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COMMUTING TOEPLITZ OPERATORS ON THE PLURIHARMONIC BERGMAN SPACE

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Abstract. We prove that two Toeplitz operators acting on the pluriharmonic Bergman space with radial symbol and pluriharmonic symbol respectively commute only in an obvious case.

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1. Introduction

Let B be the open unit ball of the complex n-space $\mathbb{C}^n$. The pluriharmonic Bergman space b^2 is the subspace of the Lebesgue space $L^2 = L^2(B, V)$ consisting of all pluriharmonic functions on B where the notation V denotes the normalized Lebesgue volume measure on B. It is known that b^2 is a closed subspace of L^2 and hence is a Hilbert space. We let Q be the Hilbert space orthogonal projection from L^2 onto b^2. It can easily be seen that the domain of Q can be extended to $L^1(B, V)$ via an integral formula; see Section 2.

For a function $u \in L^2$, the *Toeplitz operator* $T_u\colon b^2 \to b^2$ with symbol u is the linear operator defined by

$$T_u f = Q(uf), \quad f \in b^2.$$

Clearly, T_u is densely defined. In fact, for any bounded holomorphic function f on B we have $Q(uf) \in b^2$.

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In this paper, we study the problem of when two Toeplitz operators commute each other on b^2. Originally, this problem was first considered and solved on Hardy space of the unit disk [1]. Later, the same problem has been studied in the context of holomorphic Bergman space by several authors [3], [4], [5], [6], [10] and has been completely solved in case of holomorphic or pluriharmonic symbols.

Recently, the present pluriharmonic case was also studied in [2], [7] and [8] and holomorphic symbols for commuting Toeplitz operators were completely characterized. In particular, the author and K. Zhu [7] proved the following: *For nonconstant holomorphic symbols f, g, T_f and T_g commute on b^2 if and only if a nontrivial linear combination of f and g is constant on B.* So far, we believe nothing else is known for this problem in the nonholomorphic symbols case.

In this paper, we would like to offer a partial result on this problem with nonholomorphic symbols. We consider general radial symbol and pluriharmonic symbol and then characterize the symbols for which the corresponding Toeplitz operators are commuting. Our result shows that the Toeplitz operators under consideration commute only in the obvious case. The following is the main result.

Main theorem. *Let $u \in L^2$ be radial and $v \in b^2$. Then $T_uT_v = T_vT_u$ on b^2 if and only if either v or u is constant on B.*

In the next section we first collect some properties on the holomorphic Bergman projection. Our main theorem above will be restated and proved in Theorem 3.

2. Proof

For each $z \in B$ we let K_z denote the Bergman kernel at z. Thus

$$K_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}}, \quad w \in B$$

where the notation $\langle w, z\rangle = w_1\overline{z_1} + \ldots + w_n\overline{z_n}$ denotes the Hermitian inner product on $\mathbb{C}^n$. The well known Bergman projection P is then the integral operator

$$P\psi(z) = \int_B \psi\overline{K_z}\,\mathrm{d}V, \quad z \in B$$

for functions $\psi \in L^2$. See Chapter 3 of [9] for more information about the Bergman kernel and the Bergman projection.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_k is a nonnegative integer, we will write

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

and

$$\alpha! = \alpha_1! \ldots \alpha_n!.$$

We will also write $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ for $z = (z_1, \ldots, z_n) \in B$.

We first need the following calculation.

Lemma 1. *For every multi-index α, we have*

$$\int_B |w^\alpha|^2 \,\mathrm{d}V(w) = \frac{n!\alpha!}{(n+|\alpha|)!}.$$

P r o o f. See Proposition 1.4.9 of [9]. $\square$

For two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, the notation $\beta \preceq \alpha$ means that

$$\beta_k \leqslant \alpha_k, \quad k = 1, \ldots, n,$$

and for $\beta \preceq \alpha$ we define

$$\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n).$$

Note that $|\alpha - \beta| = |\alpha| - |\beta|$ for $\beta \preceq \alpha$.

Before preceding to the proof of the main theorem, we have some properties of the Bergman projection which will be useful in the proof.

Lemma 1. *Let $f \in L^2$ be holomorphic and $u \in L^2$ be radial. Suppose*

$$f(z) = \sum_\beta f_\beta z^\beta$$

is the power series representation of f. Then the following statements hold for each multi-index α and point $z \in B$.

(a) $P(\overline{f}w^\alpha)(z) = \sum\limits_{\beta \preceq \alpha} \overline{f_\beta} \dfrac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!} z^{\alpha-\beta}.$

(b) $P(f\overline{w^\alpha})(z) = \sum\limits_{\alpha \preceq \beta} f_\beta \dfrac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!} z^{\beta-\alpha}.$

(c) $P(uw^\alpha)(z) = \dfrac{(n+|\alpha|)!}{n!\alpha!} z^\alpha \displaystyle\int_B u(w)|w^\alpha|^2 \,\mathrm{d}V(w)$.

(d) *In addition, if* $\alpha \neq 0$, *then* $P(u\overline{w^\alpha}) = 0$.

Proof. Write

$$K_z(w) = \sum_\gamma \frac{(n+|\gamma|)!}{n!\,\gamma!}\, w^\gamma\, \overline{z^\gamma}, \quad z, w \in B.$$

Since holomorphic monomials are orthogonal to each other in L^2, Lemma 1 gives

$$\begin{aligned} P(w^\alpha\overline{w^\beta})(z) &= \int_B w^\alpha\overline{w^\beta}\,\overline{K_z(w)}\,\mathrm{d}V(w) \\ &= \frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!} z^{\alpha-\beta}, \quad \beta \preceq \alpha, \end{aligned}$$

and $P(w^\alpha\overline{w^\beta}) = 0$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$. Then (a) follows from the power series expansion of f and term-by-term integration. Also, by the similar argument, we also have (b).

Since u is radial, an application of integration in polar coordinates gives

$$\int_B u(w) w^\alpha \overline{w^\gamma}\,\mathrm{d}V(w) = 0, \quad \alpha \neq \gamma.$$

It follows that

$$\begin{aligned} P(uw^\alpha)(z) &= \int_B u(w) w^\alpha \overline{K_z}(w)\,\mathrm{d}V(w) \\ &= \sum_\gamma \frac{(n+|\gamma|)!}{n!\gamma!} z^\gamma \int_B u(w) w^\alpha \overline{w^\gamma}\,\mathrm{d}V(w) \\ &= \frac{(n+|\alpha|)!}{n!\alpha!} z^\alpha \int_B u(w)|w^\alpha|^2\,\mathrm{d}V(w), \end{aligned}$$

so we have (c). The similar argument can be applied to prove (d). This completes the proof. $\square$

Each point evaluation is easily verified to be a bounded linear functional on b^2. Hence, for each $z \in B$, there exists a unique function $R_z \in b^2$ which has the following reproducing property

$$u(z) = \int_B u\overline{R_z}\,\mathrm{d}V$$

for every $u \in b^2$.

As is well known, a function v in B is pluriharmonic if and only if it admits a decomposition $v = f + \overline{g}$, where the functions f and g are holomorphic. Furthermore, if v is in L^2, then the holomorphic functions f and g are all in L^2. This immediately follows from the boundedness of the Bergman projection P. As a result of this observation we see that there is a simple relation between R_z and the Bergman kernel K_z:

$$R_z = K_z + \overline{K_z} - 1.$$

More specifically,

$$R_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{n+1}} + \frac{1}{(1 - \langle z, w\rangle)^{n+1}} - 1, \quad w \in B,$$

and the orthogonal projection $Q\colon L^2 \to b^2$ admits the integral representation

$$Q\varphi(z) = \int_B \Big(\frac{1}{(1 - \langle w, z\rangle)^{n+1}} + \frac{1}{(1 - \langle z, w\rangle)^{n+1}} - 1\Big)\varphi(w)\,\mathrm{d}V(w)$$

for functions $\varphi \in L^2$. This integral formula shows that the domain of Q can be naturally extended to $L^1(B, V)$. Since

$$P\varphi(0) = \int_B \varphi\,\mathrm{d}V,$$

the projection Q can be rewritten as

$$Q\varphi = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0) \tag{1}$$

for functions $\varphi \in L^2$.

Now, we are ready to prove the main theorem.

Theorem 3. *Let $u \in L^2$ be nonconstant radial and $v \in b^2$. Then T_u and T_v commute on b^2 if and only if v is constant on B.*

Proof. Assume $T_uT_v = T_vT_u$. Write $v = f + \overline{g}$ where the functions f, g are holomorphic in L^2. Suppose

$$f(z) = \sum_\beta f_\beta z^\beta, \quad g(z) = \sum_\beta g_\beta z^\beta$$

are their power series representations of f and g, respectively. Fix a multi-index α with $|\alpha| \geqslant 1$. Note that $\overline{u}$ is still radial. By (1) and Lemma 2, we have

$$\begin{aligned} T_u(w^\alpha)(z) &= Q(uw^\alpha)(z) \\ &= P(uw^\alpha)(z) + \overline{P(\overline{uw^\alpha})(z)} - P(uw^\alpha)(0) \\ &= \frac{(n+|\alpha|)!}{n!\alpha!} z^\alpha \int_B u(w)|w^\alpha|^2\,\mathrm{d}V(w), \quad z \in B. \end{aligned}$$

For any multi-index γ, letting

$$\tilde{u}(\gamma) = \frac{(n+|\gamma|)!}{n!\gamma!} \int_B u(w)|w^\gamma|^2 \, \mathrm{d}V(w)$$

for notational simplicity, we have

$$T_u(w^\alpha)(z) = \tilde{u}(\alpha) z^\alpha, \tag{2}$$

and hence

$$T_f T_u(w^\alpha)(z) = \widetilde{u}(\alpha) z^\alpha f(z), \quad z \in B. \tag{3}$$

On the other hand, by (1) and Lemma 2 again, we see

$$\begin{aligned} T_{\overline{g}}(w^\alpha)(z) &= Q(\overline{g} w^\alpha)(z) \\ &= P(\overline{g} w^\alpha)(z) + \overline{P(g\overline{w^\alpha})(z)} - P(\overline{g} w^\alpha)(0) \\ &= \sum_{\beta \preceq \alpha} \overline{g_\beta} \frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!} z^{\alpha-\beta} \\ &\quad + \sum_{\alpha \preceq \beta} \overline{g_\beta} \frac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!} \overline{z^{\beta-\alpha}} - \overline{g_\alpha} \frac{n!\alpha!}{(n+|\alpha|)!} \end{aligned} \tag{4}$$

and hence by (2)

$$\begin{aligned} T_{\overline{g}} T_u(w^\alpha)(z) &= \tilde{u}(\alpha) T_{\overline{g}}(w^\alpha)(z) \\ &= \tilde{u}(\alpha) \Bigg(\sum_{\beta \preceq \alpha} \overline{g_\beta} \frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!} z^{\alpha-\beta} \\ &\quad + \sum_{\alpha \preceq \beta} \overline{g_\beta} \frac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!} \overline{z^{\beta-\alpha}} - \overline{g_\alpha} \frac{n!\alpha!}{(n+|\alpha|)!} \Bigg) \end{aligned}$$

for every $z \in B$. It follows from (3) that

$$\begin{aligned} T_{f+\overline{g}} T_u(w^\alpha)(z) &= T_f T_u(w^\alpha)(z) + T_{\overline{g}} T_u(w^\alpha)(z) \\ &= \tilde{u}(\alpha) \Bigg(\sum_{\beta} f_\beta z^{\alpha+\beta} + \sum_{\beta \preceq \alpha} \overline{g_\beta} \frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!} z^{\alpha-\beta} \\ &\quad + \sum_{\alpha \preceq \beta} \overline{g_\beta} \frac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!} \overline{z^{\beta-\alpha}} - \overline{g_\alpha} \frac{n!\alpha!}{(n+|\alpha|)!} \Bigg) \end{aligned} \tag{5}$$

for every $z \in B$.

On the other hand, by (2)

$$T_uT_f(w^\alpha)(z) = T_u(fw^\alpha)(z) = \sum_\beta f_\beta \tilde{u}(\alpha+\beta) z^{\alpha+\beta}, \quad z \in B.$$

Also, using (4), (2) and Lemma 2, we can see

$$\begin{aligned} T_uT_{\overline{g}}(w^\alpha)(z) = & \sum_{\beta\preceq\alpha} \overline{g}_\beta \frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!}\tilde{u}(\alpha-\beta)z^{\alpha-\beta} \\ & + \sum_{\alpha\preceq\beta} \overline{g}_\beta \frac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!}\tilde{u}(\beta-\alpha)\overline{z^{\beta-\alpha}} \\ & - \overline{g_\alpha}\frac{n!\alpha!}{(n+|\alpha|)!}\int_B u\,\mathrm{d}V, \quad z \in B. \end{aligned}$$

It follows that

$$\begin{aligned} (6) \quad & T_uT_{f+\overline{g}}(w^\alpha)(z) \\ & = T_uT_f(w^\alpha)(z) + T_uT_{\overline{g}}(w^\alpha)(z) \\ & = \sum_\beta f_\beta\tilde{u}(\alpha+\beta)z^{\alpha+\beta} + \sum_{\beta\preceq\alpha}\overline{g_\beta}\frac{(n+|\alpha|-|\beta|)!\alpha!}{(\alpha-\beta)!(n+|\alpha|)!}\tilde{u}(\alpha-\beta)z^{\alpha-\beta} \\ & \quad + \sum_{\alpha\preceq\beta}\overline{g_\beta}\frac{(n+|\beta|-|\alpha|)!\beta!}{(\beta-\alpha)!(n+|\beta|)!}\tilde{u}(\beta-\alpha)\overline{z^{\beta-\alpha}} - \overline{g_\alpha}\frac{n!\alpha!}{(n+|\alpha|)!}\int_B u\,\mathrm{d}V \end{aligned}$$

for $z \in B$. Since $T_{f+\overline{g}}T_u = T_uT_{f+\overline{g}}$ by the assumption, we have, in particular,

$$T_{f+\overline{g}}T_u(w^\alpha) = T_uT_{f+\overline{g}}(w^\alpha)$$

for every multi-index α. Hence, by (5) and (6), we have in particular

$$f_\beta\tilde{u}(\alpha) = f_\beta\tilde{u}(\alpha+\beta) \tag{7}$$

for every multi-index β.

For any multi-index γ, we recall

$$\tilde{u}(\gamma) = \frac{(n+|\gamma|)!}{n!\gamma!}\int_B u(w)|w^\gamma|^2\,\mathrm{d}V(w).$$

Since u is radial, abusing the notation $u(|z|) = u(z)$ and using the integration in polar coordinates together with Proposition 1.4.9 of [9], one can see

$$\tilde{u}(\gamma) = (2n+2|\gamma|)\int_0^1 u(r)r^{2|\gamma|+2n-1}\,\mathrm{d}r.$$

It follows from (7) that

$$f_\beta(2n+2|\alpha|)\int_0^1 u(r)r^{2|\alpha|+2n-1}\,\mathrm{d}r = f_\beta(2n+2|\alpha|+2|\beta|)\int_0^1 u(r)r^{2|\alpha|+2|\beta|+2n-1}\,\mathrm{d}r \tag{8}$$

for every multi-index β.

Now, assume v is not constant. Then, we have either f or g is not constant. First, assume f is not constant and further $f_{\beta_0}\neq 0$ for some β_0 with $|\beta_0|\geqslant 1$. Then, (8) yields

$$(2n+2|\alpha|)\int_0^1 u(r)r^{2|\alpha|+2n-1}\,\mathrm{d}r = (2n+2|\alpha|+2|\beta_0|)\int_0^1 u(r)r^{2|\alpha|+2|\beta_0|+2n-1}\,\mathrm{d}r$$

for every multi-index α with $|\alpha|\geqslant 1$. In other words, there exists a positive integer m $(=|\beta_0|)$ such that

$$(2n+2k)\int_0^1 u(r)r^{2k+2n-1}\,\mathrm{d}r = (2n+2k+2m)\int_0^1 u(r)r^{2k+2m+2n-1}\,\mathrm{d}r \tag{9}$$

for every $k=1,2,\ldots$.

Let $H=\{\xi\in\mathbb{C}\colon\ \operatorname{Re}\xi>1\}$. Consider the Mellin transform $\mathscr{M}u$ of u defined by

$$\mathscr{M}u(\xi)=\int_0^1 u(r)r^{\xi-1}\,\mathrm{d}r,\quad \xi\in H.$$

It is known that $\mathscr{M}u$ is analytic on H. Moreover, by the Cauchy-Schwarz inequality, we see

$$|\mathscr{M}u(\xi)|^2\leqslant\left(\int_0^1|u|^2\,\mathrm{d}r\right)\left(\int_0^1 r^{2(\operatorname{Re}\xi-1)}\,\mathrm{d}r\right)\leqslant\left(\int_0^1|u|^2\,\mathrm{d}r\right)$$

for every $\xi\in H$. It follows that $\mathscr{M}u$ is bounded on H.

Define a function U on H by

$$U(\xi)=\mathscr{M}u(\xi+2m)-\frac{\xi}{\xi+2m}\mathscr{M}u(\xi).$$

The observations above show U is also analytic and bounded on H. Moreover, by (9),

$$U(2n+2k)=0,\quad k=1,2,\ldots.$$

Hence, U must be constant, with value 0 (see the proof of Theorem 2 of [4]), so

$$\xi\mathscr{M}u(\xi)=(\xi+2m)\mathscr{M}u(\xi+2m),\quad \xi\in H.$$

Hence, the analytic function $\xi \mapsto \xi\mathscr{M}u(\xi)$ is periodic with period $2m$ on H. Thus, the function $\xi\mathscr{M}u$ can be extended to whole plane $\mathbb{C}$, so we can think of the function $\xi \mapsto \xi\mathscr{M}u(\xi)$ as an entire function. Note that

$$\xi\mathscr{M}u(\xi) = O(|\xi|).$$

Hence $\xi\mathscr{M}u(\xi) = \lambda\xi + \delta$ for some constants λ, δ. On the other hand, since $\xi\mathscr{M}u$ is periodic, we must have $\lambda = 0$. Hence $\xi\mathscr{M}u = \delta$ and then clearly u is constant, which is a contradiction.

Next, assume g is not constant. Taking the adjoint to $T_uT_{f+\overline{g}} = T_{f+\overline{g}}T_u$, we see

$$T_{g+\overline{f}}T_{\overline{u}} = T_{\overline{u}}T_{g+\overline{f}}.$$

Note that $\overline{u}$ is a still radial function. According to the case proved above, u must be constant, which is also a contradiction.

Therefore v is constant, as desired.

The converse implication is clear. The proof is complete. $\square$

In view of main theorem proved in this paper, one may naturally ask a question. What is the situation without the radial condition on $u \in L^2$ in Theorem 3? Specially, for nonconstant $f, g \in b^2$, we don't know whether $T_fT_g = T_gT_f$ implies $f = \alpha g + \beta$ for some constants α, β.

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