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ON THE HENSTOCK-KURZWEIL INTEGRAL FOR RIESZ-SPACE-VALUED FUNCTIONS DEFINED ON UNBOUNDED INTERVALS

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Abstract. In this paper we introduce and investigate a Henstock-Kurzweil-type integral for Riesz-space-valued functions defined on (not necessarily bounded) subintervals of the extended real line. We prove some basic properties, among them the fact that our integral contains under suitable hypothesis the generalized Riemann integral and that every simple function which vanishes outside of a set of finite Lebesgue measure is integrable according to our definition, and in this case our integral coincides with the usual one.

Keywords: Riesz spaces, Henstock-Kurzweil integral

MSC 2000: 28B15, 28B05, 28B10, 46G10

1. Introduction

The Henstock-Kurzweil integral for Riesz-space-valued functions defined on bounded subintervals of the real line and with respect to operator-valued measures was investigated in [6], [7], [9], [10], [11] with respect to (D)-convergence (that is, a kind of convergence in which the ε-technique is replaced by a technique involving double sequences, see also [3], [8]) and in [1] with respect to the order convergence. In [12] this integral was studied for functions defined on a Hausdorff, compact topological space.

In this paper we introduce a Henstock-Kurzweil-type integral for Riesz-space-valued maps, defined in (not necessarily bounded) subintervals of the extended real line, and we prove some fundamental properties. Moreover, we demonstrate that our integral contains under suitable hypothesis the improper Riemann integral and that every simple function, vanishing outside of a set of finite Lebesgue measure, is Henstock-Kurzweil integrable, and its integral coincides with the usual one.

2. Preliminaries

Let $\mathbb{N}$ be the set of all strictly positive integers, $\mathbb{R}$ the set of the real numbers, $\mathbb{R}^+$ be the set of all strictly positive real numbers, $\widetilde{\mathbb{R}}$ the set of all extended real numbers. We begin with some preliminary definitions and results.

Definition 2.1. A Riesz space R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R.

Definition 2.2. Given a sequence (r_n) in R, we say that (r_n) (D)-*converges* to an element $r \in R$ if there exists a bounded double sequence $(a_{i,j})_{i,j}$ in R, such that, for each $i \in \mathbb{N}$, $a_{i,j} \downarrow 0$, that is $a_{i,j} \geqslant a_{i,j+1}$ $\forall\, j \in \mathbb{N}$ and $\bigwedge_{j\in\mathbb{N}} a_{i,j} = 0$ (such a sequence will be called a *regulator* or (D)-*sequence* from now on), and satisfying the following condition:

$$\forall \text{ mapping } \varphi\colon\ \mathbb{N} \to \mathbb{N}, \text{ there exists an integer } n_0 \text{ such that}$$
$$|r_n - r| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \geqslant n_0$. In this case, we write $(D) \lim_n r_n = r$.

Analogously, given $l \in R$, a function $f\colon\ A \to R$, where $\emptyset \neq A \subset \widetilde{\mathbb{R}}$, and a limit point x_0 for A, we will say that $(D) \lim_{x\to x_0} f(x) = l$ if there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a neighborhood $\mathscr{U}$ of x_0 such that for all $x \in \mathscr{U} \cap A \setminus \{x_0\}$

$$|f(x) - l| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Definition 2.3. We say that R is *weakly σ-distributive* if for every (D)-sequence $(a_{i,j})$ one has:

$$\bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0. \tag{1}$$

It is easy to check that the usual order convergence implies (D)-convergence, while the converse is true in weakly σ-distributive spaces (see also [2]).

Throughout the paper, we shall always assume that R is a Dedekind complete weakly σ-distributive Riesz space.

The following lemma will be useful in the sequel (see [4], [8]).

Lemma 2.4. *Let* $\{(a^n_{i,j})_{i,j}\colon\ n \in \mathbb{N}\}$ *be any countable family of regulators. Then for each fixed element* $u \in R$, $u \geqslant 0$, *there exists a regulator* $(a_{i,j})_{i,j}$ *such that for every* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *one has*

$$u \wedge \sum_{n=1}^{\infty}\left(\bigvee_{i=1}^{\infty} a^n_{i,\varphi(i+n)}\right) \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi,(i)}.$$

3. The Henstock-Kurzweil integral

The aim of this section is to construct a type of integral for Riesz-space-valued maps (with respect to the Lebesgue measure defined on intervals, not necessarily bounded), containing the improper Riemann integral. From now on, we denote by $[A,B]$ a closed interval or halfline contained in $\widetilde{\mathbb{R}}$, or the whole of $\widetilde{\mathbb{R}}$, and by Δ the set of all positive real-valued functions, defined on $[A,B]$. Moreover, given a measurable set $E \subset \widetilde{\mathbb{R}}$, we denote by $|E|$ its Lebesgue measure (this quantity can be finite or $+\infty$). Throughout this paper, our integral deals with Riesz-space-valued functions defined on $[A,B]$, but it can be investigated analogously if we take functions defined on $\mathbb{R}$ or on halflines of the type $[a,+\infty)$ or $(-\infty,a]$, with $a \in \mathbb{R}$.

Definitions 3.1. A *subpartition* Π of $[A,B]$ is a set of pairs (I_k,ξ_k), $k=1,\dots,p$, such that $\xi_k \in I_k\ \forall\, k$, and the I_k's are non-overlapping closed intervals, contained in $[A,B]$. A *partition* $\Pi = \{(I_k,\xi_k),\ k=1,\dots,p\}$ of $[A,B]$ is a subpartition of $[A,B]$ with $\bigcup\limits_{k=1}^{p} I_k = [A,B]$.

A *gauge* is a map γ defined in $[A,B]$ and taking values in the set of all open intervals in $\widetilde{\mathbb{R}}$, such that $\xi \in \gamma(\xi)$ for every $\xi \in [A,B]$ and $\gamma(\xi)$ is a bounded open interval for every $\xi \in \mathbb{R} \cap [A,B]$. Given a gauge γ, we will say that a partition $\Pi = \{(I_k,\xi_k),\ k=1,\dots,p\}$ of $[A,B]$ is γ*-fine* if $I_k \subset \gamma(\xi_k)\ \forall\, k=1,\dots,p$. Given a bounded interval $[a,b] \subset \mathbb{R}$ and a map $\delta\colon\ [a,b] \to \mathbb{R}^+$, a partition $\Pi = \{(I_k,\xi_k),\ k=1,\dots,p\}$ of $[a,b]$ is said to be δ*-fine* if $I_k \subset (\xi_k-\delta(\xi_k),\xi_k+\delta(\xi_k))\ \forall\, k=1,\dots,p$.

We note that, if I_k is an unbounded interval, then the element ξ_k associated with I_k is necessarily $+\infty$ or $-\infty$: otherwise $\gamma(\xi_k)$ should be a bounded interval and contain an unbounded interval: a contradiction.

Given any partition $\Pi = \{(I_k,\xi_k),\ k=1,\dots,p\}$ of $[A,B]$ and a function $f\colon [A,B] \to R$, we define the *Riemann sum* of f (written $\sum\limits_{\Pi} f$) to be the quantity

$$\sum_{k=1}^{p} f(\xi_k)|I_k|, \tag{2}$$

where in the sum in (2) only the terms for which I_k is a bounded interval are included. This can be secured by simply postulating it or by defining the measure of an unbounded interval as $+\infty$, by requiring $f(+\infty) = f(-\infty) = 0$ and by means of the convention $0 \cdot (+\infty) = 0$ (see also [5], p. 65).

We now formulate our definition of Henstock-Kurzweil integral for functions defined on $[A, B]$ and taking values in a Dedekind complete weakly σ-distributive Riesz space.

Definition 3.2. We say that a function $f\colon [A, B] \to R$ is *Henstock-Kurzweil integrable* (in short HK-*integrable*) on $[A, B]$ if there exist an element $I \in R$ and a (D)-sequence $(a_{i,j})_{i,j}$ in R such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a function $\delta \in \Delta$ and a positive real number P such that

$$\left|\sum_{\Pi} f - I\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{3}$$

whenever $\Pi = \{(I_k, \xi_k),\ k = 1, \ldots, p\}$ is a δ-fine partition of any bounded interval $[a, b]$ with $[a, b] \supset [A, B] \cap [-P, P]$ and $[a, b] \subset [A, B]$. In this case we say that I is the HK-*integral of* f, and we denote the element I by the symbol $\int_A^B f$. Later we will prove that our integral is well-defined, that is such an I is uniquely determined.

We now prove the following characterization of HK-integrability.

Theorem 3.3. *A function $f\colon [A, B] \to R$ is* HK-*integrable if and only if there exist $J \in R$ and a (D)-sequence $(a_{i,j})_{i,j}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge γ such that*

$$\left|\sum_{\Pi} f - J\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{4}$$

whenever $\Pi = \{(I_k, \xi_k),\ k = 1, \ldots, p\}$ is a γ-fine partition of $[A, B]$, and in this case we have $\int_A^B f = J$.

Proof. We begin with the "only if" part. By hypothesis, we know that there exists a regulator $(a_{i,j})_{i,j}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ there exist a function $\delta \in \Delta$ and a positive real number P such that (3) holds. We now define on $[A, B]$ a gauge γ in the following way:

$$\gamma(\xi) = \begin{cases} (\xi - \delta(\xi), \xi + \delta(\xi)) & \text{if } \xi \in [A, B] \cap \mathbb{R}, \\ [-\infty, -P) & \text{if } \xi = -\infty \text{ and } A = -\infty, \\ (P, +\infty] & \text{if } \xi = +\infty \text{ and } B = +\infty. \end{cases}$$

We observe that every γ-fine partition $\Pi = \{(I_k, \xi_k),\ k = 1, \dots, p\}$ of $[A, B]$ is such that $I_k \subset \gamma(\xi_k)\ \forall\, k = 1, \dots, p$. In the case of $A = -\infty$, $B = +\infty$, the partition Π contains two unbounded intervals, which we call J and K: of course, if $\inf J = -\infty$ and $\sup K = +\infty$, then the ξ_k's associated with J and K are $-\infty$ and $+\infty$ respectively. Then, since Π is γ-fine, we have $J \subset \gamma(-\infty)$ and $K \subset \gamma(+\infty)$. Then $J \subset [-\infty, -P)$ and $K \subset (P, +\infty]$. So, if $a = \sup J$ and $b = \inf K$, then $[a, b]$ is a bounded interval, containing $[-P, P]$. If Π' is the restriction of Π to $[a, b]$, then Π' is δ-fine, and by construction we get

$$\sum_{\Pi'} f = \sum_{\Pi} f. \tag{5}$$

In this case, the assertion follows from (3) and (5).

In the case of $A \in \mathbb{R}$, $B = +\infty$, the partition Π contains only an unbounded interval K, with $\sup K = +\infty$. Let P be associated with K as above, and $b = \inf K$: we have $P \leqslant b$. We note that, without loss of generality, P can be taken greater than $|A|$. Thus, $[A, b]$ is a bounded interval, containing $[-P, P]$, and the assertion follows by proceeding as in the previous case. The case of $A = -\infty$, $B \in R$ is analogous to the previous one. Finally, if $[A, B]$ is bounded, then the assertion is straightforward, because in this case the number P can be taken greater than $\max(|A|, |B|)$ and, of course, (3) holds even in the case $[a, b] = [A, B]$. This concludes the proof of the "only if" part.

We now turn to the "if" part. By hypothesis, we know that there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, there exists a gauge γ satisfying (4). By the definition of gauge, there exist $\delta_1, \delta_2 \in \Delta$ such that

$$\gamma(\xi) = (\xi - \delta_1(\xi), \xi + \delta_2(\xi)) \quad \forall\, \xi \in [A, B] \cap \mathbb{R}.$$

For such ξ's, let $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Moreover, if $+\infty$ and $-\infty$ belong to $[A, B]$, and $\gamma(-\infty) = [-\infty, P_1^*)$, $\gamma(+\infty) = (P_2^*, +\infty]$, put $P_1 = \min\{P_1^*, -1\}$, $P_2 = \max\{P_2^*, 1\}$, $P = \max\{-P_1, P_2\}$: we note that, in case $A \in \mathbb{R}$ (resp. $B \in \mathbb{R}$), P can be chosen greater than $|A|$ (resp. $|B|$); moreover, set $\delta(-\infty) = \delta(+\infty) = P$. Let now $[a, b] \subset [A, B]$ be any bounded interval, containing $[A, B] \cap [-P, P]$, and $\Pi = \{(I_k, \xi_k) \colon k = 1, \dots, p\}$ be a δ-fine partition of $[a, b]$. Let Π' be that partition of $[A, B]$, whose elements are the ones of Π with the addition of $([A, a], A)$, if $A = -\infty$, and $([b, B], B)$, if $B = +\infty$: we note that Π' is γ-fine. This follows from the fact that, if (I_k, ξ_k) is any element of Π, then

$$I_k \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \subset (\xi_k - \delta_1(\xi_k), \xi_k + \delta_2(\xi_k)) = \gamma(\xi_k),$$

and from the following inclusions:

$$
\begin{aligned}
(b, +\infty] \subset (P, +\infty] \subset (P_2, +\infty] \subset (P_2^*, +\infty] = \gamma(+\infty), \\
[-\infty, a) \subset [-\infty, P) \subset [-\infty, P_1) \subset [-\infty, P_1^*) = \gamma(-\infty).
\end{aligned}
$$

Then, taking into account that the Riemann sum corresponding to the partition Π' is done without considering the unbounded intervals, we get $\sum\limits_{\Pi'} f = \sum\limits_{\Pi} f$. From this and (4) the assertion follows, by proceeding analogously as at the end of the proof of the converse implication. This concludes the proof of the theorem. □

Remark 3.4. We note that the Henstock-Kurzweil integral is well-defined, that is there exists at most one element I satisfying condition (4): indeed, if $\exists$ two such elements I, J, then $\exists$ two (D)-sequences $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ $\exists$ two gauges γ_1, γ_2 such that, for each γ_1-fine partition Π and for every γ_2-fine partition Π' of $[A, B]$ we have

$$
\left| \sum_{\Pi} f - I \right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}
$$

and

$$
\left| \sum_{\Pi'} f - J \right| \leqslant \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}
$$

respectively. Let now $\gamma(\xi) = \gamma_1(\xi) \cap \gamma_2(\xi)$, $\forall\, \xi \in [A, B]$ and take any γ-fine partition Π'': then Π'' is both γ_1- and γ_2-fine, and thus we have

$$
0 \leqslant |I - J| \leqslant \bigvee_{i=1}^{\infty} c_{i,\varphi(i)},
$$

where $c_{i,j} = 2(a_{i,j} + b_{i,j})\ \forall\, i, j \in \mathbb{N}$. By the arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$, we get

$$
0 \leqslant |I - J| \leqslant \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \right) = 0,
$$

since $(c_{i,j})_{i,j}$ is a (D)-sequence and thanks to the weak σ-distributivity of R. Thus $I = J$, and so our HK-integral is well-defined. □

We now state the main properties of the HK-integral.

Proposition 3.5. *If f_1, f_2 are* HK-*integrable on $[A,B]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is* HK-*integrable on $[A,B]$ and*

$$\int_A^B (c_1 f_1 + c_2 f_2) = c_1 \int_A^B f_1 + c_2 \int_A^B f_2.$$

Proof. The proof is similar to the one of [5], Theorems 2.5.1 and 2.5.3. □

Proposition 3.6. *If f and g are* HK-*integrable on $[A,B]$ and $f \leqslant g$, then*

$$\int_A^B f \leqslant \int_A^B g.$$

Proof. By hypothesis, there exist two (D)-sequences $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ such that, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$, there exist two gauges γ_1, γ_2 such that, whenever Π is a γ_1-fine partition of $[A,B]$ and Π' is a γ_2-fine partition of $[A,B]$, we have

$$\int_A^B f - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leqslant \sum_{\Pi} f \leqslant \int_A^B f + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

and

$$\int_A^B g - \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leqslant \sum_{\Pi'} g \leqslant \int_A^B g + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

respectively. For every $\xi \in [A,B]$, let $\gamma(\xi) = \gamma_1(\xi) \cap \gamma_2(\xi)$, and take any γ-fine partition Π'' of $[A,B]$: then Π'' is both γ_1- and γ_2-fine. Thus we get

$$\int_A^B f - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leqslant \sum_{\Pi''} f \leqslant \sum_{\Pi''} g \leqslant \int_A^B g + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$$

and hence, $\forall \varphi \in \mathbb{N}^{\mathbb{N}}$,

$$\int_A^B f - \int_A^B g \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leqslant \bigvee_{i=1}^{\infty} c_{i,\varphi(i)},$$

where $c_{i,j} = 2(a_{i,j} + b_{i,j})\ \forall i,j \in \mathbb{N}$. By the arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$, since $(c_{i,j})_{i,j}$ is a (D)-sequence and taking into account the weak σ-distributivity of R, we get

$$\int_A^B f - \int_A^B g \leqslant \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \right) = 0,$$

that is $\int_A^B f \leqslant \int_A^B g$. This concludes the proof. □

Corollary 3.7. *If both f and $|f|$ are* HK*-integrable in $[A,B]$, then*

$$\left|\int_A^B f\right| \leqslant \int_A^B |f|.$$

Proposition 3.8. *Let $A,B \in \widetilde{\mathbb{R}}$, and c be such that $A < c < B$. If $f\colon [A,B] \to R$ is* HK*-integrable both on $[A,c]$ and on $[c,B]$, then f is* HK*-integrable on $[A,B]$ and*

$$\int_A^B f = \int_A^c f + \int_c^B f.$$

Proof. In view of the HK-integrability of f on $[A,c]$ and $[c,B]$ there exist two (D)-sequences $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exist two mappings $\underline{\delta}\colon [A,c] \to \mathbb{R}^+$, $\overline{\delta}\colon [c,B] \to \mathbb{R}^+$, and two positive real numbers $\underline{P}$ and $\overline{P}$ (without loss of generality, $\underline{P} > |c|$, $\overline{P} > |c|$) such that, if $\underline{\Pi}$ is any $\underline{\delta}$-fine partition of any bounded interval $[a_1,b_1] \subset [A,c]$, $[a_1,b_1] \supset [A,c] \cap [-\underline{P},\underline{P}]$ and $\overline{\Pi}$ is any $\overline{\delta}$-fine partition of any bounded interval $[a_2,b_2] \subset [c,B]$, $[a_2,b_2] \supset [c,B] \cap [-\overline{P},\overline{P}]$, then

$$\left|\sum_{\underline{\Pi}} f - \int_{a_1}^{b_1} f\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

and

$$\left|\sum_{\overline{\Pi}} f - \int_{a_2}^{b_2} f\right| \leqslant \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.$$

If $A = -\infty$, let $\delta(-\infty) = \underline{\delta}(-\infty)$; if $B = +\infty$, let $\delta(+\infty) = \overline{\delta}(+\infty)$. Moreover, set

$$\delta(x) = \begin{cases} \min\{\underline{\delta}(x), \frac{1}{2}(c-x)\}, & \text{if } x \in [A,c) \cap \mathbb{R}, \\ \min\{\overline{\delta}(x), \frac{1}{2}(x-c)\} & \text{if } x \in (c,B] \cap \mathbb{R}, \\ \min\{\underline{\delta}(c), \overline{\delta}(c)\} & \text{if } x = c, \end{cases}$$

and $P = \max\{\underline{P}, \overline{P}\}$. Take now an arbitrary bounded interval $[a,b] \subset [A,B]$, $[a,b] \supset [A,B] \cap [-P,P]$, and any δ-fine partition $\Pi = \{([u_k,v_k],\xi_k),\ k = 1,\dots,p\}$ of $[a,b]$. Then necessarily $c \in (a,b)$. We now claim that there exists $k \in \{1,\dots,p\}$ such that $c = \xi_k$, or $c = u_k$, or $c = v_k$. Otherwise there would be an interval $[u_j,v_j]$ such that $u_j < c < v_j$ and either $c < \xi_j < v_j$ or $u_j < \xi_j < c$. Since Π is δ-fine, we would get $[u_j,v_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ and thus $v_j - u_j < 2\delta(\xi_j)$. So $v_j - u_j < \xi_j - c$ if

$\xi_j > c$ or $v_j - u_j < c - \xi_j$ if $\xi_j < c$. This would imply that ξ_j is outside (u_j, v_j), a contradiction. Thus we have:

$$(6) \qquad \sum_{\Pi} f = \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(v_j - u_j) + \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l)$$
$$= \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(\xi_j - u_j) + f(\xi_j)(v_j - \xi_j)$$
$$+ \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l).$$

The quantity $S_a^c = \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(\xi_j - u_j)$ is a Riemann sum for a suitable $\underline{\delta}$-fine partition of $[a, c]$, which is a bounded interval contained in $[A, c]$ and containing $[A, c] \cap [-\underline{P}, \underline{P}]$, by construction.

Analogously, the quantity $S_c^b = f(\xi_j)(v_j - \xi_j) + \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l)$ is a Riemann sum for a suitable $\overline{\delta}$-fine partition of $[c, b]$, which is a bounded interval contained in $[c, B]$ and containing $[c, B] \cap [-\overline{P}, \overline{P}]$. Thus we have:

$$\left| S_a^c - \int_A^c f \right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, \qquad \left| S_c^b - \int_c^B f \right| \leqslant \bigvee_{i=1}^{\infty} b_{i,\varphi(i)},$$

and hence

$$\left| \sum_{\Pi} f - \int_A^c f - \int_c^B f \right| \leqslant \bigvee_{i=1}^{\infty} c_{i,\varphi(i)},$$

where $c_{i,j} = 2(a_{i,j} + b_{i,j})$, $\forall\, i, j \in \mathbb{N}$. Since the double sequence $(c_{i,j})_{i,j}$ is a (D)-sequence, the assertion follows. □

We now state two versions of the Cauchy criterion.

Theorem 3.9. *A map $f\colon [A, B] \to R$ is* HK*-integrable if and only if there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists$ a gauge γ such that for every γ-fine partition Π_1, Π_2 of $[A, B]$ we have*

$$\left| \sum_{\Pi_1} f - \sum_{\Pi_2} f \right| \leqslant \bigvee_{n=1}^{\infty} a_{i,\varphi(i)}.$$

Theorem 3.10. *A map $f\colon [A,B] \to R$ is* HK*-integrable if and only if there exists a (D)-sequence $(a_{i,j})_{i,j}$ in R such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists$ a map $\delta \in \Delta$ and a positive real number P such that*

$$\Big|\sum_{\Pi_1} f - \sum_{\Pi_2} f\Big| \leqslant \bigvee_{n=1}^{\infty} a_{i,\varphi(i)}$$

whenever Π_1, Π_2 are δ-fine partitions of any bounded interval $[a,b]$, with $[a,b] \subset [A,B]$ and $[a,b] \supset [A,B] \cap [-P,P]$.

Proof. The proof is similar to the one of Theorem 5.2.9, p. 77, of [8]. □

We now prove a result about HK-integrability on subintervals.

Theorem 3.11. *Let $f\colon [A,B] \to R$ be* HK*-integrable, and $A < c < B$. Then $f_{|[A,c]}$ and $f_{|[c,B]}$ are* HK*-integrable too, and*

$$\int_A^B f = \int_A^c f + \int_c^B f. \tag{7}$$

Proof. By virtue of Theorem 3.9, there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ $\exists$ a gauge γ on $[A,B]$ such that for all γ-fine partitions Π_1 and Π_2 of $[A,B]$ we have

$$\Big|\sum_{\Pi_1} f - \sum_{\Pi_2} f\Big| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{8}$$

Set $\gamma_0 = \gamma_{|[A,c]}$ and let Π, Π' be any two γ_0-fine partitions of $[A,c]$. By virtue of the Cousin Lemma there exists a γ-fine partition Π_0 of $[c,B]$. Put $\Pi_1 = \Pi \cup \Pi_0$, $\Pi_2 = \Pi' \cup \Pi_0$. Then Π_1 and Π_2 are γ-fine partitions of $[A,B]$. Moreover, we get

$$\sum_{\Pi_1} f = \sum_{\Pi} f + \sum_{\Pi_0} f, \qquad \sum_{\Pi_2} f = \sum_{\Pi'} f + \sum_{\Pi_0} f. \tag{9}$$

From (8) and (9) we have

$$\Big|\sum_{\Pi} f - \sum_{\Pi'} f\Big| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{10}$$

From (10) and Theorem 3.9 it follows that $f_{|[A,c]}$ is HK-integrable. The proof of HK-integrability of $f_{|[c,B]}$ is analogous. The equality (7) follows from this and Proposition 3.8. □

We now prove the following:

Theorem 3.12. *Let $f\colon [A,B] \to R$ be an* HK*-integrable function. Let $A < c < B$. Then the function $g = f\chi_{[A,c]}$ is* HK*-integrable on $[A,B]$, and $\int_A^c f = \int_A^B g$.*

P r o o f. First of all, we note that $c \in \mathbb{R}$, and g is HK-integrable on $[A,c]$, because g coincides with f in $[A,c]$ and, by virtue of Theorem 3.11, f is HK-integrable on $[A,c]$. Moreover, it is easy to see that g is HK-integrable on $[c,B]$ and $\int_c^B g = 0$. So, by virtue of Proposition 3.8, we get that g is HK-integrable on $[A,B]$ and

$$\int_A^B g = \int_A^c g + \int_c^B g = \int_A^c f. \tag{11}$$

This concludes the proof. □

Remark 3.13. In an analogous way it is possible to prove that $h = f\chi_{[c,B]}$ is HK-integrable on $[A,B]$ and $\int_c^B f = \int_A^B h$.

Corollary 3.14. *Let $f\colon [A,B] \to R$ be* HK*-integrable on $[A,B]$, and let $A < c < c' < B$. Then the map $l = f\chi_{[c,c']}$ is* HK*-integrable on $[A,B]$, and $\int_c^{c'} f = \int_A^B l$.*

P r o o f. First of all, we note that $c, c' \in \mathbb{R}$. Let $k = f_{|[A,c']}$: by virtue of Theorem 3.11, k is HK-integrable on $[A,c']$, and by Theorem 3.12, where the rôle of A, B, c, is played by A, c', c, respectively, the function

$$l' = k\chi_{[c,c']} = f_{|[A,c']}\chi_{[c,c']}$$

is HK-integrable on $[A,c']$, and $\int_c^{c'} f = \int_c^{c'} k = \int_A^{c'} l'$. Moreover, since l coincides with l' on $[A,c']$ and vanishes on $(c',B]$, then, thanks to Proposition 3.8, we get that l is HK-integrable on $[A,B]$ and $\int_A^B l = \int_A^{c'} l'$. From this the assertion follows. □

Now, given an interval $[a,b] \subset \mathbb{R}$, a partition $\Pi = \{([x_{k-1},x_k],\xi_k),\ k = 1,2,\dots p\}$ and a point $c \in (a,b)$, if c coincides with some x_k, let $\Pi_1(\Pi_2)$ be the partition of all elements of Π which are contained in $[a,c]$ ($[c,b]$) respectively, and put

$$\sum_{\Pi}{}_a^c f = \sum_{\Pi_1} f, \qquad \sum_{\Pi}{}_c^b f = \sum_{\Pi_2} f.$$

If $c \in (x_{k-1},x_k)$ for some $k = 1,\dots,p$, then put

$$\sum_{\Pi}{}_a^c f = \sum_{l=1}^{k-1} f(\xi_l)(x_l - x_{l-1}) + f(c)(c - x_{k-1});$$

$$\sum_{\Pi}{}_c^b f = f(c)(x_k - c) + \sum_{l=k+1}^{p} f(\xi_l)(x_l - x_{l-1}).$$

In the sequel, when we will deal with the interval $[a,b]$ or $[A,B]$, sometimes we will write $\sum_{\Pi}{}_a^b f$, or $\sum_{\Pi}{}_A^B f$, respectively, instead of $\sum_{\Pi} f$, in order to avoid confusion. We now prove the following theorem (for the proof in the case $R = \mathbb{R}$, see [5], Lemma 2.8.1, pp. 56–57):

Theorem 3.15. *Let $[a,b] \subset \mathbb{R}$ be a bounded interval, $f\colon [a,b] \to R$ be a* HK-*integrable function, and suppose that there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $\delta\colon [a,b] \to \mathbb{R}^+$ such that, for every δ-fine partition Π' of $[a,b]$,*

$$\left|\sum_{\Pi'}{}_a^b f - \int_a^b f\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{12}$$

Then δ is such that, $\forall\, c \in (a,b)$ and for every δ-fine partition Π of $[a,b]$,

$$\left|\sum_{\Pi}{}_a^c f - \int_a^c f\right| \leqslant 2\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}, \qquad \left|\sum_{\Pi}{}_c^b f - \int_c^b f\right| \leqslant 2\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{13}$$

P r o o f. Let Π be a δ-fine partition of $[a,b]$. By virtue of Theorem 3.11, f is HK-integrable in $[a,c]$ with respect to the same regulator $(a_{i,j})_{i,j}$ as in the hypotheses of the theorem. Fix arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then there exists a function $\delta_c\colon [a,c] \to \mathbb{R}^+$ such that for every δ_c-fine partition Π'_c of $[a,c]$ we have:

$$\left|\sum_{\Pi'_c}{}_a^c f - \int_a^c f\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{14}$$

Let now Π_c be a δ- and δ_c-fine partition of $[a,c]$. Moreover, let Π_0 be the partition of $[c,b]$ consisting of those elements $([x_{l-1},x_l],\xi_l)$ of Π such that the intervals $[x_{l-1},x_l]$ are contained in $[c,b]$ and, as the case may be, of (J,c), where J is the intersection of $[c,b]$ with that interval $[x_{k-1},x_k]$ for which $x_{k-1} < c < x_k$ (if there is one). Let Π' be the partition consisting of the "union" of Π_c and Π_0: Π' is δ-fine, and we have:

$$\begin{aligned}\sum_{\Pi}{}_c^b f - \int_c^b f = \sum_{\Pi_0}{}_c^b f - \int_c^b f &= \sum_{\Pi'}{}_c^b f - \int_c^b f\\ &= \sum_{\Pi'}{}_a^b f - \int_a^b f - \left(\sum_{\Pi'}{}_a^c f - \int_a^c f\right)\\ &= \sum_{\Pi'}{}_a^b f - \int_a^b f - \left(\sum_{\Pi_c}{}_a^c f - \int_a^c f\right).\end{aligned}$$

By virtue of (12) and (14) we get:

$$\Big|\sum_{\Pi}{}_c^b f - \int_c^b f\Big| \leqslant \Big|\sum_{\Pi'}{}_a^b f - \int_a^b f\Big| + \Big|\sum_{\Pi_c}{}_a^c f - \int_a^c f\Big| \leqslant 2\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

This proves the second inequality of (13). The proof of the first inequality of (13) is analogous. □

We now prove that the HK-integral contains under suitable hypothesis the improper Riemann integral (for the real case, see [5, Theorem 2.9.3, pp. 61–63]).

Theorem 3.16. *Let $a \in \mathbb{R}$, $f\colon [a,+\infty] \to R$ be* HK*-integrable on $[a,+\infty]$. Then f is* HK*-integrable on every interval $[a,b]$ with $a < b < +\infty$, and*

$$(D)\lim_{b\to+\infty}\int_a^b f = \int_a^{+\infty} f.$$

Conversely, let $f\colon [a,+\infty] \to R$ be HK*-integrable on every interval $[a,b]$ with $a < b < +\infty$ and let there exist in R the limit $l = (D)\lim\limits_{b\to+\infty}\int_a^b f$. Moreover, suppose that*

3.16.1) *there exist $u \in R$, $u \geqslant 0$, and a map $\delta_0\colon [a,+\infty] \to \mathbb{R}^+$, such that for every b with $a < b < +\infty$ and for every δ_0-fine partition Π of $[a,b]$, we have:*

$$\Big|\sum_{\Pi}{}_a^b f - \int_a^b f\Big| \leqslant u.$$

Then f is HK*-integrable on $[a,+\infty]$ and $\int_a^{+\infty} f = l$.*

P r o o f. We begin with the first part of the theorem. Since $f\colon [a,+\infty] \to R$ is HK-integrable, there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that, $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists\,\delta\colon [a,+\infty] \to \mathbb{R}^+$ and $\exists\,P > |a|$, such that for each bounded interval $[d_1,d_2]$ with $[d_1,d_2] \subset [a,+\infty]$, $[d_1,d_2] \supset [a,+\infty] \cap [-P,P]$, and for every δ-fine partition Π of $[d_1,d_2]$ we have:

$$\Big|\sum_{\Pi} f - \int_a^{+\infty} f\Big| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{15}$$

Now, we get that f is HK-integrable on $[a,b]$ for every $b \in (a,+\infty]$ with respect to the *same* regulator $(a_{i,j})_{i,j}$, and hence we get that $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}$, $\forall\, b \in (a,+\infty]$, $\exists\,\delta_1\colon [a,b] \to \mathbb{R}^+$ such that for each δ_1-fine partition Π' of $[a,b]$ we have:

$$\Big|\sum_{\Pi'} f - \int_a^b f\Big| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{16}$$

Let us define $\delta_2\colon\ [a,b] \to \mathbb{R}^+$ by setting $\delta_2(x) = \min\{\delta(x), \delta_1(x)\}$, and let Π be a δ_2-fine partition of $[a,b]$, $b > P$. Then, thanks to (15) and (16), $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}\ \ \exists\,P > 0\colon\ \forall\,b > P$,

$$\left|\int_a^b f - \int_a^{+\infty} f\right| \leqslant \left|\sum_{\Pi} f - \int_a^b f\right| + \left|\sum_{\Pi} f - \int_a^{+\infty} f\right| \leqslant 2\bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Thus the first part is completely proved.

We now turn to the second part. By hypothesis, there exists a (D)-sequence $(a_{i,j})_{i,j}$ such that, $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}$, $\exists\,P > 0\colon\ \forall\,b > P$ we get

$$\left|\int_a^b f - l\right| \leqslant \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \tag{17}$$

Let now $(b_n)_n$ be a strictly increasing sequence of real numbers, such that $\lim_n b_n = +\infty$ and $b_1 = a$. We observe that f is HK-integrable in $[b_n, b_{n+1}]$ for each n (with respect to the same regulator $(a^{(n)}_{i,j})_{i,j}$, which is the one "associated" to the interval $[a, b_{n+1}]$). So, $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}$ and $\forall\,n \in \mathbb{N}$, $\exists$ a function $\delta_n\colon\ [b_n, b_{n+1}] \to \mathbb{R}^+$ such that

$$\left|\sum_{\Pi_n} f - \int_{b_n}^{b_{n+1}} f\right| \leqslant \bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i+n)} \tag{18}$$

whenever Π_n is any δ-fine partition of $[b_n, b_{n+1}]$. Let now $(b_{i,j})_{i,j}$ be a (D)-sequence such that

$$u \bigwedge\left(\sum_{n=1}^{\infty}\left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i+n)}\right)\right) \leqslant \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}, \quad \forall\,\varphi \in \mathbb{N}^{\mathbb{N}}, \tag{19}$$

where u is as in 3.16.1): such a sequence does exist, by virtue of Lemma 2.4.

Let now $\delta\colon\ [a,+\infty] \to \mathbb{R}^+$ be such that $\delta \leqslant \delta_0$, where δ_0 is as in 3.16.1), and moreover such that, $\forall\,n \in \mathbb{N}$,

$$\begin{cases} \delta(\xi) \leqslant \delta_n(\xi) & \text{if } \xi \in [b_n, b_{n+1}], \\ [\xi - \delta(\xi), \xi + \delta(\xi)] \subset (b_n, b_{n+1}) & \text{if } \xi \in (b_n, b_{n+1}), \\ (b_n - \delta(b_n), b_n + \delta(b_n)) \subset (b_{n-1}, b_{n+1}). \end{cases} \tag{20}$$

Choose now arbitrarily $b > P$. If $b_N < b \leqslant b_{N+1}$ and $\Pi = \{([x_{k-1}, x_k], \xi_k),\ k = 1, 2, \dots p\}$ is a partition of $[a,b]$, then each b_n, with $n \leqslant N$, must belong to some interval $[x_{k-1}, x_k]$. So, either b_n coincides with some x_k's, or $b_n \in (x_{k-1}, x_k)$. In this last case, from (20) and the fact that Π is δ-fine it follows that $\xi_k \notin (b_n, b_{n+1})$, otherwise

$$[x_{k-1}, x_k] \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \subset (b_n, b_{n+1}),$$

which is a contradiction. Analogously, $\xi_k \notin (b_{n-1}, b_n)$, and in general, if $j \in \mathbb{N}$ is such that $b_j \in (x_{k-1}, x_k)$, we have necessarily $\xi_k \notin (b_{j-1}, b_j)$, $\xi_k \notin (b_j, b_{j+1})$: otherwise $[x_{k-1}, x_k] \subset (b_{j-1}, b_j)$ or $[x_{k-1}, x_k] \subset (b_j, b_{j+1})$, which is absurd. Thus ξ_k does coincide with some b_{j_0}. From the third condition in (20) and the fact that Π is δ-fine it follows that

$$
\begin{aligned}
(21) \qquad [x_{k-1}, x_k] &\subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \\
&= (b_{j_0} - \delta(b_{j_0}), b_{j_0} + \delta(b_{j_0})) \subset (b_{j_0-1}, b_{j_0+1}).
\end{aligned}
$$

But we know that, by hypothesis, $b_n \in (x_{k-1}, x_k)$, and from (21) it follows that $j_0 = n$ and that no b_j but b_n belongs to (x_{k-1}, x_k). So, all the b_n's do coincide either with some x_k or with some ξ_k. Let Π be the partition of $[a, b]$ determined by the x_k's and the b_n's. We have:

$$
(22) \qquad \sum_{\Pi} {}_a^b f = \sum_{n=1}^{N-1} \left(\sum_{\Pi} {}_{b_n}^{b_{n+1}} f \right) + \sum_{\Pi} {}_{b_N}^{b} f.
$$

Since the restriction of Π to $[b_N, b]$ is δ_N-fine, then Π can be "extended" to a δ_N-fine partition Π' of $[b_N, b_{N+1}]$. By (18) and Theorem 3.15, where the roles of $[a, b]$ and c are played by $[b_N, b_{N+1}]$ and b respectively, we get

$$
(23) \qquad \left| \sum_{\Pi} {}_{b_N}^{b} f - \int_{b_N}^{b} f \right| \leqslant 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i+N)}^{(N)}.
$$

Since the restriction of Π to $[b_n, b_{n+1}]$ is δ_n-fine, from (18) and (23) it follows that

$$
(24) \quad \sum_{n=1}^{N-1} \left| \sum_{\Pi} {}_{b_n}^{b_{n+1}} f - \int_{b_n}^{b_{n+1}} f \right| + \left| \sum_{\Pi} {}_{b_N}^{b} f - \int_{b_N}^{b} f \right| \leqslant 2 \left(\sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \right).
$$

From 3.16.1), (17), (19), (22) – (24) we get:

$$
\left| \sum_{\Pi} {}_a^b f - l \right| \leqslant \left| \sum_{\Pi} {}_a^b f - \int_a^b f \right| + \left| \int_a^b f - l \right| \leqslant 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.
$$

Thus the assertion follows. □

Remark 3.17. We observe that theorems similar to Theorem 3.16 hold even if we consider open, semi-open and/or left halflines, $\mathbb{R}$ or $\widetilde{\mathbb{R}}$, instead of $[a, +\infty]$.

We now prove that every simple measurable function defined on $\mathbb{R}$, and assuming values different from zero only on a set of finite Lebesgue measure, is HK-integrable according to our definition, and in this case our integral coincides with the usual one. To do this, thanks to Proposition 3.5, it is sufficient to prove the following:

Theorem 3.18. *Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $|E| < +\infty$, $r \in R$, and χ_E be the characteristic function associated with E. Then the function $\chi_E\, r$ is* HK*-integrable, and $\int_{-\infty}^{\infty} \chi_E\, r = |E| r$.*

P r o o f. Without loss of generality, we can suppose that $r \geqslant 0$: indeed every element r of a Riesz space R is the difference between r^+ and r^-, which are two positive elements of R. In order to demonstrate the theorem, we prove that $\forall\, \varepsilon > 0$ there exists a gauge γ, defined on $\mathbb{R}$, such that for all γ-fine partitions Π of $\mathbb{R}$ we get

$$\left|\sum_{\Pi} \chi_E\, r - |E| r\right| \leqslant \varepsilon r; \tag{24}$$

from (24) it will follow that there exists a (D)-sequence $(d_{i,j})_{i,j}$ such that $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a gauge γ, defined on $\mathbb{R}$, such that for each γ-fine partition Π of $\mathbb{R}$ we have

$$\left|\sum_{\Pi} \chi_E\, r - |E| r\right| \leqslant \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}. \tag{25}$$

Indeed, for every $i, j \in \mathbb{N}$, put $d_{i,j} = r/j$. It is easy to check that the double sequence $(d_{i,j})_{i,j}$ is a (D)-sequence. Fix arbitrarily a map $\varphi \in \mathbb{N}^{\mathbb{N}}$, and set $i_0 = \min\{\varphi(i) \colon i \in \mathbb{N}\}$, $\varepsilon = 1/i_0$. Then we get:

$$\bigvee_{i=1}^{\infty} d_{i,\varphi(i)} = \bigvee_{i=1}^{\infty} \frac{1}{\varphi(i)} r = \left(\bigvee_{i=1}^{\infty} \frac{1}{\varphi(i)}\right) r = \frac{1}{i_0} r = \varepsilon r. \tag{26}$$

So the assertion of the theorem, that is (25), will follow from (24) and (26). Thus, for our purposes, it will be enough to prove (24). By virtue of [5], p. 136, we know that the theorem is true in the particular case $R = \mathbb{R}$ and $r = 1$. Thus for every $\varepsilon > 0$ there exists a gauge γ, defined on $\mathbb{R}$, such that for each γ-fine partition Π of $\mathbb{R}$ we get

$$\left|\sum_{\Pi} \chi_E - |E|\right| \leqslant \varepsilon. \tag{27}$$

Moreover, it is easy to see that for each partition Π of $\mathbb{R}$ we have

$$\sum_{\Pi} \chi_E\, r = \left(\sum_{\Pi} \chi_E\right) r. \tag{28}$$

Thus (24) follows from (27) and (28). This concludes the proof. □

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