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ON EQUITORSION HOLOMORPHICALLY PROJECTIVE MAPPINGS OF GENERALIZED KÄHLERIAN SPACES

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Abstract. In this paper we investigate holomorphically projective mappings of generalized Kählerian spaces. In the case of equitorsion holomorphically projective mappings of generalized Kählerian spaces we obtain five invariant geometric objects for these mappings.

Keywords: Generalized Riemannian space, Kählerian space, generalized Kählerian space, holomorphically projective mapping, equitorsion holomorphically projective mapping, holomorphically projective parameter, holomorphically projective tensor

MSC 2000: 53B05

1. Introduction

A generalized Riemannian space $GR_N$ in the sense of Eisenhart’s definition [1] is a differentiable $N$-dimensional manifold, equipped with a nonsymmetric basic tensor $g_{ij}$. Connection coefficients of this space are generalized Cristoffel’s symbols of the second kind. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

In a generalized Riemannian space one can define four kinds of covariant derivatives [3], [4]. For example, for a tensor $a^i_j$ in $GR_N$ we have

$$a^i_j|_1 = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{jm}a^i_p, \quad a^i_j|_2 = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{mj}a^i_p.$$  

$$a^i_j|_3 = a^i_{j,m} + \Gamma^i_{pm}a^p_j - \Gamma^p_{mj}a^i_p, \quad a^i_j|_4 = a^i_{j,m} + \Gamma^i_{mp}a^p_j - \Gamma^p_{jm}a^i_p.$$  

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In the case of the space $GR_N$ we have five independent curvature tensors [6] (in [6] $\tilde{R}$ is denoted by $\tilde{R}$):

$$
R_{1 jmn}^i = \Gamma_{jm,n}^i - \Gamma_{jn,m}^i + \Gamma_{j}^p \Gamma_{pn}^i - \Gamma_{jn}^p \Gamma_{pm}^i,
$$

$$
R_{2 jmn}^i = \Gamma_{mj,n}^i - \Gamma_{nj,m}^i + \Gamma_{m}^p \Gamma_{np}^i - \Gamma_{nj}^p \Gamma_{mp}^i,
$$

$$
R_{3 jmn}^i = \Gamma_{jm,n}^i - \Gamma_{nj,m}^i + \Gamma_{j}^p \Gamma_{np}^i - \Gamma_{nj}^p \Gamma_{pm}^i + \Gamma_{nm}^p (\Gamma_{pj}^i - \Gamma_{jp}^i),
$$

$$
R_{4 jmn}^i = \Gamma_{jm,n}^i - \Gamma_{nj,m}^i + \Gamma_{j}^p \Gamma_{np}^i - \Gamma_{nj}^p \Gamma_{pm}^i + \Gamma_{mn}^p (\Gamma_{pj}^i - \Gamma_{jp}^i),
$$

$$
R_{5 jmn}^i = \frac{1}{2} (\Gamma_{jm,n}^i + \Gamma_{mj,n}^i - \Gamma_{jn,m}^i - \Gamma_{nj,m}^i + \Gamma_{jm}^p \Gamma_{pn}^i + \Gamma_{mj}^p \Gamma_{np}^i - \Gamma_{jn}^p \Gamma_{mp}^i - \Gamma_{nj}^p \Gamma_{pm}^i).
$$

Kählerian spaces and their mappings were investigated by many authors, for example K. Yano [11], [12], M. Prvanović [9], N. S. Sinyukov [10], J. Mikeš [2] and many others.

In [7] we defined a generalized Kählerian space $GK_N$ as a generalized $N$-dimensional Riemannian space with a (nonsymmetric) metric tensor $g_{ij}$ and an almost complex structure $F_{i}^{h}(x)$ such that

$$
F_{p}^{h}(x)F_{i}^{p}(x) = -\delta_{i}^{h},
$$

$$
g_{pq}F_{i}^{p}F_{j}^{q} = g_{ij}, \quad g^{ij} = g_{pq}F_{i}^{p}F_{j}^{q},
$$

$$
F_{i}^{h}|_{\theta}^{j} = 0 \quad (\theta = 1, 2),
$$

where $|_{\theta}$ denotes the covariant derivative of the kind $\theta$ with respect to the metric tensor $g_{ij}$, and $ij$ denotes the symmetrization over $i, j$.

Generalizing the concept of an analytic planar curve in a Kählerian space [8], [10] we obtained in [7] an analogous notion for a generalized Kählerian space.

A curve

$$
l: \ x^{h} = x^{h}(t) \quad (h = 1, 2, \ldots, N)
$$

is said to be analytic planar if the relation

$$
\frac{d\lambda^{h}}{dt} + \Gamma_{pq}^{h} \lambda^{p} \lambda^{q} = a(t)\lambda^{h} + b(t)F_{i}^{h} \lambda^{p}
$$

is satisfied, where $\lambda^{h} = dx^{h}/dt$ and $a(t), b(t)$ are functions of the parameter $t$.

Let us consider two $N$-dimensional generalized Kählerian spaces $GK_N$ and $G\overline{K}_N$ with almost complex structures $F_{i}^{h}$ and $F_{i}^{h}$, respectively. We introduce a common
by the mapping $f$ coordinate system, i.e. if $M \in GK_N$, $\overline{M} = f(M) \in G\overline{K}_N$ and at the local carte $(\mathcal{U}, \varphi)$ we have $\varphi(M) = (x^1, x^2, \ldots, x^N) \in R^N$ then at the local carte $(\mathcal{U}, \overline{\varphi} = \varphi \circ f^{-1})$ we have

$$\overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})(\overline{M}) = \varphi(M) = (x^1, x^2, \ldots, x^N).$$

We suppose that

$$F^h_i = \overline{F}^h_i$$

in the common by the mapping $f : GK_N \rightarrow G\overline{K}_N$ coordinate system.

Diffeomorfism $f : GK_N \rightarrow G\overline{K}_N$ is holomorphically projective or analytic planar [7] if it maps analytic planar curves of the space $GK_N$ into analytic planar curves of the space $G\overline{K}_N$. In this case, Cristoffel’s symbols of the second kind of these spaces satisfy the relations [7]

$$(1.7) \quad \Gamma^h_{ij} = \Gamma^h_{ij} + \psi(i\delta^h_j) + \sigma(iF^h_j) + \xi^h_{ij},$$

where $(ij)$ denotes a symmetrization without division by indices $i$, $j$ and $\xi^h_{ij}$ is an anti-symmetric tensor. In (1.7) the vector $\sigma_i$ can be selected such that $\sigma_i = -\psi_pF^p_i$. Then we have

$$(1.8) \quad \Gamma^h_{ij} = \Gamma^h_{ij} + \psi(i\delta^h_j) - \psi_pF^p_iF^h_j + \xi^h_{ij}.$$

2. Equitorsion holomorphically projective mappings

Let $f : GK_N \rightarrow G\overline{K}_N$ be a holomorphically projective mapping, and let the torsion tensor $\Gamma^h_{ij} \big|_{\overline{\mathcal{U}}}$ and $\Gamma^h_{ij} \big|_{\mathcal{U}}$ of the spaces $GK_N$ and $G\overline{K}_N$ satisfy the condition

$$(2.1) \quad \overline{\Gamma}^h_{ij} = \Gamma^h_{ij}.$$

In this case the mapping $f$ is called an equitorsion holomorphically projective mapping of the spaces $GK_N$ and $G\overline{K}_N$. Then (1.7) implies

$$(2.2) \quad \xi^h_{ij} = 0.$$

2.1. Holomorphically projective parameters of the first kind.

Curvature tensors of the first kind $R$ and $\overline{R}$ of the spaces $GK_N$ and $G\overline{K}_N$, respectively, are connected by the relation [5]

$$(2.3) \quad \overline{R}^i_{jmn} = R^i_{jmn} + P^i_{jm|n} - P^i_{jn|m} + P^p_{jm} P^i_{pn} - 2\Gamma^p_{mn} P^i_{jp}.$$
where $P_{ij}^h = \Gamma_{ij}^h - \Gamma_{ij}^h$ is a deformation tensor. Substituting (1.8) and (2.2) into (2.3) we get

\begin{equation}
\frac{\overline{R}}{R} = R_{jmn} + \delta_m^{[i} \psi_{j\nu]n} + \delta_i^{[i} \psi_{jmn] \delta_j^{(pq)} F_p^i F_q^j = R_{1 (ij)} - 2 \Gamma_{jc}^{(ij)} F_p^c F_q^c F_p^c F_q^c,
\end{equation}

where we denote

\begin{equation}
\psi_{ij}^1 = \psi_{ij}^1 - \psi_i^1 \psi_j^1 + \psi_p^1 F_q^p F_q^q.
\end{equation}

Contracting with respect to indices $i, n$ in (2.4) we obtain

\begin{equation}
\overline{R}_{jm} = R_{jm} + \psi_{[mj]}^1 - N \psi_{jm} - F_p^q F_m^q \psi_{(pq)} + 2 \Gamma_{mc}^p \psi_p^2 F_q^q F_p^c - 2 \Gamma_{mc}^p \psi_p^2 F_q^q F_p^c.
\end{equation}

Anti-symmetrization without division in (2.6) with respect to indices $j, m$ yields

\begin{equation}
(N + 2) \psi_{jm} = R_{[jm]} - \overline{R}_{[jm]} + 4 \Gamma_{mc}^p \psi_p^q F_q^q F_p^c - 2 \Gamma_{mc}^p \psi_p^q F_q^q F_p^c + 2 \Gamma_{mc}^p \psi_p^q F_q^q F_p^c.
\end{equation}

Symmetrizing without division with respect to $j, m$ in (2.6) we arrive at

\begin{equation}
\overline{R}_{(jm)} = R_{(jm)} - N \psi_{(jm)} - F_p^q F_m^q \psi_{(pq)} - 2 \Gamma_{mc}^p \psi_p^q F_q^q F_p^c - 2 \Gamma_{mc}^p \psi_p^q F_q^q F_p^c.
\end{equation}

For the spaces $GK_N$ and $\overline{GK}_N$ the relations [7]

\begin{equation}
R_{(pq)} F_i^p F_j^q = R_{(ij)} - 2 \Gamma_{rqs} F_j^r F_m^s + 2 \Gamma_{jq}^r \Gamma_{pm}^q
\end{equation}

and

\begin{equation}
\overline{R}_{(pq)} F_i^p F_j^q = \overline{R}_{(ij)} - 2 \Gamma_{rqs} F_j^r F_m^s + 2 \Gamma_{jq}^r \Gamma_{pm}^q
\end{equation}

are valid, respectively.
By composition with \( F^j_p F^m_q \), contraction with respect to \( j, m \), and using the conditions (2.9), (2.10) we get from (2.8)

\[
(2.11) \quad \overline{R}_{1(jm)} = R_{1(jm)} - N \psi_{(pq)} F^p_j F^m_q - 2\psi_{(jm)} + 2\Gamma^p_{qr} \psi_j F^q_p F^m_q \\
+ 2\Gamma^p_{jr} \psi_m F^r_j + 2\Gamma^p_{rq} \psi_j F^q_p F^r_m + 2\Gamma^p_{qr} \psi_m F^q_p F^r_j.
\]

From (2.8) and (2.11) we have

\[
(2.12) \quad (N - 2) F^p_j F^m_q \psi_{(pq)} = (N - 2) \psi_{(jm)} + 2\Gamma^p_{mr} \psi_q F^q_p F^r_j \\
+ 2\Gamma^p_{jr} \psi_q F^q_m F^r_p + 2\Gamma^p_{mr} \psi_q F^q_p F^r_j + 2\Gamma^p_{qr} \psi_m F^q_p F^r_j.
\]

Substituting (2.12) into (2.9) we obtain

\[
(2.13) \quad (N + 2) \psi_{1(jm)} = R_{1(jm)} - \overline{R}_{1(jm)} - \frac{2}{N - 2} (N \Gamma^p_{mr} \psi_q F^q_j F^r_p \\
+ N \Gamma^p_{jr} \psi_q F^q_m F^r_p + 2\Gamma^p_{mr} \psi_q F^q_j F^r_j + 2\Gamma^p_{qr} \psi_m F^q_p F^r_j) \\
- 2\Gamma^p_{mr} \psi_q F^q_j F^r_p - 2\Gamma^p_{jr} \psi_q F^q_m F^r_p
\]

Using (2.7) and (2.13) we get

\[
(2.14) \quad (N + 2) \psi_{1 jm} = R_{1 jm} - \overline{R}_{1 jm} + 2\Gamma^p_{qr} \psi_q F^q_m F^r_p \\
- \frac{2}{N - 2} \Gamma^p_{jr} \psi_q F^q_m F^r_p - \frac{2}{N - 2} \Gamma^p_{mr} \psi_q F^q_j F^r_p \\
- \frac{2}{N - 2} \Gamma^p_{mr} \psi_q F^q_m F^r_p - 2\Gamma^p_{mr} \psi_q F^q_p F^r_j.
\]

Eliminating \( \psi_i \) by using the condition

\[
(2.15) \quad \overline{\Gamma}_{pj}^p - \Gamma_{pj}^p = (N + 2) \psi_j
\]

we reduce the equation (2.14) to the form

\[
(2.16) \quad (N + 2) \psi_{1 jm} = R_{1 jm} - \overline{R}_{1 jm} + \overline{P}_{1 jm} - P_{1 jm},
\]

where we denote

\[
(2.17) \quad P_{1 jm} = \frac{2}{N + 2} \left( \Gamma^p_{mr} \Gamma^q_{sp} - \frac{N - 1}{N - 2} \Gamma^p_{mr} \Gamma^q_{sp} F^r_p F^q_j - \frac{1}{N - 2} \Gamma^p_{qr} \Gamma^q_{sm} F^r_p F^q_j \\
- \frac{1}{N - 2} \Gamma^p_{qr} \Gamma^q_{sp} F^r_p F^q_m - \frac{1}{N - 2} \Gamma^p_{qr} \Gamma^q_{sm} F^r_p F^q_j - \frac{1}{N - 2} \Gamma^p_{mr} \Gamma^q_{sp} F^r_p F^q_m \\
- \frac{1}{N - 2} \Gamma^p_{qr} \Gamma^q_{sp} F^r_p F^q_j \right).
\]

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In the same manner the geometric objects $P_{1 \, jm}$ of the space $GKN$ is defined. Eliminating $\psi_{1 \, jm}$ from (2.4) we obtain

\begin{equation}
HPW_1 \, i \, jmn = HPW_1^i \, jmn,
\end{equation}

where the magnitude

\begin{equation}
HPW_1^i \, jmn = R_1^i \, jmn + \frac{1}{N + 2} \left[ \delta^i_m (R_1^m - P_1^m) + \delta^j_n (R_1^mn - P_1^mn) \right] \\
- \delta^i_n (R_1^m - P_1^m) + P_p^p F_n^i (R_1^pm - P_1^pm) - F_j^p F_m^i (R_1^pm - P_1^pm) \\
+ F_j^p F_n^i (R_1^pm - P_1^pm) - F_j^p F_m^i (R_1^pm - P_1^pn) - 2 \Gamma_{n^m}^q \Gamma_{qj}^p - 2 \delta^i_j \Gamma_{mn}^p \Gamma_{qp} \\
+ 2 \Gamma_{n^m}^s \Gamma_{sq}^s F_{j}^q F_{i}^p + 2 \Gamma_{n^m}^s \Gamma_{sq}^s F_{j}^q F_{i}^p
\end{equation}

is expressed by geometric objects of the space $GKN$. In the same manner the magnitude $HPW_1^i \, jmn$ is expressed by geometric objects of the space $GKN$. The magnitude $HPW_1^i \, jmn$ is not a tensor, and we call it the equitorsion holomorphically projective parameter of the first kind of the space $GKN$. From the facts given above, we have

\textbf{Theorem 2.1.} The equitorsion holomorphically projective parameter (2.19) is an invariant of equitorsion holomorphically projective mappings $f : GKN \to GKN$.

\subsection*{2.2. Holomorphically projective parameters of the second kind.}

For the curvature tensors $R_2$ and $\overline{R}$ of the spaces $GKN$ and $GKN$ the relation

\begin{equation}
\overline{R}^i_{2 \, jmn} = \overline{R}^i_{2 \, jmn} + P^n_{m \, j^2} - P^n_{n \, j^2} + P^n_{m \, j^2} P^n_{n \, p} - P^n_{p \, m} P^n_{n \, p} + 2 \Gamma_{n^m}^q P^n_{p \, j^q}
\end{equation}

is valid [5], where $P^i_{jm}$ is a deformation tensor. Substituting (1.8) and (2.2) into (2.20) we get

\begin{equation}
\overline{R}^i_{2 \, jmn} = R^i_{2 \, jmn} + \delta^i_m \psi_{jn} + \delta^i_n \psi_{jm} - \delta^i_n \psi_{jm} \\
+ F^n_j (F^n_i \psi_{jn} - F^n_i \psi_{jn}) + F^n_j (F^n_i \psi_{jn} - F^n_i \psi_{jn}) \\
+ 2 \Gamma_{n^m}^q \psi_{p} \psi_j + 2 \Gamma_{n^m}^q \psi_{p} \psi_j - 2 \Gamma_{n^m}^q \psi_{p} \psi_j F^n_{j^q} F^n_{i^p} - 2 \Gamma_{n^m}^q \psi_{p} \psi_j F^n_{j^q} F^n_{i^p}
\end{equation}

where we denote

\begin{equation}
\psi_{ij} = \psi_{i^j} - \psi_{i^j} \psi_{j} + \psi_{p} F^n_{i^p} \psi_{q} F^n_{j^q}.
\end{equation}
Contracting by indices $i$, $n$ in (2.21) we arrive at

\[
(2.23) \quad \overline{R}_{jm}^2 = R_{jm}^2 + \psi_j^{(pq)} - N\psi_j^{(pq)} - F_{jm}^p F_{m}^q \\
+ 2\Gamma_{jm}^q \psi_q - 2\Gamma_{jm}^q \psi_q F_{jm}^q F_{m}^q - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p.
\]

Anti-symmetrization without division in (2.23) with respect to indices $j$, $m$ yields

\[
(2.24) \quad (N + 2)\psi_j^{(pq)} = R_{jm}^2 - \overline{R}_{jm}^2 + 4\Gamma_{jm}^q \psi_p - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p \\
+ 2\Gamma_{jm}^q \psi_q F_{jm}^r F_{m}^r - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p + 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p.
\]

Symmetrizing with respect to indices $j$, $m$ in (2.23) we get

\[
(2.24') \quad \overline{R}_{(jm)}^2 = R_{(jm)}^2 - N\psi_j^{(pq)} - 2\Gamma_{jm}^q F_{jm}^m F_{m}^q - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p \\
- 2\Gamma_{jm}^q \psi_q F_{jm}^r F_{m}^r - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p + 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p.
\]

For the spaces $GK_N$ and $G\overline{K}_N$ the following relations are valid [7]:

\[
(2.25) \quad R_{(pq)}^2 F_j^p F_j^q = R_{(ij)}^2 - 2\Gamma_{pq}^q \psi_j^F F_j^m F_j^m + 2\Gamma_{pq}^q \psi_j^F F_j^m F_j^m
\]

and

\[
(2.26) \quad \overline{R}_{(pq)}^2 F_j^p F_j^q = \overline{R}_{(ij)}^2 - 2\Gamma_{pq}^q \psi_j^F F_j^m F_j^m + 2\Gamma_{pq}^q \psi_j^F F_j^m F_j^m.
\]

By composition with $F_{jm}^p F_{m}^p$, contracting with respect to indices $j$, $m$, and using (2.25), (2.26) we obtain from (2.24') the relation

\[
(2.27) \quad \overline{R}_{(jm)}^2 = R_{(jm)}^2 - N\psi_j^{(pq)} F_{jm}^p F_{m}^q - 2\psi_j^{(pq)} + 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p \\
+ 2\Gamma_{jm}^q \psi_q F_{jm}^r F_{m}^r + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m.
\]

From (2.24') and (2.27) we get

\[
(2.28) \quad (N - 2)F_j^p F_{m}^p \psi_j^{(pq)} = (N - 2)\psi_j^{(pq)} + 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p \\
+ 2\Gamma_{jm}^q \psi_q F_{jm}^r F_{m}^r + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m.
\]

Substituting (2.28) in (2.27) we conclude that

\[
(2.29) \quad (N + 2)\psi_j^{(pq)} = R_{(jm)}^2 - \overline{R}_{(jm)}^2 - \frac{2}{N - 2}(N\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p \\
+ N\Gamma_{jm}^q \psi_q F_{jm}^r F_{m}^r + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m + 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m \\
- 2\Gamma_{jm}^q \psi_j^F F_j^m F_j^m - 2\Gamma_{jm}^q \psi_q F_{jm}^m F_{m}^p).
\]

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From (2.24) and (2.29) we get

\[(N + 2)\psi_{jm} = R_{2jm} - \overline{R}_{2jm} + 2\Gamma_j^p \psi_p + 2\Gamma_j^p \psi_p F^q_r F^r_q - 2N - \Gamma_j^p \psi_j F^q_r F^r_q - 2\Gamma_j^p \psi_j F^q_r F^r_q.
\]

Eliminating \(\psi_i\) by condition (2.15) we reduce the last equation to the form

\[(N + 2)\psi_{jm} = R_{2jm} - \overline{R}_{2jm} + P_{2jm} - P_{2jm},\]

where we denote

\[(2.32) P_{2jm} = \frac{2}{N + 2}\left(\Gamma_j^p \Gamma_{qp} \Gamma^s_{sq} F^q_p F^r_q - \frac{N - 1}{N - 2} \Gamma_j^p \Gamma^s_{sq} F^q_p F^r_q - \frac{1}{N - 2} \Gamma_j^p \Gamma^s_{sq} F^q_p F^r_q - \frac{1}{N - 2} \Gamma_j^p \Gamma^s_{sm} F^q_p F^r_q \right).
\]

Eliminating \(\psi_{jm}\) from (2.21) we get

\[(2.33) HPW^i_{jm} = HPW^i_{jm},\]

where

\[(2.34) HPW^i_{jm} = R_{jm}^i + \frac{1}{N + 2} \left[\delta^i_{jm}(R_{jm} - P_{jm}) + \delta^i_{jm}(R_{jm} - P_{jm}) \right] - \delta^i_{jm}(R_{jm} - P_{jm}) + F^i_{jm} R_{jm} - P_{jm} \right] - F^i_{jm} R_{jm} - P_{jm} \right] + F^i_{jm} R_{jm} - P_{jm} \right] + 2\delta^i_{jm} \Gamma^q_{jm} F^q_j - 2\Gamma^q_{jm} F^q_j.
\]

The magnitude \(HPW^i_{jm}\) is not a tensor, and we call it the equitorsion holomorphically projective parameter of the second kind of the space \(G_{KN}\). From the facts given above, we have

**Theorem 2.2.** The equitorsion holomorphically projective parameter of the second kind is an invariant of equitorsion holomorphically projective mappings of the spaces \(G_{KN}\) and \(\overline{G_{KN}}\).
2.3. Holomorphically projective parameters of the third kind.

The curvature tensors $R$ and $\overline{R}$ of the space $GKN$ and $G\overline{K}N$ satisfy the relation [5]

$$R_{3jnm}^i = R_{3jmn}^i + P_{jm|n}^i - P_{nj|m}^i + P_{jm}^p P_{np}^i - P_{nj}^p P_{pm}^i$$

$$+ 2P_{nm}^p \Gamma_{pj}^i + 2\Gamma_{nm}^p P_{pm}^i.$$  (2.35)

Substituting (1.8) and (2.2) in (2.35) we get

$$\overline{R}_{3jmn}^i = R_{3jmn}^i + \delta_m^i \psi_{jn} + \delta_j^i (\psi_{mn} - \psi_{nm}) - \delta_n^i \psi_{jm}$$

$$+ F_j^l (F_{n1}^i \psi_{pm} - F_{m1}^i \psi_{pn}) + F_j^l (F_{n1}^p \psi_{pm} - F_{m1}^p \psi_{pn})$$

$$+ 2\Gamma_{mj}^i \psi_n + 2\Gamma_{nj}^i \psi_m - 2\Gamma_{pj}^i \psi_q F_{n1}^q F_{m1}^p - 2\Gamma_{pj}^i \psi_q F_{m1}^q F_{n1}^p,$$  (2.36)

where we denote

$$\psi_{ij} = \psi_{ij\theta} - \psi_i \psi_j + \psi_p F_{iq}^p \psi_q F_{j\theta}^q, \quad (\theta = 1, 2).$$

The following equality is also valid [5]:

$$\psi_{[mn]} = \psi_{[mn]} + 2\Gamma_{mn}^p \psi_p.$$  (2.37)

From the equation (2.36) we get

$$\overline{R}_{3jmn}^i = R_{3jmn}^i + \delta_m^i \psi_{jn} + \delta_j^i (\psi_{mn} - \psi_{nm}) - \delta_n^i \psi_{jm}$$

$$+ F_j^l (F_{n1}^i \psi_{pm} - F_{m1}^i \psi_{pn}) + F_j^l (F_{n1}^p \psi_{pm} - F_{m1}^p \psi_{pn})$$

$$+ 2\Gamma_{mj}^i \psi_n + 2\Gamma_{nj}^i \psi_m - 2\Gamma_{pj}^i \psi_q F_{n1}^q F_{m1}^p - 2\Gamma_{pj}^i \psi_q F_{m1}^q F_{n1}^p.$$  (2.38)

Contracting with respect to $i, n$ in (2.38) we arrive at

$$\overline{R}_{3jm} = R_{3jm} + \psi_{[mj]} - N \psi_{jm} - F_j^q F_{m1}^q \psi_{(pq)}$$

$$+ 2\Gamma_{mj}^p \psi_p - 2\Gamma_{pm}^r \psi_q F_{r1}^q F_{m1}^p - 2\Gamma_{pj}^r \psi_q F_{m1}^q F_{r1}^p.$$  (2.39)

By anti-symmetrization without division (2.39) with respect to $j, m$ we get

$$(N + 2) \psi_{[jm]} = R_{3jm} - \overline{R}_{3jm} + 4\Gamma_{mj}^p \psi_p - 2\Gamma_{pj}^r \psi_q F_{r1}^q F_{m1}^p$$

$$+ 2\Gamma_{pm}^q \psi_q F_{r1}^q F_{j1}^r - 2\Gamma_{pj}^r \psi_q F_{m1}^q F_{r1}^p + 2\Gamma_{pm}^q \psi_q F_{j1}^q F_{r1}^p.$$  (2.40)

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Symmetrizing with respect to $j, m$ in (2.39) we obtain

\begin{equation}
R_{3(jm)} = R_{3(jm)} - N\psi_{1(jm)} - 2F_{j}^{p}F_{m}^{q}\psi_{1(pq)} - 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{m}^{p} \\
- 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{j}^{p} - 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} - 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r}.
\end{equation}

By composition with $F_{j}^{p}F_{m}^{q}$, contraction with respect to $j, m$, and using the conditions [7], one obtains

\begin{equation}
R_{3(pq)}F_{i}^{p}F_{j}^{q} = R_{3(ij)} - 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{m}^{s} + 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{m}^{s}.
\end{equation}

From (2.41) we have

\begin{equation}
R_{3(jm)} = R_{3(jm)} - N\psi_{1(pq)}F_{j}^{p}F_{m}^{q} - 2\psi_{1(jm)} + 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{j}^{p} \\
+ 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{m}^{s} + 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{j}^{p} + 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{m}^{s}.
\end{equation}

and (2.41) and (2.44) imply

\begin{equation}
(N - 2)F_{j}^{p}F_{m}^{q}\psi_{1(pq)} = (N - 2)\psi_{1(jm)} + 2\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{j}^{p} \\
+ 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{m}^{s} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r}.
\end{equation}

Substituting (2.45) into (2.44) we have

\begin{equation}
(N + 2)\psi_{1(jm)} = R_{3(jm)} - \overline{R}_{3(jm)} - \frac{2}{N - 2}(N\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{p}^{p} \\
+ NT_{pq}^{q}\psi_{m}F_{r}^{r}F_{p}^{p} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} \\
- 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{m}^{s} - 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r}).
\end{equation}

From (2.40) and (2.46) we get

\begin{equation}
(N + 2)\psi_{jm} = R_{3(jm)} - \overline{R}_{3(jm)} + 2\Gamma_{pq}^{q}\psi_{m} - \frac{2}{N - 2}(N\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{p}^{p} \\
+ N\Gamma_{pq}^{q}\psi_{m}F_{r}^{r}F_{p}^{p} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} + 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r} \\
- 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{m}^{s} - 2\Gamma_{pq}^{q}\psi_{m}F_{p}^{p}F_{r}^{r}).
\end{equation}
Eliminating $\psi_i$ from the last equation we have

(2.48) \[(N + 2)\psi_{jm} = R_{jm} - \overline{R}_{jm} + \overline{P}_{jm} - P_{jm},\]

where we denote

(2.49) \[P^{\text{jm}}_{3} = \frac{2}{N + 2} \left( \Gamma^{p}_{mj} \Gamma^{q}_{qr} - \frac{N - 1}{N - 2} \Gamma^{p}_{pj} \Gamma^{s}_{mq} F^{q}_{m} F^{r}_{r} - \frac{1}{N - 2} \Gamma^{r}_{pm} \Gamma^{s}_{sq} F^{q}_{r} F^{p}_{m} \right).\]

Eliminating $\psi_{jm}$ from (2.36) we obtain

(20.50) \[HPW^{i}_{jm} = HPW^{i}_{jm},\]

where

(2.51) \[HPW^{i}_{jm} = R^{i}_{jm} + \frac{1}{N + 2} \left[ \delta^{i}_{m} (R_{jn} - P_{jm}) + \delta^{i}_{j} (R_{[m]} - P_{[mn]}) \right. \]
\[\left. - \delta^{i}_{j} (R_{3jm} - P_{jm}) + F^{p}_{j} F^{i}_{n} (R_{3pm} - P_{3pm}) - F^{p}_{j} F^{i}_{m} (R_{3pm} - P_{3pm}) \right. \]
\[+ F^{i}_{j} F^{p}_{n} (R_{3pm} - P_{3pm}) - F^{i}_{j} F^{p}_{m} (R_{3pm} - P_{3pm}) - 2\Gamma^{p}_{jn} \Gamma^{q}_{qp} \delta^{i}_{m} \]
\[+ 2\Gamma^{p}_{pj} \Gamma^{s}_{mq} F^{p}_{m} + 2\Gamma^{p}_{pj} \Gamma^{s}_{mq} F^{p}_{m} - 2\Gamma^{i}_{mj} \Gamma^{p}_{pm} \]
\[+ 2\Gamma^{i}_{nj} \Gamma^{p}_{pm} + 2\Gamma^{i}_{nj} \Gamma^{s}_{mq} F^{p}_{m} + 2\Gamma^{i}_{nj} \Gamma^{s}_{mq} F^{p}_{m} \right].\]

The magnitude $HPW^{i}_{jm}$ is not a tensor and we call it the equitorsion holomorphically projective parameter of the third kind of the space $GK_{N}$. By virtue of the facts given above we have the following result:

**Theorem 2.3.** The equitorsion holomorphically projective parameter of the third kind is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces.

### 2.4. Holomorphically projective parameters of the fourth kind.

For curvature tensors $R^{i}_{4}$ and $\overline{R}^{i}_{4}$ of spaces $GK_{N}$ and $GK_{N}$ the following relation is valid [5]:

(2.52) \[\overline{R}^{i}_{jm} = R^{i}_{jm} + P^{i}_{jm|n} - P^{i}_{nj|m} + P^{i}_{jm} P^{i}_{nm} - P^{i}_{nj} P^{i}_{pm} \]
\[+ 2P^{p}_{mn} \Gamma^{i}_{pj} + 2\Gamma^{p}_{mn} P^{i}_{pj}.\]
where $P_{jm}^{i}$ is the deformation tensor. Substituting (1.8) and (2.2) in (2.52) we have

$$\overline{R}_{4}^{i}_{jm} = \frac{R_{4}^{i}_{jm}}{4} + \frac{1}{2} \left[ \delta_{m}(R_{4}^{i}_{jn} - P_{3}^{i}_{jm}) + \delta_{j}(R_{4}^{i}_{mn} - P_{3}^{i}_{jm}) \right] - \delta_{n}(R_{4}^{i}_{jm} - P_{3}^{i}_{jm}) + F^{p}_{j} F^{i}_{n}(R_{4}^{i}_{pm} - P_{3}^{i}_{pm}) - F^{p}_{j} F^{i}_{m}(R_{4}^{i}_{pn} - P_{3}^{i}_{pm})$$

$$+ 2\Gamma^{i}_{m\nu} \psi_{n} + 2\Gamma^{i}_{n\nu} \psi_{m} - 2\Gamma^{i}_{p\nu} \psi_{q} F^{q}_{m} F^{i}_{n} + 2\Gamma^{i}_{p\nu} \psi_{q} F^{q}_{m} F^{i}_{n}.$$  

In the same manner as in the previous cases we get

$$HPW_{4}^{i}_{jm} = HPW_{4}^{i}_{jm},$$

where we denote

$$HPW_{4}^{i}_{jm} = \frac{1}{N + 2} \left[ \delta^{i}_{m}(R_{4}^{i}_{jn} - P_{3}^{i}_{jm}) + \delta^{i}_{j}(R_{4}^{i}_{mn} - P_{3}^{i}_{jm}) \right] - \delta^{i}_{n}(R_{4}^{i}_{jm} - P_{3}^{i}_{jm}) + F^{p}_{j} F^{i}_{n}(R_{4}^{i}_{pm} - P_{3}^{i}_{pm}) - F^{p}_{j} F^{i}_{m}(R_{4}^{i}_{pn} - P_{3}^{i}_{pm})$$

$$+ 2\Gamma^{i}_{m\nu} \Gamma^{q}_{nq} \delta^{i}_{j} + 2\Gamma^{i}_{n\nu} \Gamma^{q}_{m} F^{q}_{j} F^{i}_{m} + 2\Gamma^{i}_{p\nu} \Gamma^{q}_{m} F^{q}_{j} F^{i}_{n} - 2\Gamma^{i}_{p\nu} \Gamma^{q}_{m} F^{q}_{n} F^{i}_{n}.$$  

The magnitude $HPW_{4}^{i}_{jm}$ is not a tensor either, and we call it the equitorsion projective parameter of the fourth kind of the space $GK_{N}$. In this case we have

**Theorem 2.4.** The equitorsion holomorphically projective parameter of the fourth kind is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces $GK_{N}$ and $GK_{N}$.

### 2.5. Holomorphically projective tensor.

For curvature tensors of the fifth kind $R_{5}$ and $\overline{R}$ of the spaces $GK_{N}$ and $GK_{N}$ the following relation is valid:

$$\overline{R}_{5}^{i}_{jm} = \frac{R_{5}^{i}_{jm}}{5} + \frac{1}{2} \left( P_{5}^{i}_{jm} \right)_{m} - \frac{1}{2} \left( P_{5}^{i}_{jm} \right)_{m} + P_{5}^{i}_{mj} \frac{1}{2} - \frac{1}{2} P_{5}^{i}_{nj} \frac{1}{2} + P_{5}^{i}_{jm} P_{5}^{i}_{pn}$$

$$- P_{5}^{i}_{jn} P_{5}^{i}_{np} + P_{5}^{i}_{mj} P_{5}^{i}_{np} - P_{5}^{i}_{nj} P_{5}^{i}_{pn} + 4\Gamma^{i}_{j} \Gamma^{i}_{mn}.$$  

Substituting (1.8) and (2.2) in (2.56) we have

$$\overline{R}_{5}^{i}_{jm} = \frac{R_{5}^{i}_{jm}}{5} + \delta_{n} \psi_{jm} + \delta_{j} \psi_{jm} - \delta_{n} \psi_{jm}$$

$$+ F^{p}_{j} \left( F_{5}^{i}_{pm} - F_{5}^{i}_{pm} \right) + F^{p}_{j} \left( F_{5}^{i}_{pm} - F_{5}^{i}_{pm} \right).$$
where we denote

\[
\psi_{jm}^{12} = \frac{1}{2}(\psi_{j|m} + \psi_{j|m}^{2}) - \psi_{j}\psi_{m} + \psi_{p}F_{j}^{p}\psi_{q}F_{m}^{q}.
\]

Contracting with respect to \(i, n\) in (2.57) we get

\[
\overline{R}_{5jm} = R_{5jm} - \psi_{[jm]}^{12} - N\psi_{jm}^{12} - F_{j}^{m}F_{m}^{q}\psi_{(pq)}^{12}.
\]

Using anti-symmetrization without division in (2.59) with respect to \(j, m\) we get

\[
(N + 2)\psi_{12[jm]} = \overline{R}_{5[jm]} - \overline{R}_{5[jm]}^{12}.
\]

Symmetrizing without division with respect to \(j, m\) in (2.59) we have

\[
\overline{R}_{5(jm)} = R_{5(jm)} - N\psi_{(jm)}^{12} - 2F_{j}^{m}F_{m}^{q}\psi_{(pq)}^{12}.
\]

By composition with \(F_{j}^{i}F_{m}^{j}\), contracting with respect to \(j, m\), and using the relations

\[
R_{5(pq)}^{i}F_{i}^{p}F_{j}^{q} = R_{5(ij)}^{p} + 2\Gamma_{pq}^{r}\Gamma_{ps}^{q}F_{j}^{r}F_{m}^{s} - 2\Gamma_{pq}^{r}\Gamma_{pm}^{q}
\]

and

\[
\overline{R}_{5(pq)}^{i}F_{i}^{p}F_{j}^{q} = \overline{R}_{5(ij)}^{p} + 2\Gamma_{pq}^{r}\Gamma_{ps}^{q}F_{j}^{r}F_{m}^{s} - 2\Gamma_{pq}^{r}\Gamma_{pm}^{q},
\]

one obtains from (2.61) the equality

\[
\overline{R}_{5(jm)} = R_{5(jm)} - N\psi_{(pq)}^{12}F_{j}^{p}F_{m}^{q} - 2\psi_{12(jm)}^{12}.
\]

From (2.61) and (2.64) we obtain

\[
F_{j}^{p}F_{m}^{q}\psi_{(pq)}^{12} = \psi_{(jm)}^{12}.
\]

Substituting (2.65) in (2.64) we have

\[
(N + 2)\psi_{12(jm)} = R_{5(jm)} - \overline{R}_{5(jm)}^{12}.
\]

From (2.61) and (2.66) one gets

\[
(N + 2)\psi_{jm}^{12} = R_{5jm} - \overline{R}_{5jm}^{12}.
\]
Eliminating $\psi_{jm}$ from (2.57) we have

\begin{equation}
HPW^i_{5 jmn} = HPW^i_{5 jmn},
\end{equation}

where

\begin{equation}
HPW^i_{5 jmn} = R^i_{5 jmn} + \frac{1}{N + 2} [\delta^i_m R_{jn} + \delta^i_j R_{mj} - \delta^i_n R_{jm} \\
+ F^p_j (F^i_n R_{pn} - F^i_m R_{pm}) + F^i_j (F^p_n R_{pm} - F^p_m R_{pm})].
\end{equation}

Contrary to the previous cases the magnitude $HPW^i_{5 jmn}$ is a tensor and we call it the \textit{equitorsion holomorphically projective tensor} of the space $GK_N$.

Based on the facts given above, we have the

\textbf{Theorem 2.5.} The equitorsion holomorphically projective tensor is an invariant of equitorsion holomorphically projective mappings of generalized Kählerian spaces.

\section{2.6. The case of Kählerian spaces.}

In the case of holomorphically projective mappings of Kählerian spaces the magnitudes $HPW^i_{\theta jmn}$, $(\theta = 1, \ldots, 5)$, given by (2.19, 34, 51, 55, 69) reduce to the holomorphically projective curvature tensor [10]

\begin{equation}
HPW^i_{\theta jmn} = R^i_{\theta jmn} + \frac{1}{N + 2} (R_{j[n} \delta^i_m + F^p_j R_{[m} F^i_n] + 2 F^i_j F^p_n R_{pm}).
\end{equation}

\textbf{References}


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