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A FULL CHARACTERIZATION OF MULTIPLIERS FOR THE STRONG ϱ-INTEGRAL IN THE EUCLIDEAN SPACE

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Abstract. We study a generalization of the classical Henstock-Kurzweil integral, known as the strong ϱ-integral, introduced by Jarník and Kurzweil. Let $(\mathcal{S}_\varrho(E), \|\cdot\|)$ be the space of all strongly ϱ-integrable functions on a multidimensional compact interval E, equipped with the Alexiewicz norm $\|\cdot\|$. We show that each element in the dual space of $(\mathcal{S}_\varrho(E), \|\cdot\|)$ can be represented as a strong ϱ-integral. Consequently, we prove that fg is strongly ϱ-integrable on E for each strongly ϱ-integrable function f if and only if g is almost everywhere equal to a function of bounded variation (in the sense of Hardy-Krause) on E.

Keywords: strong ϱ-integral, multipliers, dual space

MSC 2000: 26A39, 46E99

1. Introduction

It is well-known that if f is Denjoy-Perron integrable on a compact interval $[a, b]$ of $\mathbb{R}$ and g is of bounded variation on $[a, b]$, then fg is Denjoy-Perron integrable on $[a, b]$ and the integration by parts formula holds. See, for example, [2, Chapter 11]. As a result, every function g of bounded variation on $[a, b]$ induces a bounded linear functional on $\mathcal{D}^*([a, b])$, namely the space of all Denjoy-Perron integrable functions on $[a, b]$. In [1], it is shown that if T is a bounded linear functional on $\mathcal{D}^*([a, b])$, then T can be represented as a Denjoy-Perron integral, and the proof of this result uses integration by parts for the Denjoy-Perron integral and the Riesz representation theorem. Since the one-dimensional Henstock-Kurzweil integral is equivalent to the Denjoy-Perron integral [6], this representation theorem is also obtained from the one-dimensional integration by parts for the Henstock-Kurzweil integral. In higher dimensions, the corresponding integration by parts formula for the Henstock-Kurzweil integral is much more difficult to prove. Kurzweil [4] used the definition of the Henstock-Kurzweil integral to prove that if f is Henstock-Kurzweil integrable on a compact interval E of the multidimensional Euclidean space, and g is of bounded variation (in the sense of Hardy-Krause) on E, then fg is Henstock-Kurzweil integrable on E and the integration by parts formula holds. Here g is known as a *multiplier* for $\mathcal{HK}(E)$, the space of all Henstock-Kurzweil integrable functions on E. Moreover, each multiplier g for $\mathcal{HK}(E)$ induces a bounded linear functional on $\mathcal{HK}(E)$. This led Piotr Mikusiński and K. Ostaszewski [12, Remark 2.15] to ask whether each element in the dual space of $\mathcal{HK}(E)$ can be represented by Henstock-Kurzweil integration involving a suitable multiplier for $\mathcal{HK}(E)$, and the problem was solved independently in [8, 10]. However, those proofs depend strongly on Kurzweil's result [4, Theorem 2.10], whose proof is long and involved compared to the corresponding one-dimensional case. By using Young's multidimensional integration by parts formula for the Lebesgue integral [16], we generalize the above results to the strong ϱ-integral ([3, Definition 4.1] or [5, Definition 1.1]), which coincides with the Henstock-Kurzweil integral when $\varrho \equiv 0$. Consequently, we prove that fg is strongly ϱ-integrable on E for each strongly ϱ-integrable function f if and only if g is almost everywhere equal to a function of bounded variation (in the sense of Hardy-Krause) on E. In other words, we have characterized the multipliers for the strong ϱ-integral. Moreover, our method also offers a transparent way of extending Young's results [16] to non-absolute integrals, which he did mention in his paper without proof.

2. Preliminaries

Unless stated otherwise, the following conventions and notation will be used. The set of all real numbers is denoted by $\mathbb{R}$, and the ambient space of this paper is $\mathbb{R}^m$, where m is a fixed positive integer. The norm in $\mathbb{R}^m$ is the maximum norm $\|\cdot\|_0$. Let $E = \prod_{i=1}^{m} [a_i, b_i]$ be a fixed interval in $\mathbb{R}^m$. For a set $A \subset E$, we denote by χ_A and $\operatorname{diam}(A)$ the characteristic function and diameter of A, respectively. If $Z \subseteq E$, we denote its interior with respect to the subspace topology of E by $\operatorname{int}(Z)$. The expressions "measure", "measurable", "almost all", "almost everywhere" refer to the m-dimensional Lebesgue measure μ_m. A set $Z \subset E$ is called *negligible* whenever $\mu_m(Z) = 0$. Given two subsets X, Y of E, the symmetric difference of X and Y is denoted by $X \Delta Y$. We say that X and Y are nonoverlapping if their intersection is negligible. A function is always real-valued. When no confusion is possible, we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$. If Z is a measurable subset of E, $\mathcal{L}(Z)$ will denote the space of all Lebesgue integrable functions on Z. If $f \in \mathcal{L}(Z)$, the Lebesgue integral of f over Z will be denoted by $(L)\int_Z f$.

An *interval* is a compact nondegenerate subinterval of E. $\mathcal{I}$ denotes the family of all nondegenerate subintervals of E. If $I \in \mathcal{I}$, we will write $\mu_m(I)$ as $|I|$. For each $J \in \mathcal{I}$, the *regularity* of an m-dimensional interval $J \subseteq E$, denoted by $\operatorname{reg}(J)$, is the ratio of its shortest and longest sides. A function F defined on $\mathcal{I}$ is said to be additive if $F(I \cup J) = F(I) + F(J)$ for each nonoverlapping intervals $I, J \in \mathcal{I}$ with $I \cup J \in \mathcal{I}$. In particular, it is shown in [7, Corollary 6.2.4] that if F is an additive interval function on $\mathcal{I}$ with $J \in \mathcal{I}$ and $\{K_1, K_2, \dots, K_r\}$ is a collection of nonoverlapping subintervals of J with $\bigcup_{i=1}^{r} K_i = J$, then

$$F(J) = \sum_{i=1}^{r} F(K_i).$$

For each $x \in E$ and $r > 0$, set

$$B(x, r) = \{y \in \mathbb{R}^m : \ \|x - y\|_0 < r\}.$$

A positive function δ on a set $Z \subseteq E$ is called a *gauge* on Z. A *partition* is a finite collection $P = \{(I_i, \xi_i)\}_{i=1}^{p}$, where $I_1, I_2, \dots, I_p$ are pairwise nonoverlapping intervals, and $\xi_i \in I_i$ for each $i \in \{1, 2, \dots, p\}$. Given $Z \subseteq E$, a gauge δ on Z and $\varrho\colon\ Z \times (0, \infty) \to [0, 1)$, we say that P is

(i) a partition *in* Z if $\bigcup_{i=1}^{p} I_i \subset Z$;
(ii) a partition *of* Z if $\bigcup_{i=1}^{p} I_i = Z$;
(iii) *anchored* in Z if $\{\xi_1, \xi_2, \dots, \xi_p\} \subset Z$;
(iv) *δ-fine* if it is anchored in Z with $I_i \subset B(\xi_i, \delta_i(\xi))$ for each $i \in \{1, 2, \dots, p\}$;
(v) *ϱ-regular* if $\operatorname{reg}(I_i) > \varrho(\xi_i, \operatorname{diam}(I_i))$ for each $i \in \{1, 2, \dots, p\}$.

Lemma 2.1 [7, Lemma 6.2.6]**.** *Given a gauge δ on E, δ-fine partitions of E exist.*

Definition 2.2. A function $f\colon\ E \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on E if there exists $A \in \mathbb{R}$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on E such that

$$\left|\sum_{i=1}^{p} f(\xi_i)|I_i| - A\right| < \varepsilon \tag{1}$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ of E. Here A is called the Henstock-Kurzweil integral of f over E, and we write A as $(HK)\int_E f$.

Remarks 2.3.

(a) The linear space of all Henstock-Kurzweil integrable functions on E is denoted by $\mathcal{HK}(E)$.

(b) It follows from [7, Theorem 6.4.2] that if $f \in \mathcal{HK}(E)$, then $f \in \mathcal{HK}(J)$ for each subinterval J of E. The interval function $F\colon J \mapsto (HK)\int_J f$ is known as the *indefinite Henstock-Kurzweil integral*, or in short the indefinite $\mathcal{HK}$-integral, of f. By [7, Theorem 6.4.1], F is an additive interval function on $\mathcal{I}$.

(c) By [7, p. 228] and [7, Theorem 3.13.3], we see that $\mathcal{L}(E) \subset \mathcal{HK}(E)$. Furthermore, $(L)\int_E f = (HK)\int_E f$ for each $f \in \mathcal{L}(E)$.

(d) If f is a non-negative, Henstock-Kurzweil integrable function on E, then it follows from [7, p. 228] that $f \in \mathcal{L}(E)$.

By specializing [3, Lemma 1.7] to the case of the Henstock-Kurzweil integral (see [3, Note 1.5]), we have the following important Saks-Henstock Lemma.

Theorem 2.4 (Saks-Henstock)**.** *Let $f \in \mathcal{HK}(E)$ and let F be the indefinite $\mathcal{HK}$-integral of f on E. Then given $\varepsilon > 0$ there exists a gauge δ on E such that*

$$\sum_{i=1}^{p} \bigl| f(\xi_i)|I_i| - F(I_i) \bigr| < \varepsilon.$$

3. The strong ϱ-integral

Unless otherwise stated, throughout this paper we shall assume that $\varrho\colon E \times (0,\infty) \longrightarrow [0,1)$ satisfies the following conditions:

$$\limsup_{t \to 0^+} \varrho(x,t) < 1 \text{ for each } x \in E, \tag{2}$$

$$\inf\{\varrho(x,t)\colon\ x \in E, t > 0\} > 0. \tag{3}$$

The following lemma, due to Jarník and Kurzweil, generalizes Lemma 2.1.

Lemma 3.1 [3, Lemma 1.1]**.** *Assuming that $\varrho\colon E \times (0,\infty) \longrightarrow [0,1)$ satisfies* (2) *and* (3)*, then for any gauge δ and every interval J of E there exists a δ-fine, ϱ-regular partition of J.*

In view of Lemma 3.1, we have the following definition.

Definition 3.2. A function $f\colon E \longrightarrow \mathbb{R}$ is said to be *strongly ϱ-integrable* if there exists an additive interval function F on $\mathcal{I}$ with the following property: given $\varepsilon > 0$ there exists a gauge δ on E such that

$$\sum_{i=1}^{p} \bigl| f(\xi_i)|J_i| - F(J_i) \bigr| < \varepsilon$$

for each δ-fine ϱ-regular partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E, and J_i is a subinterval of I_i for each $i \in \{1, 2, \ldots, p\}$. For each $J \in \mathcal{I}$, we write $F(J)$ as $\int_J f$.

Remarks 3.3.

(a) The linear space of all strongly ϱ-integrable functions on E is denoted by $\mathcal{S}_\varrho(E)$.

(b) If $f \in \mathcal{S}_\varrho(E)$, then $f \in \mathcal{S}_\varrho(J)$ for each subinterval J of E.

(c) If $\varrho \equiv 0$ or $m = 1$, then each strongly ϱ-integrable function is also Henstock-Kurzweil integrable [3, Note 1,5].

(d) If $\{f_1, f_2\} \subset \mathcal{S}_\varrho(E)$ and $f_1 \geqslant f_2$ almost everywhere on E, then $\int_E f_1 \geqslant \int_E f_2$.

We shall next prove that if $f \in \mathcal{HK}(E)$, then $f \in \mathcal{S}_\varrho(E)$ for every ϱ satisfying (2) and (3). Moreover, the indefinite integrals coincide. First we need the following Strong Saks-Henstock Lemma.

Theorem 3.4 (Strong Saks-Henstock Lemma)**.** *If $f \in \mathcal{H}K(E)$, then for $\varepsilon > 0$ there exists a gauge δ on E such that*

$$\sum_{i=1}^{p} \left| f(\xi_i)|J_i| - (HK)\int_{J_i} f \right| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E, and J_i is a subinterval of I_i for each $i \in \{1, 2, \ldots, p\}$.

Proof. By the Saks-Henstock Lemma, there exists a gauge δ on E such that

$$\sum_{i=1}^{p} \left| f(\xi_i)|I_i| - (HK)\int_{I_i} f \right| < \frac{\varepsilon}{2^m} \tag{4}$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E.

For each $i \in \{1, 2, \ldots, p\}$, we define a function $g_i\colon I_i \longrightarrow \mathbb{R}$ by $g_i(x) = f(\xi_i) - f(x)$. Let $\{v_{i,1}, v_{i,2}, \ldots, v_{i,2^m}\}$ denote the vertices of J_i, and $\langle \alpha, \beta \rangle$ denotes the subinterval of E with α, β as opposite vertices, we have

$$\begin{aligned}
&\left|f(\xi_i)|J_i| - (HK)\int_{J_i} f\right| = \left|(HK)\int_{J_i} g_i(x)\right| \\
&= \left|\sum_{k=1}^{2^m}(-1)^{\gamma(k)}(HK)\int_{\langle \xi_i, v_{i,k}\rangle} g_i(x)\right| \text{ for some positive integers } \gamma(1), \dots, \gamma(2^m) \\
&\leqslant \sum_{k=1}^{2^m}\left|(HK)\int_{\langle \xi_i, v_{i,k}\rangle} g_i(x)\right|,
\end{aligned}$$

which implies that

$$\left|f(\xi_i)|J_i| - (HK)\int_{J_i} f\right| \leqslant \sum_{k=1}^{2^m}\left|(HK)\int_{\langle \xi_i, v_{i,k}\rangle} g_i(x)\right| \tag{5}$$

for each $i \in \{1, 2, \dots, p\}$. Observe that we have

$$(HK)\int_{\langle \xi_i, v_{i,k}\rangle} g_i(x) = 0 \text{ whenever } \langle \xi_i, v_{i,k}\rangle \text{ is a degenerate subinterval of } E. \tag{6}$$

For each $k \in \{1, 2, \dots, 2^m\}$, the finite collection

$$\{(\langle \xi_i, v_{i,k}\rangle, \xi_i)\colon \langle \xi_i, v_{i,k}\rangle \text{ is a subinterval of } E\}_{i=1}^{p} \text{ is a } \delta\text{-fine partition in } E \tag{7}$$

provided that it is nonempty. By (5), (6), (7) and (4), we have

$$\sum_{i=1}^{p}\left|f(\xi_i)|J_i| - (HK)\int_{J_i} f\right| \leqslant \sum_{i=1}^{p}\sum_{k=1}^{2^m}\left|(HK)\int_{\langle \xi_i, v_{i,k}\rangle} g_i(x)\right| < \varepsilon.$$

The proof is complete. □

The next theorem, together with Remark 3.3(c), shows that the strong ϱ-integral coincides with the Henstock-Kurzweil integral when $\varrho \equiv 0$. In view of Remark 2.3(c), it is a mild generalization of [5, Lemma 2.8].

Theorem 3.5. *If $f \in \mathcal{HK}(E)$, then $f \in \mathcal{S}_\varrho(E)$ for every ϱ satisfying* (2) *and* (3).

Proof. This follows from Remark 2.3(b) and Theorem 3.4. □

Our next aim is to show that $\mathcal{S}_\varrho(E)$, like the space $\mathcal{HK}(E)$, can be equipped with the Alexiewicz norm. The next crucial lemma sharpens [3, Theorem 2.1] for the strong ϱ-integral.

Lemma 3.6. *If f is strongly ϱ-integrable on E, then given $\varepsilon > 0$ there exists $\eta > 0$ such that $\left|\int_{E_1} f - \int_{E_2} f\right| < \varepsilon$ whenever E_1, E_2 are subintervals of E with $|E_1 \Delta E_2| < \eta$.*

Proof. Since $f \in \mathcal{S}_\varrho(E)$, for each $j = 1, 2$ there exists a gauge δ on E such that

$$\sum_{i=1}^{p}\left|f(\xi_i)|I_i \cap E_j| - \int_{I_i \cap E_j} f\right| < \frac{\varepsilon}{3}$$

for each δ-fine ϱ-regular partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ in E.

By Lemma 3.1, we may fix a δ-fine ϱ-regular partition $\{(I_i', \xi_i')\}_{i=1}^{p_0}$ of E. Put $M = \max\{|f(\xi_i')|\colon\ i = 1, 2, \ldots, p_0\}$. Then whenever $|E_1 \Delta E_2| < \eta = \varepsilon/(3M+1)$, we have

$$\begin{aligned}\left|\int_{E_1} f - \int_{E_2} f\right| &\leqslant \sum_{i=1}^{p_0}\left|f(\xi_i')|I_i' \cap E_1| - \int_{I_i' \cap E_1} f\right| + \sum_{i=1}^{p_0}\left|f(\xi_i')|I_i' \cap E_2| - \int_{I_i' \cap E_2} f\right| \\ &\quad + \sum_{i=1}^{p_0} |f(\xi_i')\{|I_i' \cap E_1| - |I_i' \cap E_2|\}| \\ &\leqslant \frac{2\varepsilon}{3} + M\sum_{i=1}^{p_0}\{|(I_i' \cap E_1)\Delta(I_i' \cap E_2)|\} \\ &\leqslant \frac{2\varepsilon}{3} + M\sum_{i=1}^{p_0}|I_i' \cap (E_1 \Delta E_2)| \leqslant \frac{2\varepsilon}{3} + M|E_1 \Delta E_2| \\ &< \frac{2\varepsilon}{3} + M\frac{\varepsilon}{3M+1} < \varepsilon.\end{aligned}$$

The proof is complete. □

If f is strongly ϱ-integrable on E and F denotes the indefinite integral of f, then it follows from Lemma 3.6 that F is continuous in the sense that $F(I) \to 0$ as the measure of the interval I tends to zero. Denoting the distribution function of the indefinite strong ϱ-integral F of f by

$$\widetilde{F}(x) = \begin{cases} \int_{[a_1,x_1]\times[a_2,x_2]\times\ldots\times[a_m,x_m]} f & \text{if } a_i < x_i \leqslant b_i \quad \text{for all } i \in \{1, 2, \ldots, m\}, \\ 0 & \text{if } x_i = a_i \quad \text{for some } i \in \{1, 2, \ldots, m\} \end{cases}$$

we see that the continuity of $\widetilde{F}$ on E follows from the continuity of F. Note that we may convert $\widetilde{F}$ into F and vice versa [7, p. 231]. Thus we may equip the space $\mathcal{S}_\varrho(E)$ with the Alexiewicz norm $\|\cdot\|_H$, where

$$\|f\|_H := \sup_{(x_1,x_2,\ldots,x_m)\in E}\left|\widetilde{F}(x_1, x_2, \ldots, x_m)\right|,$$

and the supremum is taken over all points $(x_1, x_2, \ldots, x_m) \in E$. Letting $\|f\| := \sup\limits_{I\subseteq E}\left|\int_I f\right|$, where the supremum is taken over all subintervals I of E, then we have

$$\|f\|_H \leqslant \|f\| \leqslant 2^m \|f\|_H.$$

For this paper, we shall equip the space of all strongly ϱ-integrable functions on E with the norm $\|\cdot\|$.

By repeating the proof of [3, Theorems 2.8–2.9], we see that if f is strongly ϱ-integrable on E, then f is measurable.

Theorem 3.7. *If $f \in \mathcal{S}_\varrho(E)$ and $f \geqslant 0$ almost everywhere on E, then $f \in \mathcal{L}(E)$.*

Proof. Since $f \geqslant 0$ almost everywhere on E, the Monotone Convergence Theorem, Remark 2.3(c), Theorem 3.5 and Remark 3.3(d) yield

$$\begin{aligned}(L)\int_E f &= \lim_{n\to\infty} (L)\int_E \min\{n, f\}\\ &= \lim_{n\to\infty} \int_E \min\{n, f\} \leqslant \int_E f < \infty,\end{aligned}$$

proving that $f \in \mathcal{L}(E)$. □

Theorem 3.8. *The space of all step functions on E is $\|\cdot\|$-dense in $\mathcal{S}_\varrho(E)$.*

Proof. Fix $f \in \mathcal{S}_\varrho(E)$. Given $\varepsilon > 0$, there exists a gauge δ on E such that

$$\sum_{i=1}^{p} |f(\xi_i)|J_i| - F(J_i)| < \frac{\varepsilon}{2}$$

for each δ-fine ϱ-regular partition $P = \{(I_i, \xi_i)\}_{i=1}^p$ in E, and J_i is a subinterval I_i for each $i \in \{1, 2, \ldots, p\}$.

In view of Lemma 3.1, we may fix a δ-fine ϱ-regular partition $Q = \{(L_i, x_i)\}_{i=1}^p$ of E. Set

$$\varphi(x) = \begin{cases} f(x_i) & \text{if } x \in \operatorname{int}(L_i) \text{ with } (L_i, x_i) \in Q,\\ 0 & \text{otherwise.}\end{cases}$$

Let J be any subinterval of E. Then we have

$$\begin{aligned}\left|\int_J \varphi - \int_J f\right| &= \left|\sum_{i=1}^{q} \int_{J\cap L_i} (f(x_i) - f)\right|\\ &\leqslant \sum_{i=1}^{q}\left|f(x_i)|J\cap L_i| - \int_{J\cap L_i} f\right| < \frac{\varepsilon}{2},\end{aligned}$$

which implies that

$$\|\varphi - f\| \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

The proof is complete. □

Remark 3.9. By using Theorem 3.8 and [9, Theorem 3.6], it can be shown that the Uniform Boundedness Theorem holds for $(\mathcal{S}_\varrho(E), \|\cdot\|)$. In particular, the space $(\mathcal{S}_\varrho(E), \|\cdot\|)$ is barrelled, but not complete. However, we do not need this result in this paper.

4. Functions of strongly bounded variation

In this section, we shall prove that if T is a $\|\cdot\|$-bounded linear functional on $\mathcal{L}(E)$, then T can be represented by Lebesgue integration (Theorem 4.7). Since the norm $\|\cdot\|$ is not equivalent to the L^1-norm $\|\cdot\|_1$, the dual space of $(\mathcal{L}(E), \|\cdot\|)$ need not be equal to $L^\infty(E)$, the space of all essentially bounded, measurable functions on E. It turns out that the dual space of $(\mathcal{L}(E), \|\cdot\|)$ is the space of all functions of bounded variation in the sense of Hardy-Krause, or equivalently strongly bounded variation [4], on E. We need some definitions.

Definition 4.1. Let $g\colon E \longrightarrow \mathbb{R}$ and let $I = \prod_{i=1}^{m} [\alpha_i, \beta_i]$ be a subinterval of E. We define

$$\Delta_g(I) = \Delta_1 \Delta_2 \dots \Delta_m g$$

where

$$\Delta_k g = g(\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \beta_k, \alpha_{k+1}, \dots, \alpha_m) - g(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Theorem 4.2 [11, Section 45.2s, Section 45, p. 242]**.** *Let $g\colon E \longrightarrow \mathbb{R}$ be a function. If $I = \prod_{i=1}^{m} [\alpha_i, \beta_i]$ is a subinterval of E and $\{v^{(k)}\}_{k=1}^{2^m}$ denotes the vertices of I, then*

$$\Delta_g(I) = \sum_{k=1}^{2^m} (-1)^{\gamma(k)} g(v^{(k)})$$

where $\gamma(k)$ is the cardinality of the set $\{i\colon v_i^{(k)} = \alpha_i\}$.

Folowing [7, p. 204–205] we say that $\{I_i\}_{i=1}^{p}$ is a division of E if $I_1, I_2, \dots, I_p$ are pairwise nonoverlapping intervals with $\bigcup_{i=1}^{p} I_i = E$.

Definition 4.3 [4, Definition 1.14]**.** Let $g\colon E \to \mathbb{R}$. Put

$$\operatorname{Var}(g, I) := \sup \sum_{i=1}^{p} |\Delta_g(I_i)|,$$

where the supremum is taken over all divisions of E. g is said to be of bounded variation of E if $\operatorname{Var}(g, E)$ is finite.

The space of all functions of bounded variation on E will be denoted by $\mathcal{BV}(E)$. Let

$$\mathcal{BV}_0(E) := \{g \in \mathcal{BV}(E)\colon\ g(x_1, x_2, \dots, x_m) = 0 \\ \text{whenever } x_i = a_i \text{ for some } i \in \{1, 2, \dots, m\}\}.$$

The next definition is equivalent to [4, Definition 1.14].

Definition 4.4. A function $g\colon E \longrightarrow \mathbb{R}$ is said to be of strongly bounded variation on E if

(i) $g \in \mathcal{BV}(E)$;

(ii) for each $x_1 \in [a_1, b_1]$, the function $g(x_1, \cdot, \cdot, \dots, \cdot)$ is of strongly bounded variation on $[a_2, b_2] \times [a_3, b_3] \times \dots \times [a_m, b_m]$;

(iii) for each $x_m \in [a_m, b_m]$, the function $g(\cdot, \cdot, \cdot, \dots, \cdot, x_m)$ is of strongly bounded variation on $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_{m-1}, b_{m-1}]$;

(iv) for each $i \in \{2, \dots, m-1\}$, and $x_i \in [a_i, b_i]$, the function $g\,(\cdot, \cdot, \dots, \cdot, x_i, \cdot, \dots, \cdot, \cdot)$ is of strongly bounded variation on $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_{i-1}, b_{i-1}] \times [a_{i+1}, b_{i+1}] \times \dots \times [a_m, b_m]$.

The class of all functions of strongly bounded variation on E will be denoted by $\mathcal{SBV}(E)$. It is well-known that if $\widetilde{F}$ is continuous on E and $g \in \mathcal{SBV}(E)$, then the Riemann-Stieltjes integral of $\widetilde{F}$ with respect to g over E, denoted by $\int_E \widetilde{F}\,\mathrm{d}g$, exists. See, for example, [16]. Before we state and prove the next lemma, we need a notation.

Let $g\colon E \longrightarrow \mathbb{R}$. For each $(x_1, x_2, \dots, x_m) \in E$ and positive integers $i_1, i_2, \dots, i_l$ with $1 \leqslant i_1 < i_2 < \dots < i_l \leqslant m$, we define

$$g_{i_1, i_2, \dots, i_l}(x_1, x_2, \dots, x_m) = g(z_1, z_2, \dots, z_p, \dots, z_m)$$

where

$$z_p = \begin{cases} x_p & \text{if } p \in \{i_1, i_2, \dots, i_l\}; \\ b_p & \text{otherwise.} \end{cases}$$

We are now ready to use Young's result to prove the next lemma, which says that if $g \in \mathcal{SBV}(E)$, then g induces a bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$.

Lemma 4.5. *Let $f \in \mathcal{L}(E)$ and let $\widetilde{F}$ be the distribution function of the indefinite Lebesgue integral of f. If $g \in \mathcal{SBV}(E)$, then $fg \in \mathcal{L}(E)$ and*

$$
\begin{aligned}
(L)\int_E fg &= \widetilde{F}(b_1, b_2, \dots, b_m) g(b_1, b_2, \dots, b_m) \\
&\quad - \sum_i \int_{a_i}^{b_i} \widetilde{F}(b_1, b_2, \dots, b_{i-1}, x_i, b_{i+1}, \dots, b_m) \, \mathrm{d}g_i(x) \\
&\quad + \sum_{i,j} \int_{[a_i,b_i]\times[a_j,b_j]} \widetilde{F}(b_1, b_2, \dots, b_{i-1}, x_i, b_{i+1}, \dots, b_{j-1}, x_j, b_{j+1}, \dots, b_m) \, \mathrm{d}g_{i,j}(x) \\
&\quad - \sum_{i,j,k} + \sum_{i,j,k,l} + \dots \\
&\quad + (-1)^{m-1} \sum_k \int_{\prod\limits_{i=1}^{k-1}[a_i,b_i]\times \prod\limits_{i=k+1}^{m}[a_i,b_i]} \widetilde{F}(x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_m) \, \mathrm{d}g_{(k)}(x) \\
&\quad + (-1)^m \int_E \widetilde{F}(x) \, \mathrm{d}g(x)
\end{aligned}
$$

where $g_{(k)} = g_{1,2,\dots,k-1,k+1,\dots,m}$. In particular, g induces a bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$.

P r o o f. The integration by parts formula follows from [16]. Define $T\colon (\mathcal{L}(E), \|\cdot\|) \longrightarrow \mathbb{R}$ by

$$T(f) = (L)\int_E fg.$$

Then T is linear and it remains to prove that T is a bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$. Since

$$|T(f)| = \left|(L)\int_E fg\right|,$$

by putting all summands on the right hand side of the integration by parts formula in absolute values we obtain

$$|T(f)| \leqslant M\|f\|$$

where

$$
\begin{aligned}
M = |g(b_1, b_2, \dots, b_m)| &+ \sum_i \operatorname{Var}(g_i, [a_i, b_i]) + \sum_{i,j} \operatorname{Var}(g_{i,j}, [a_i, b_i] \times [a_j, b_j]) + \dots \\
&+ \sum_k \operatorname{Var}\Bigl(g_{1,2,\dots,k-1,k+1,\dots,m}, \prod_{i=1}^{k-1}[a_i, b_i] \times \prod_{i=k+1}^{m}[a_i, b_i]\Bigr) + \operatorname{Var}(g, E).
\end{aligned}
$$

Since the finiteness of M follows from the assumption that $g \in \mathcal{SBV}(E)$, we see that T is $\|\cdot\|$-bounded. The proof is complete. □

Our objective is to prove that every bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$ can be represented as an integral similar to the one given in Lemma 4.5. We need a lemma, which is proven in [8, Theorem 3.1] by means of the Fubini's theorem. In this case, we show that it can be deduced directly from Lemma 4.5.

Lemma 4.6. *If $g \in \mathcal{SBV}(E)$, then there exists $g_0 \in \mathcal{SBV}(E)$ such that* $\operatorname{Var}(g_0, E) = \operatorname{Var}(g, E)$. *Moreover, the equality*

$$\int_E \widetilde{F}\,\mathrm{d}g = (\mathcal{L})\int_E fg_0$$

holds whenever $f \in \mathcal{L}(E)$ and $\widetilde{F}$ is the distribution function of the indefinite Lebesgue integral of f.

Proof. We observe that the function $g_0\colon E \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
&g_0(x_1,\ldots,x_m) = \Delta_g\biggl(\prod_{i=1}^m [x_i,b_i]\biggr)\\
&= g(b_1,\ldots,b_m) - \sum_i g(b_1,\ldots,b_{i-1},x_i,b_{i+1},\ldots,b_m)\\
&\quad + \sum_{i,j} g(b_1,\ldots,b_{i-1},x_i,b_{i+1},\ldots,b_{j-1},x_j,b_{j+1},\ldots,b_m)\\
&\quad - \sum_{i,j,k} + \sum_{i,j,k,l} + \ldots\\
&\quad + (-1)^{m-1}\sum_k g(x_1,\ldots,x_{k-1},b_k,x_{k+1},\ldots,x_m) + (-1)^m g(x_1,\ldots,x_m)
\end{aligned}$$

satisfies the following conditions:

(a) the equality $g_0(x_1,x_2,\ldots,x_m) = 0$ holds whenever $x_i = b_i$ for some $i \in \{1,2,\ldots,m\}$;

(b) the equality $\Delta_{g_0}(I) = \Delta_{(-1)^m g}(I)$ holds for each subinterval I of E;

(c) $g_0 \in \mathcal{SBV}(E)$ and $\operatorname{Var}(g_0, E) = \operatorname{Var}(g, E)$.

It follows from (a), (b), (c) and Lemma 4.5 that $fg_0 \in \mathcal{L}(E)$ and

$$\begin{aligned}
(L)\int_E fg_0 &= \widetilde{F}(b_1,b_2,\ldots,b_m)g_0(b_1,b_2,\ldots,b_m)\\
&\quad - \sum_i \int_{a_i}^{b_i} \widetilde{F}(b_1,b_2,\ldots,b_{i-1},x_i,b_{i+1},\ldots,b_m)\,\mathrm{d}(g_0)_i(x)\\
&\quad + \sum_{i,j} \int_{[a_i,b_i]\times[a_j,b_j]} \widetilde{F}(b_1,b_2,\ldots,b_{i-1},x_i,b_{i+1},\ldots,b_{j-1},x_j,b_{j+1},\ldots,b_m)\,\mathrm{d}(g_0)_{i,j}(x)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i,j,k} + \sum_{i,j,k,l} + \dots \\
&+ (-1)^{m-1} \sum_k \int_{\prod_{i=1}^{k-1}[a_i,b_i] \times \prod_{i=k+1}^{m}[a_i,b_i]} \widetilde{F}(x_1, x_2, \dots, x_{k-1}, b_k, x_{k+1}, \dots, x_m) \,\mathrm{d}(g_0)_{(k)}(x) \\
&+ (-1)^m \int_E \widetilde{F}(x) \,\mathrm{d}g_0(x) \\
&= (-1)^m \int_E \widetilde{F}(x) \,\mathrm{d}g_0(x) = (-1)^m \int_E \widetilde{F}(x) d((-1)^m g(x)) = \int_E \widetilde{F} \,\mathrm{d}g
\end{aligned}
$$

where $(g_0)_{(k)} := (g_0)_{1,2,\dots,k-1,k+1,\dots,m}$. The proof is complete. $\square$

We are now ready to prove the main result of this section, namely that every bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$ can be represented as a Lebesgue integral similar to the one given in Lemma 4.5. Since the norm $\|\cdot\|$ is not equivalent to the L^1-norm $\|\cdot\|_1$, the dual space of $(\mathcal{L}(E), \|\cdot\|)$ need not be equal to $L^\infty(E)$. It turns out that the dual space of $(\mathcal{L}(E), \|\cdot\|)$ is the space of all functions of strongly bounded variation on E.

Theorem 4.7. *Let T be a bounded linear functional on $(\mathcal{L}(E), \|\cdot\|)$. Then there exists a function $g_0 \in \mathcal{SBV}(E)$ such that*

$$T(f) = (L)\int_E f g_0$$

for every $f \in \mathcal{L}(E)$. Moreover, $\|T\| = \operatorname{Var}(g_0, E)$.

P r o o f. By following the proofs of [13, Proposition 3], [12, Proposition 2.6] and [12, Propositions 2.11–2.13], we conclude that there exists $g \in \mathcal{BV}_0(E)$ such that

$$T(f) = \int_E \widetilde{F} \,\mathrm{d}g$$

for all $f \in \mathcal{L}(E)$, where $\widetilde{F}$ denotes the distribution function of the indefinite Lebesgue integral of f. Moreover, $\|T\| = \operatorname{Var}(g, E)$.

It follows from Lemma 4.6 that there exists $g_0 \in \mathcal{SBV}(E)$ such that

$$\int_E \widetilde{F} \,\mathrm{d}g = (L)\int_E f g_0$$

and $\operatorname{Var}(g_0, E) = \operatorname{Var}(g, E)$. Thus the equality $\|T\| = \operatorname{Var}(g, E) = \operatorname{Var}(g_0, E)$ follows. The proof is complete. $\square$

5. Main results

Lemma 5.1. *Let $g_0 \in L^\infty(E)$ and let G be the indefinite integral of g. Then there exists a negligible set $Z \subset E$ with the following property: given $\varepsilon_0 > 0$ there exists a gauge δ_1 on $E - Z$ such that*

$$\bigl|g_0(\xi)|J| - G(J)\bigr| < \varepsilon_0 |I|$$

whenever $\xi \in I \setminus Z$, $I \in \mathcal{I}$, $I \subset B(\xi, \delta_1(\xi))$ and J is any subinterval of I.

Proof. Since $g_0 \in L^\infty(E)$ and G is the indefinite integral of g, it follows from [14] that given $\varepsilon_0 > 0$ there exists a gauge δ_1 on $E - Z$ such that

$$\bigl|g_0(\xi)|I| - G(I)\bigr| < \frac{\varepsilon_0}{2^m}|I| \tag{8}$$

whenever $\xi \in I \setminus Z$, $I \in \mathcal{I}$ and $I \subset B(\xi, \delta_1(\xi))$.

Let J be any given subinterval of I. If $\{v^{(1)}, v^{(2)}, \ldots, v^{(2^m)}\}$ denote the vertices of J, then

$$\begin{aligned}\bigl|g_0(\xi)|J| - G(J)\bigr| &= \left|\sum_{k=1}^{2^m} (-1)^{\gamma(k)} (g_0(\xi)|\langle \xi, v^{(k)}\rangle| - G(\langle \xi, v^{(k)}\rangle))\right| \text{ for some} \\ &\qquad\qquad \text{positive integers } \gamma(1), \ldots, \gamma(2^m) \\ &< \frac{\varepsilon_0}{2^m}\sum_{k=1}^{2^m} |\langle \xi, v^{(k)}\rangle| \leqslant \varepsilon_0 |I|\end{aligned}$$

by (8), proving that assertion (ii) holds. The proof is complete. □

We are now ready to prove the integral representation theorem for bounded linear functionals on $(\mathcal{S}_\varrho(E), \|\cdot\|)$. Observe that the proof does not use Kurzweil's result [4].

Theorem 5.2. *If T is a bounded linear functional on $(\mathcal{S}_\varrho(E), \|\cdot\|)$, then there exists a function $g_0 \in \mathcal{SBV}(E)$ such that*

$$T(f) = \int_E f g_0$$

for every $f \in \mathcal{S}_\varrho(E)$. Moreover, $\|T\| = \operatorname{Var}(g_0, E)$.

Proof. We shall first use Theorem 4.7 to obtain the required function g_0. Let $T|_{(\mathcal{L}(E), \|\cdot\|)}$ be the restriction of T to $(\mathcal{L}(E), \|\cdot\|)$. It follows from Theorem 4.7 that there exists $g_0 \in \mathcal{SBV}(E)$ such that

$$T(f) = (L)\int_E f g_0$$

for all $f \in \mathcal{L}(E)$. Let G denote the indefinite integral of g_0. By Lemma 5.1, there exists a negligible subset Z of E with the following property: given $\varepsilon > 0$, there exists a gauge δ_1 on $E - Z$ such that

$$|g_0(\xi)|J| - G(J)| < \frac{\varepsilon}{3(|f(\xi)|+1)} \frac{|I|}{1+|E|}$$

whenever $\xi \in I \setminus Z$, $I \in \mathcal{I}$ and $I \subset B(\xi, \delta_1(\xi))$.

Since f is strongly ϱ-integrable on E, there exists a gauge δ_2 on E such that

$$\sum_{i=1}^{p} |f(\xi_i)|J_i'| - F(J_i')| < \frac{\varepsilon}{3(\|T\|+1)}$$

for each δ_2-fine ϱ-regular partition $\{(I_i, \xi_i)\}_{i=1}^p$ in E, and $\{J_i'\}_{i=1}^p$ is any finite collection of subintervals of E such that $J_i' \subseteq I_i$ for each $i \in \{1, 2, \dots, p\}$.

Without loss of generality, we may assume that $f \equiv 0$ on Z. Define a gauge δ on E by

$$\delta(\xi) = \begin{cases} \min\{\delta_1(\xi), \delta_2(\xi)\} & \text{if } \xi \in E - Z, \\ \delta_2(\xi) & \text{if } \xi \in Z. \end{cases}$$

Consider any δ-fine ϱ-regular partition $\{(I_i, \xi_i)\}_{i=1}^p$ in E with $S_1 = \{i\colon \xi_i \notin Z\}$ and $S_2 = \{i\colon \xi_i \in Z\}$. If J_i is a subinterval of I_i for each $i \in \{1, 2, \dots, p\}$, then

$$\begin{aligned}
&\sum_{i=1}^{p} |f(\xi_i) g_0(\xi_i)|J_i| - T(f\chi_{J_i})| \\
&= \sum_{i\in S_1} |f(\xi_i) g_0(\xi_i)|J_i| - T(f\chi_{J_i})| + \sum_{i\in S_2} |T(f\chi_{J_i})| \\
&\leqslant \sum_{i\in S_1} \left| f(\xi_i) \int_{J_i} g_0 - T(f\chi_{J_i}) \right| + \sum_{i\in S_1} |f(\xi_i)| \left| g_0(\xi_i)|J_i| - \int_{J_i} g_0 \right| + \sum_{i\in S_2} \|T\| \|f\chi_{J_i}\| \\
&< \sum_{i\in S_1} |T(f(\xi_i)\chi_{J_i}) - T(f\chi_{J_i})| + \frac{\varepsilon}{3} + \sum_{i\in S_2} \|T\| \|f\chi_{I_i}\| \\
&\leqslant \sum_{i\in S_1} \|T\| \cdot \|f(\xi_i)\chi_{I_i} - f\chi_{I_i}\| + \frac{\varepsilon}{3} + \sum_{i\in S_2} \|T\| \|f\chi_{I_i}\| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},
\end{aligned}$$

which yields

$$\sum_{i=1}^{p} \left| f(\xi_i) g_0(\xi_i)|J_i| - T(f\chi_{J_i}) \right| < \varepsilon.$$

Since T induces an additive interval function $T(f\chi)$ on $\mathcal{I}$, it follows from Definition 3.2 that fg_0 is strongly ϱ-integrable on E and

$$\int_E f g_0 = T(f).$$

Finally, the equality $\|T\| = \mathrm{Var}(g_0, E)$ follows from Theorems 3.8 and 4.7. The proof is complete. □

Recall that when $\varrho \equiv 0$, the strong ϱ-integral reduces to the Henstock-Kurzweil integral. In view of this observation, we are now ready to generalize a remarkable result of Kurzweil [4, Theorem 2.10] which says that every function $g \in \mathcal{SBV}(E)$ is a multiplier for $\mathcal{HK}(E)$.

Theorem 5.3. *Let $f \in \mathcal{S}_\varrho(E)$ and let $\widetilde{F}$ be the distribution function of the indefinite strong ϱ-integral of f. If $g \in \mathcal{SBV}(E)$, then $fg \in \mathcal{S}_\varrho(E)$ and*

$$\begin{aligned}\int_E fg &= \widetilde{F}(b_1, b_2, \ldots, b_m) g(b_1, b_2, \ldots, b_m) \\ &- \sum_i \int_{a_i}^{b_i} \widetilde{F}(b_1, b_2, \ldots, b_{i-1}, x_i, b_{i+1}, \ldots, b_m)\,\mathrm{d}g_i(x) \\ &+ \sum_{i,j} \int_{[a_i,b_i]\times[a_j,b_j]} \widetilde{F}(b_1, b_2, \ldots, b_{i-1}, x_i, b_{i+1}, \ldots, b_{j-1}, x_j, b_{j+1}, \ldots, b_m)\,\mathrm{d}g_{i,j}(x) \\ &- \sum_{i,j,k} + \sum_{i,j,k,l} + \ldots \\ &+ (-1)^{m-1} \sum_k \int_{\prod\limits_{i=1}^{k-1}[a_i,b_i]\times \prod\limits_{i=k+1}^{m}[a_i,b_i]} \widetilde{F}(x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_m)\,\mathrm{d}g_{(k)}(x) \\ &+ (-1)^m \int_E \widetilde{F}(x)\,\mathrm{d}g(x)\end{aligned}$$

where $g_{(k)} := g_{1,2,\ldots,k-1,k+1,\ldots,m}$.

Proof. We shall first obtain a bounded linear functional T_0 on $(\mathcal{S}_\varrho(E), \|\cdot\|)$. Define $T\colon \mathcal{L}(E) \longrightarrow \mathbb{R}$ by

$$T(f) = (L)\int_E fg.$$

It follows from the Hahn-Banach Theorem that T can be extended to a bounded linear functional T_0 on $(\mathcal{S}_\varrho(E), \|\cdot\|)$. By Theorem 5.2, there exists $g_0 \in \mathcal{SBV}(E)$ such that

$$T_0(f) = \int_E fg_0$$

for every $f \in \mathcal{S}_\varrho(E)$. In order to prove that g is a multiplier for $\mathcal{S}_\varrho(E)$, it suffices to prove that $g = g_0$ almost everywhere on E. Since T is extended to a bounded linear functional T_0, we have

$$(L)\int_E fg = \int_E fg_0$$

for all $f \in \mathcal{L}(E)$. Consequently, $g = g_0$ almost everywhere on E, proving that $fg \in \mathcal{S}_\varrho(E)$. In view of Theorem 3.8 and the uniform convergence theorem for the Riemann-Stieltjes integral, the integration by parts formula follows from Lemma 4.5. □

Remark 5.4. It was first shown in [8, Theorem 5.1] that functions of strongly bounded variation and those equivalent to them are the only multipliers for $\mathcal{HK}(E)$. We can now generalize this result from the modern point of view. By following the proof of [9, Theorem 4.4], we see that if g is a multiplier for $\mathcal{S}_\varrho(E)$, then the linear functional $T\colon (\mathcal{S}_\varrho(E), \|\cdot\|) \longrightarrow \mathbb{R}$ defined by

$$T(f) = \int_E fg$$

for every $f \in \mathcal{S}_\varrho(E)$ must be $\|\cdot\|$-bounded. Consequently, it follows from Theorem 5.2 that there exists $g_0 \in \mathcal{SBV}(E)$ such that $g = g_0$ almost everywhere on E.

Alternatively, according to [9, Theorem 4.7], every multiplier for $\mathcal{S}_\varrho(E)$ is almost everywhere equal to some function of strongly bounded variation on E.

In conclusion, fg is strongly ϱ-integrable on E for each strongly ϱ-integrable function f if and only if g is almost everywhere equal to a function of strongly bounded variation on E. This extends the corresponding one-dimensional result of Sargent [15]. In other words, we have characterized the multipliers for the strong ϱ-integral.

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