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ON TOPOLOGICAL CLASSIFICATION OF NON-ARCHIMEDEAN FRÉCHET SPACES

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Abstract. We prove that any infinite-dimensional non-archimedean Fréchet space E is homeomorphic to $D^{\mathbb{N}}$ where D is a discrete space with $\operatorname{card}(D) = \operatorname{dens}(E)$. It follows that infinite-dimensional non-archimedean Fréchet spaces E and F are homeomorphic if and only if $\operatorname{dens}(E) = \operatorname{dens}(F)$. In particular, any infinite-dimensional non-archimedean Fréchet space of countable type over a field $\mathbb{K}$ is homeomorphic to the non-archimedean Fréchet space $\mathbb{K}^{\mathbb{N}}$.

Keywords: non-archimedean Fréchet spaces, homeomorphisms

MSC 2000: 46S10

1. Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|\colon\ \mathbb{K} \to [0,\infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [3], [6] and [5].

Any finite-dimensional lcs E is linearly homeomorphic to the Banach space $\mathbb{K}^{\dim E}$ and any infinite-dimensional Banach space of countable type is linearly homeomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero (with the sup-norm) ([5], Theorem 3.16). Nevertheless, there exist Fréchet spaces of countable type without a Schauder basis ([7]).

Van Rooij proved that any infinite-dimensional Banach space E is homeomorphic to $D^{\mathbb{N}}$ where D is a discrete space with $\operatorname{card}(D) = \operatorname{dens}(E)$ ([4], Theorem 3.8 (ii)).

In this note we extend this result to infinite-dimensional Fréchet spaces:

Any infinite-dimensional Fréchet space E is homeomorphic to $D^{\mathbb{N}}$ where D is a discrete space with $\operatorname{card}(D) = \operatorname{dens}(E)$ (Theorem 3).

It follows that infinite-dimensional Fréchet spaces E and F are homeomorphic if and only if $\mathrm{dens}(E) = \mathrm{dens}(F)$ (Corollary 4). In particular, any infinite-dimensional Fréchet space of countable type (over $\mathbb{K}$) is homeomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (Corollary 5).

On the other hand any finite-dimensional Fréchet space E (over $\mathbb{K}$) with $E \neq \{0\}$ is homeomorphic to $\mathbb{K}$ (Proposition 6) (see also [4], Theorem 3.8 (i)).

Finally, we show that any non-compact absolutely convex open subset U in a Fréchet space E is homeomorphic to E (Proposition 9).

2. Preliminaries

$\mathbb{N}$ is the set of all positive integers. The cardinality of a set D is denoted by $\mathrm{card}(D)$. The smallest of the cardinalities of the dense subsets of a topological space X is denoted by $\mathrm{dens}(X)$. The smallest among the cardinalities of the linearly dense subsets of a lcs E is denoted by $t(E)$. If topological spaces X and Y are homeomorphic we write $X \sim Y$.

A subset U in a lcs E is *absolutely convex* if $\alpha x + \beta y \in U$ for all $x, y \in U$ and $\alpha, \beta \in \mathbb{K}$ with $|\alpha|, |\beta| \leqslant 1$.

Any open absolutely convex subset in a lcs E is a closed subgroup of E. Hence for any two open absolutely convex subsets A and B in a lcs E with $A \supset B \neq \emptyset$ the topological quotient group (A/B) is discrete.

Any metrizable lcs E possesses a decreasing sequence (U_n) of absolutely convex open subsets which forms a base of neighborhoods of zero in E.

A metrizable lcs is *of countable type* if $t(E) \leqslant \aleph_0$. A *Fréchet space* is a metrizable complete lcs.

Let (x_n) be a sequence in a Fréchet space F. The series $\sum\limits_{n=1}^{\infty} x_n$ is convergent in F if and only if $\lim x_n = 0$.

For all $\alpha, \beta \in \mathbb{K}$ we have $|\alpha\beta| = |\alpha||\beta|$ and $|\alpha + \beta| \leqslant \max\{|\alpha|, |\beta|\}$; if $|\alpha| < |\beta|$ then $|\alpha + \beta| = |\beta|$. The set $J = \{\alpha \in \mathbb{K}\colon\ |\alpha| \leqslant 1\}$ is a subring of $\mathbb{K}$ and $I = \{\alpha \in \mathbb{K}\colon\ |\alpha| < 1\}$ is a maximal ideal in J. The field $k = (J/I)$ is the *residue class field* of $\mathbb{K}$.

We will need two lemmas.

Lemma 1. *Let E be a Fréchet space and let (U_n) be a decreasing sequence of open absolutely convex subsets of E which forms a base of neighborhoods of zero in E. Then E is homeomorphic to the product space $\prod\limits_{n=0}^{\infty}(U_n/U_{n+1})$ where $U_0 = E$.*

Proof. Let $n \geqslant 0$. Denote by π_n the quotient map $U_n \to (U_n/U_{n+1})$ and let $\psi_n\colon (U_n/U_{n+1}) \to U_n$ be a map with $\pi_n(\psi_n(z)) = z$ for any $z \in (U_n/U_{n+1})$. Put $V_n = \psi_n(U_n/U_{n+1})$. Clearly,

$$(*)\qquad \forall x, y \in V_n\colon\ [(x-y) \in U_{n+1} \Rightarrow x = y].$$

It follows that the set V_n is discrete, so it is homeomorphic to (U_n/U_{n+1}).

Let $x \in U_0$. Since $\forall n \geqslant 0\ \forall y \in U_n\ \exists z \in V_n\colon (y-z) \in U_{n+1}$, we can construct inductively a sequence $(\varphi_n^x) \in \prod\limits_{n=0}^{\infty} V_n$ with $\Big(x - \sum\limits_{n=0}^{k} \varphi_n^x\Big) \in U_{k+1}$ for any $k \geqslant 0$.

Clearly, $x = \sum\limits_{n=0}^{\infty} \varphi_n^x$. By induction one can show easily that $\forall n \geqslant 0\colon x_n = \varphi_n^x$ for any $(x_n) \in \prod\limits_{n=0}^{\infty} V_n$ with $\sum\limits_{n=0}^{\infty} x_n = x$. Thus the map $\varphi\colon U_0 \to \prod\limits_{n=0}^{\infty} V_n$, $x \mapsto (\varphi_n^x)$ is a bijection.

Let $n \in \mathbb{N}$ and $x, y \in U_0$, $(x-y) \in U_n$. Then $\sum\limits_{i=0}^{k}(\varphi_i^x - \varphi_i^y) \in U_{k+1}$ for $k = 0, 1, \ldots, n-1$. Using $(*)$ we obtain in turn $\varphi_0^x = \varphi_0^y$, …, $\varphi_{n-1}^x = \varphi_{n-1}^y$. Thus the map φ is continuous.

If $n \in \mathbb{N}$, $x, y \in U_0$ and $\varphi_k^x = \varphi_k^y$ for $k = 0, 1, \ldots, n-1$, then $(x-y) \in U_n$. Hence φ^{-1} is continuous.

We have proved that the spaces E and $\prod\limits_{n=0}^{\infty}(U_n/U_{n+1})$ are homeomorphic. □

Lemma 2. *Let $S_1, S_2, \ldots, S$ be infinite discrete topological spaces with $\operatorname{card}(S_n) \leqslant \operatorname{card}(S_{n+1})$, $n \in \mathbb{N}$, and $\operatorname{card}(S) = \sup\limits_n \operatorname{card}(S_n)$. Then the product spaces $\prod\limits_{n=1}^{\infty} S_n$ and $S^{\mathbb{N}}$ are homeomorphic.*

Proof. Let $\Phi\colon \mathbb{N} \to S_1$ be an injective map such that $\Phi(\mathbb{N}) \neq S_1$. The sets $(S_1 \setminus \Phi(\mathbb{N})) \times \prod\limits_{n=2}^{\infty} S_n$ and $\{\Phi(i)\} \times \prod\limits_{k=2}^{i+2}\{s_k\} \times \prod\limits_{n=i+3}^{\infty} S_n$ for $i \in \mathbb{N}$, $(s_2, \ldots, s_{i+2}) \in S_2 \times \ldots \times S_{i+2}$ form an open covering of the space $\prod\limits_{n=1}^{\infty} S_n$. These sets are pairwise disjoint and each of them is homeomorphic to $\prod_{n=1}^{\infty} S_n$, since $(S_1 \setminus \Phi(\mathbb{N})) \times S_2 \sim S_1 \times S_2$ and $S_{i+3} \sim S_1 \times \ldots \times S_{i+3}$, $i \in \mathbb{N}$. The cardinality of this covering is equal to $\operatorname{card}(S)$, because $\sum_{n=1}^{\infty} \operatorname{card}(S_n) = \operatorname{card}(S)$. It follows that $\prod_{n=1}^{\infty} S_n \sim S \times \prod_{n=1}^{\infty} S_n$.

Since $\prod_{n=1}^{\infty} S_n \sim \prod_{n=1}^{\infty} (S_1 \times \ldots \times S_n) \sim \prod_{n=1}^{\infty} S_n^{\mathbb{N}} \sim \left(\prod_{n=1}^{\infty} S_n\right)^{\mathbb{N}}$, it follows that $\prod_{n=1}^{\infty} S_n \sim \left(S \times \prod_{n=1}^{\infty} S_n\right)^{\mathbb{N}} \sim S^{\mathbb{N}} \times \left(\prod_{n=1}^{\infty} S_n\right)^{\mathbb{N}} \sim S^{\mathbb{N}} \times \prod_{n=1}^{\infty} S_n \sim \prod_{n=1}^{\infty} (S \times S_n) \sim S^{\mathbb{N}}$. □

Now we can prove our main result.

Theorem 3. *Any infinite-dimensional Fréchet space E is homeomorphic to $D^{\mathbb{N}}$ where D is a discrete space with* $\operatorname{card}(D) = \operatorname{dens}(E)$.

P r o o f. Since E is not locally compact, there exists a decreasing sequence of open absolutely convex subsets of E which forms a base of neighborhoods of zero in E such that $\operatorname{card}(U_n/U_{n+1}) \geqslant \aleph_0$ for any $n \geqslant 0$ (where $U_0 = E$).

Let $n \geqslant 0$ and $(\alpha_k) \subset \mathbb{K}$ with $|\alpha_k| \to \infty$. Then $E = \bigcup_{k=1}^{\infty} \alpha_k U_n$. If A is a dense subset of U_n, then $\{\alpha_k a\colon\ k \in \mathbb{N},\ a \in A\}$ is dense in E. Thus $\operatorname{dens}(E) \leqslant \aleph_0 \operatorname{dens}(U_n)$.

We have $\sup_{m>n} \operatorname{card}(U_n/U_m) = \operatorname{dens}(E)$ for any $n \geqslant 0$. Indeed,

$$\begin{aligned}\sup_{m>n} \operatorname{card}(U_n/U_m) &\leqslant \operatorname{dens}(U_n) \leqslant \operatorname{dens}(E) \leqslant \aleph_0 \operatorname{dens}(U_n) = \operatorname{dens}(U_n)\\ &\leqslant \sum_{m>n} \operatorname{card}(U_n/U_m) \leqslant \aleph_0 \sup_{m>n} \operatorname{card}(U_n/U_m) = \sup_{m>n} \operatorname{card}(U_n/U_m).\end{aligned}$$

Let D be a discrete space with $\operatorname{card}(D) = \operatorname{dens}(E)$.

If $\forall n \geqslant 0\ \exists m > n\colon \operatorname{card}(U_n/U_m) = \operatorname{dens}(E)$, then there is an increasing sequence $(n_k) \subset \mathbb{N}$ such that $\operatorname{card}(U_{n_k}/U_{n_{k+1}}) = \operatorname{dens}(E)$ for all $k \geqslant 0$ and $n_0 = 0$. Thus, by Lemma 1, E is homeomorphic to $D^{\mathbb{N}}$.

If $\exists n \geqslant 0\ \forall m > n\colon \operatorname{card}(U_n/U_m) < \operatorname{dens}(E)$, then there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $\sup_k \operatorname{card}(U_{n_k}/U_{n_{k+1}}) = \operatorname{dens}(E)$ such that the sequence $(\operatorname{card}(U_{n_k}/U_{n_{k+1}}))_{k=1}^{\infty}$ is increasing. Using Lemmas 1 and 2 we get $E \sim (U_0/U_{n_1}) \times \prod_{k=1}^{\infty} (U_{n_k}/U_{n_{k+1}}) \sim (U_0/U_{n_1}) \times D^{\mathbb{N}} \sim D^{\mathbb{N}}$, since $(U_0/U_{n_1}) \times D \sim D$. □

Because $\operatorname{dens}(A^{\mathbb{N}}) = \operatorname{card}(A)$ for any infinite discrete space A, we obtain

Corollary 4. *Infinite-dimensional Fréchet spaces E and F are homeomorphic if and only if* $\operatorname{dens}(E) = \operatorname{dens}(F)$.

For any infinite-dimensional Fréchet space E of countable type we have $\operatorname{dens}(E) = \operatorname{dens}(\mathbb{K}) = \operatorname{dens}(\mathbb{K}^{\mathbb{N}})$. Thus we get

Corollary 5. *Any infinite-dimensional Fréchet space E of countable type is homeomorphic to* $\mathbb{K}^{\mathbb{N}}$.

For finite-dimensional Fréchet spaces we have the following (compare with [4], Theorem 3.8 (i)).

Proposition 6. *Any finite-dimensional Fréchet space E with $E \neq \{0\}$ is homeomorphic to $\mathbb{K}$. If $\mathbb{K}$ is locally compact, then it is homeomorphic to $\mathbb{N} \times k^{\mathbb{N}}$ where k is the residue class field of $\mathbb{K}$. If $\mathbb{K}$ is not locally compact, then it is homeomorphic to $K^{\mathbb{N}}$ where K is a discrete space with* $\operatorname{card}(K) = \operatorname{dens}(\mathbb{K})$.

Proof. First, assume that $\mathbb{K}$ is locally compact. Then the set $I = \{\alpha \in \mathbb{K} \colon |\alpha| < 1\}$ is compact. Let $\beta \in \mathbb{K}$ with $|\beta| = \max\{|\alpha| \colon \alpha \in I\}$. Put $U_n = \{\alpha \in \mathbb{K} \colon |\alpha| \leqslant |\beta|^{n-1}\}$, $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ the map

$$\Phi_n \colon (U_n/U_{n+1}) \to (U_1/U_2), \quad \alpha + U_{n+1} \mapsto \beta^{1-n}\alpha + U_2$$

is a homeomorphism. By Lemma 1 we have $\mathbb{K} \sim (\mathbb{K}/U_1) \times (U_1/U_2)^{\mathbb{N}}$. Since $I = U_2$, (U_1/U_2) is the residue class field k of $\mathbb{K}$. Clearly, k is finite. Moreover, $(\mathbb{K}/U_1) \subset \bigcup_{n=1}^{\infty} (\beta^{-n}U_1/U_1)$ and $\operatorname{card}(\beta^{-n}U_1/U_1) < \aleph_0$, $n \in \mathbb{N}$, so $(\mathbb{K}/U_1) \sim \mathbb{N}$. Thus $\mathbb{K} \sim \mathbb{N} \times k^{\mathbb{N}}$.

Next, assume that $\mathbb{K}$ is not locally compact. As in the proof of Theorem 3, we show that $\mathbb{K}$ is homeomorphic to $K^{\mathbb{N}}$, where K is a discrete space with $\operatorname{card}(K) = \operatorname{dens}(\mathbb{K})$.

It follows that any finite-dimensional Fréchet space E with $E \neq \{0\}$ is homeomorphic to $\mathbb{K}$, since E is linearly homeomorphic to $\mathbb{K}^{\dim E}$. □

By Corollary 5 and Proposition 6 we get

Corollary 7. *If $\mathbb{K}$ is not locally compact then any Fréchet space of countable type is homeomorphic to $\mathbb{K}$.*

For any $n \in \mathbb{N}$ the space $\{0, 1, \ldots, n\}^{\mathbb{N}}$ is homeomorphic to $\{0, 1\}^{\mathbb{N}}$ (see [1]). Thus we obtain the following (see [4], Theorem 3.8 (i)).

Corollary 8. *If $\mathbb{K}$ is locally compact then it is homeomorphic to $\mathbb{N} \times C$, where C is the Cantor set.*

Finally we show

Proposition 9. *Let E be a Fréchet space with $E \neq \{0\}$. Then we have*

(a) *Any non-compact absolutely convex open subset U in E is homeomorphic to E.*

(b) $\operatorname{dens}(E) = \operatorname{dens}(\mathbb{K}) t(E)$.

(c) *If E is infinite-dimensional then* $\dim(E) = \operatorname{card}(E) = (\operatorname{dens}(E))^{\aleph_0}$ *and E is homeomorphic to the Banach space $c_0(D)$ where D is a discrete space with* $\operatorname{card}(D) = \operatorname{dens}(E)$.

Proof. (a) If $\dim E \geqslant \aleph_0$ or $\mathbb{K}$ is not locally compact, then as in the proof of Theorem 3 we show that $U \sim D^{\mathbb{N}}$ where D is a discrete space with $\operatorname{card}(D) = \operatorname{dens}(E)$. Hence $U \sim E$.

If $\dim(E) < \aleph_0$ and $\mathbb{K}$ is locally compact, then $U \sim \mathbb{N} \times k^{\mathbb{N}}$. Indeed, without loss of generality we can assume that $E = \mathbb{K}^m$ where $m = \dim(E)$. Put $U_n = \{(\alpha_1, \ldots, \alpha_m) \in E\colon \max\limits_{1 \leqslant i \leqslant m} |\alpha_i| \leqslant |\beta|^{n-1}\}$, $n \in \mathbb{N}$, where β is defined in the proof of Proposition 6. For any $n \in \mathbb{N}$ the map

$$\Phi_n\colon (U_n/U_{n+1}) \to (U_1/U_2), \quad (\alpha_1, \ldots, \alpha_m) + U_{n+1} \mapsto \beta^{1-n}(\alpha_1, \ldots, \alpha_m) + U_2$$

is a homeomorphism. For some $t \in \mathbb{N}$ we have $U_t \subset U$. As in the proof of Lemma 1 we get $U \sim (U/U_t) \times \prod\limits_{n=t}^{\infty} (U_n/U_{n+1}) \sim (U/U_t) \times (U_1/U_2)^{\mathbb{N}}$. It is easy to check that $(U/U_t) \sim \mathbb{N}$ and $(U_1/U_2) \sim k^m$. Thus $U \sim \mathbb{N} \times k^{\mathbb{N}}$. Hence $U \sim E$.

(b) Let K be a dense subset of $\mathbb{K}$ with $\operatorname{card}(K) = \operatorname{dens}(\mathbb{K})$ and let X be a linearly dense subset of E with $\operatorname{card}(X) = t(E)$.

Since the set $A = \left\{ \sum\limits_{i=1}^{n} \alpha_i x_i\colon\ n \in \mathbb{N},\ \alpha_i \in K,\ x_i \in X \right\}$ is dense in E, we see that

$$\operatorname{dens}(\mathbb{K}) t(E) = \max\{\operatorname{dens}(\mathbb{K}),\ t(E)\} \leqslant \operatorname{dens}(E) \leqslant \operatorname{card}(A) = \operatorname{dens}(\mathbb{K}) t(E).$$

(c) Since $\operatorname{card}(\mathbb{K}) \leqslant \dim(E)$ ([2], Proposition 2.2), it follows that

$$\dim(E) = \operatorname{card}(\mathbb{K}) \dim(E) = \operatorname{card}(E).$$

By Theorem 3, $\operatorname{card}(E) = (\operatorname{dens}(E))^{\aleph_0}$.

It is known that $t(c_0(D)) = \operatorname{card}(D)$ ([4], Corollary 3.3). Using (b) we get $\operatorname{dens}(c_0(D)) = \operatorname{dens}(E)$. By Corollary 4, $c_0(D) \sim E$. $\square$

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