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SEQUENTIALLY COMPLETE INDUCTIVE LIMITS AND REGULARITY

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Abstract. A notion of an almost regular inductive limits is introduced. Every sequentially complete inductive limit of arbitrary locally convex spaces is almost regular.

Keywords: sequential completeness, regular, resp. almost regular, inductive limit of locally convex spaces

MSC 2000: 46A13, 46A30

1. Introduction

Throughout the paper $E_1 \subset E_2 \subset \ldots$ is a sequence of Hausdorff locally convex spaces with respective topologies τ_n and continuous identity maps $E_n \rightarrow E_{n+1}$, $n \in \mathbb{N}$. Their locally convex inductive limit $\operatorname{ind} E_n$, resp. inductive topology $\operatorname{ind} \tau_n$, is for brevity denoted by E, resp. τ. We also assume E to be Hausdorff.

If X is locally convex space with a topology α and $A \subset X$, we denote the closure of A in X by $\operatorname{cl}_\alpha A$ or $\operatorname{cl}_X A$, and strong dual of X by X'.

Definition. An inductive limit $\operatorname{ind} E_n$ is called *almost regular* if for any set B, bounded in $\operatorname{ind} E_n$, there exists $n \in \mathbb{N}$ such that for any 0-nbhd $U \in \tau_n$, the closure $\operatorname{cl}_\tau U$ absorbs B.

Lemma 1. *Let X be a locally convex space, Y its completion, U a closed 0-nbhd in X, $V = \operatorname{cl}_Y U$, $x \in X$, $x \notin U$, and $B \subset X$. Then:*

(a) $x \notin V$,

(b) *B is bounded in X iff it is bounded in Y.*

P r o o f. (a) Take $f \in X'$ such that $f(x) > 1$ and $f(U) \subset (-\infty, 1]$. Let $g \in Y'$ be the continuous extension of f to Y. Then $g(x) = f(x) > 1$ and $g(V) \subset (-\infty, 1]$. Hence $x \notin V$.

(b) Any set bounded in X is also bounded in Y. Let a set $B \subset X$ be bounded in Y and U be a 0-nbhd in X. Then $V = \mathrm{cl}_Y U$ is a 0-nbhd in Y and there exists $\lambda > 0$ such that $B \subset \lambda V$. This implies $B = B \cap X \subset \lambda V \cap X = \lambda U$. Hence B is absorbed by U. $\square$

Lemma 2. *Given* $\operatorname{ind} E_n$ *and for any* $n \in \mathbb{N}$, *a* 0*-nbhd* $U_n \in \tau_n$. *Put* $V_n = \mathrm{cl}_\tau U_n$ *and assume that for any* $k, n \in \mathbb{N}$, *there is* $x_{kn} \in E$ *such that* $x_{kn} \notin kV_n$. *For any* $k, n \in \mathbb{N}$, *pick a* τ*-closed* 0*-nbhd* $W_{kn} \in \tau$ *such that* $(x_{kn} + W_{kn}) \cap kV_n = \emptyset$. *Put* $\mathscr{V}_n = \{V_m ;\ m \geqslant n\}$, $\mathscr{W} = \{W_{kn} ;\ k, n \in \mathbb{N}\}$, *and* $M = \bigcap\{(1/n)W ;\ n \in \mathbb{N}, W \in \mathscr{W}\}$. *For any* $n \in \mathbb{N}$, *denote by* X_n *the vector space* $\mathrm{cl}_\tau E_n$ *equipped with the topology generated by the subbasis, (see* [1]*),* $\mathscr{V}_n \cup \mathscr{W}$, *by* Y_n *the quotient space* X_n/M, *and by* π_n *the canonical projection* $\mathrm{cl}_\tau E_n \to \mathrm{cl}_\tau E_n/M$. *Then for any* $k, n \in \mathbb{N}$, *the space* Y_n *is a metrizable locally convex space and* $(x_{kn} + M) \cap k\pi_n V_n = \emptyset$.

P r o o f. For any $n \in \mathbb{N}$, denote by F_n the vector space $\mathrm{cl}_\tau E_n$ with the topology generated by the subbasis $\mathscr{W}$. Then each quotient space F_n/M is Hausdorff. The space Y_n is also Hausdorff since its topology is stronger than that of F_n/M. The topology of Y_n has a countable subbasis, hence Y_n is metrizable.

The last statement in the lemma is evident. $\square$

Lemma 3. *Let* $\operatorname{ind} E_n$, *of arbitrary locally convex spaces, be sequentially complete and* B *an absolutely convex, bounded, and closed set in* $\operatorname{ind} E_n$. *Then there exist* $\lambda > 0$ *and* $m \in \mathbb{N}$ *such that* $B \subset \lambda\, \mathrm{cl}_\tau(B \cap E_m)$.

P r o o f. Let $B_n = \mathrm{cl}_\tau(B \cap E_n)$, $n \in \mathbb{N}$. Denote by F, resp. F_n, the linear span of B, resp. B_n, equipped with the topology generated by the basis $\{k^{-1}B ;\ k \in \mathbb{N}\}$, resp. $\{k^{-1}B_n ;\ k \in \mathbb{N}\}$. By [4, Prop. 1], the space F, as well as all spaces F_n, are Banach. The topology of each F_n is the same as that inherited from F and $F = \bigcup\{F_n ;\ n \in \mathbb{N}\}$. Hence $F = \operatorname{ind} F_n$ is a strict inductive limit and the identity map $F \to \operatorname{ind} F_n$ is continuous. By [3, cor. IV, 6.5], there exists $m \in \mathbb{N}$ such that $F = F_m$ and both spaces have the same topology. Since the set B is bounded in F, there exists $\lambda > 0$ such that $B \subset \lambda B_m$. $\square$

Theorem. *Any sequentially complete* $\operatorname{ind} E_n$ *of arbitrary locally convex spaces* E_n, $n \in \mathbb{N}$, *is almost regular.*

P r o o f. Assume that $\operatorname{ind} E_n$ is sequentially complete but not almost regular. Then there exists a set B, bounded in $\operatorname{ind} E_n$, such that for any $n \in \mathbb{N}$ there is a 0-nbhd $U_n \in \tau_n$ whose closure $\mathrm{cl}_\tau U_n$ does not absorb B. We may assume that B is absolutely convex and τ-closed. By Lemma 3, there exists $m \in \mathbb{N}$ such that $\mathrm{cl}_\tau(B \cap E_m)$ absorbs B. Without loss of generality we may assume $m = 1$.

Since $\mathrm{cl}_\tau U_n$ does not absorb B, there exist, for any $k \in \mathbb{N}$, a point $x_{kn} \in B$ and a τ-closed 0-nbhd $W_{kn} \in \tau$ such that $(x_{kn} + W_{kn}) \cap k\,\mathrm{cl}_\tau U_n = \emptyset$. Further, we use the same notation as in Lemma 2.

For any $n \in \mathbb{N}$, the completion Z_n of Y_n is a Fréchet space, $Z_1 \subset Z_2 \subset \ldots$, and the identity maps $Z_n \to Z_{n+1}$ are continuous. The projection $\pi\colon E_n \to Y_n$, $n \in \mathbb{N}$, is continuous. Hence $\pi\colon \operatorname{ind} E_n \to \operatorname{ind} Y_n$ is continuous, too, and the set πB is bounded in $\operatorname{ind} Y_n$ as well as in $\operatorname{ind} Z_n$.

By [3, cor. IV, 6.5] the closure of πB in the topology of $\operatorname{ind} Z_n$ is bounded in some space Z_m. Hence πB is also bounded in Z_m. By Lemma 1, πB is bounded in Y_m. This implies that πB is absorbed by πV_m. But it follows from Lemma 2, that for any $k \in \mathbb{N}$, $\pi x_{km} \in \pi B \setminus k\pi V_m$. We got a contradiction. $\square$

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