i, x_j)] = \prod_{i=1}^n \left(f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i, y_{i_1}, \dots, y_{i_t}) \right),$$

where $f(x_i, y_{i_1}, \dots, y_{i_t})$ denotes f evaluated at the greatest common divisor $(x_i, y_{i_1}, \dots, y_{i_t})$ of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S .

Proof. Let $(P, \leq) = (\mathbb{Z}^+, |)$. Then this theorem follows from Theorem 3.6. \square

Theorem 5.2. Let $S = \{x_1, \dots, x_n\}$ be a gcd-closed set. If f is a semi-multiplicative arithmetical function satisfying $f(x_i) \neq 0$ for all $1 \leq i \leq n$, then

$$\det(f[x_i, x_j]) = \prod_{i=1}^n [f(x_i)]^2 \left(\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} \frac{1}{f(x_i, y_{i_1}, \dots, y_{i_t})} \right),$$

where $f(x_i, y_{i_1}, \dots, y_{i_t})$ denotes f evaluated at the greatest common divisor $(x_i, y_{i_1}, \dots, y_{i_t})$ of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S .

Proof. Let $(P, \leq) = (\mathbb{Z}^+, |)$. Then this theorem follows from Theorem 3.8. \square

Theorem 5.3. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n distinct positive integers and f an arithmetical function. If S is gcd-closed, then the matrix $[f(x_i, x_j)]$ defined on S is nonsingular if and only if for all $1 \leq i \leq n$ one has

$$f(x_i) + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} f(x_i, y_{i_1}, \dots, y_{i_t}) \neq 0,$$

where $f(x_i, y_{i_1}, \dots, y_{i_t})$ denotes f evaluated at the greatest common divisor of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S .

Proof. Let $(P, \leq) = (\mathbb{Z}^+, |)$. Then this theorem follows immediately from Theorem 4.1. \square

Note that Theorem 5.3 gives an answer to the problem raised by Bourque and Ligh in [4].

Theorem 5.4. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n distinct positive integers and f a semi-multiplicative arithmetical function. If S is gcd-closed, then the matrix $(f[x_i, x_j])$ defined on S is nonsingular if and only if for all $1 \leq i \leq n$ one has $f(x_i) \neq 0$ and

$$\frac{1}{f(x_i)} + \sum_{t=1}^{n(x_i)} (-1)^t \sum_{1 \leq i_1 < \dots < i_t \leq n(x_i)} \frac{1}{f(x_i, y_{i_1}, \dots, y_{i_t})} \neq 0,$$

where $f(x_i, y_{i_1}, \dots, y_{i_t})$ denotes f evaluated at the greatest common divisor of $x_i, y_{i_1}, \dots, y_{i_t}$, $n(x_i)$ equals the cardinality of the set of the greatest-type divisors of x_i in S , and $\{y_1, y_2, \dots, y_{n(x_i)}\}$ equals the set of the greatest-type divisors of x_i in S .

Proof. Let $(P, \leq) = (\mathbb{Z}^+, |)$. Then this theorem follows immediately from Theorem 4.2. \square

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