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RIEMANN-TYPE DEFINITION OF THE IMPROPER INTEGRALS

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Abstract. Riemann-type definitions of the Riemann improper integral and of the Lebesgue improper integral are obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

Keywords: McShane's partition, Kurzweil-Henstock's partition

MSC 2000: 26A36, 26A39

1. INTRODUCTION

Let $F: [a, b] \rightarrow \mathbb{R}$ be a differentiable function and let f be its derivative. The problem of recovering F from f is called *the problem of primitives*.

In 1912, the problem of primitives was solved by A. Denjoy with an integration process (called *totalization*) that includes the Lebesgue integral and the Lebesgue improper integral. Equivalent solutions are due to O. Perron, J. Kurzweil and R. Henstock (see for example [4], [6], [7], and [8]).

In 1986, A. M. Bruckner, R. J. Fleissner and J. Foran [3] remarked that the solution provided by Denjoy, Perron, Kurzweil and Henstock possesses a generality which is not needed for the problem of primitives. In fact the function $F(x) = x \sin(1/x^2)$ for $x \in (0, 1]$ and $F(0) = 0$ is ACG^* (i.e. a primitive for the Denjoy-Perron-Kurzweil-Henstock integral) but, for any absolutely continuous function G , the function $F - G$ is not differentiable at $x = 0$.

Note that the function $f(x) = F'(x)$ for $x \in (0, 1]$ and $f(0) = 0$ is a Riemann improper integrable function. So the minimal integral which includes Lebesgue integrable functions and derivatives (defined descriptively in [3]) does not contain the

Riemann improper integral. Now this minimal integral can be obtained from McShane's definition of the Lebesgue integral (see for example [4], [5], and [7]) by imposing a mild regularity condition on McShane's partitions (see [1] and [2]).

In this note we prove that the Riemann improper integral and the Lebesgue improper integral can be also obtained from McShane's definition of the Lebesgue integral by imposing a Kurzweil-Henstock's condition on McShane's partitions.

2. PRELIMINARIES

The set of all real numbers is denoted by \mathbb{R} . If $E \subset \mathbb{R}$ then $|E|$ denotes the Lebesgue measure of E , and \overline{E} its closure. In this paper $[a, b]$ is a fixed interval of \mathbb{R} . By *McShane's partition* of $[a, b]$ we mean any finite collection $\{(A_h, x_h)\}_{h=1}^p$ of pairwise disjoint intervals A_h and points $x_h \in [a, b]$ such that $[a, b] = \bigcup_h A_h$. We say that a McShane's partition $\{(A_h, x_h)\}_{h=1}^p$ satisfies the Kurzweil-Henstock condition (br. K-H condition) on a set $E \subset [a, b]$ whenever $x_h \in A_h$ if $x_h \in E$.

A *McShane's partition* satisfying the K-H condition on $[a, b]$ is called a *Kurzweil-Henstock's partition* of $[a, b]$.

Let δ be a gauge (i.e. a positive function) on $[a, b]$. A *McShane's partition* of $[a, b]$, say $\{(A_h, x_h)\}_{h=1}^p$, is said to be δ -fine whenever $A_h \subset (x_h - \delta(x_h), x_h + \delta(x_h))$, for each h .

We recall that a function $f: [a, b] \rightarrow \mathbb{R}$ is said to be McShane (resp. Kurzweil-Henstock) integrable on $[a, b]$ if there is a real number I satisfying the following condition: for each $\varepsilon > 0$ there exists a gauge δ such that

$$(1) \quad \left| \sum_{h=1}^p f(x_h)|A_h| - I \right| < \varepsilon,$$

for each δ -fine McShane's (resp. Kurzweil-Henstock's) partition P of $[a, b]$ (see [4], [5], [6] and [7]).

I is called the McShane (resp. Kurzweil-Henstock) integral of f on $[a, b]$. The McShane integral is equivalent to the Lebesgue integral, and the Kurzweil-Henstock integral is equivalent to the Denjoy-Perron integral (see [4], [7]).

In this paper the Kurzweil-Henstock integral of f on $[a, b]$ is denoted by $\int_a^b f$. Since the Riemann integral, the Riemann improper integral, the Lebesgue integral and the Lebesgue improper integral are contained in the Kurzweil-Henstock integral, then by $\int_a^b f$ we also denote each of the mentioned integrals on $[a, b]$.

For simplicity, in the sequel we set $\sum_P f = \sum_{h=1}^p f(x_h)|A_h|$.

We also recall that a function $f: [a, b] \rightarrow \mathbb{R}$ is said to be Lebesgue (resp. Riemann) improper integrable on $[a, b]$ if there exist $a_i \in [a, b]$, $i = 0, 1, \dots, k$ with $a = a_0 < a_1 < \dots < a_k = b$ such that f is Lebesgue (resp. Riemann) integrable on each compact subinterval $[\alpha, \beta]$ of (a_i, a_{i+1}) , and the limits

$$(2) \quad \lim_{\substack{\alpha \searrow a_i \\ \beta \nearrow a_{i+1}}} \int_{\alpha}^{\beta} f$$

exist finite, for $i = 0, 1, \dots, k - 1$.

3. THE LEBESGUE IMPROPER INTEGRAL

In this section we prove

Theorem 1. *A function $f: [a, b] \rightarrow \mathbb{R}$ is Lebesgue improper integrable on $[a, b]$ if and only if there exists a finite set $E \subset [a, b]$ and a real number I such that:*

(LI) *for each $\varepsilon > 0$ there exists a gauge δ so that $\left| \sum_P f - I \right| < \varepsilon$ for each δ -fine McShane's partition P of $[a, b]$ satisfying the K-H condition on E .*

Proof. Assume that f is Lebesgue improper integrable on $[a, b]$. Then there exist $a = a_0 < a_1 < \dots < a_k = b$, such that f is Lebesgue integrable on each compact subinterval $[\alpha, \beta]$ of (a_i, a_{i+1}) , and the limits (2) exist finite for each i .

Given $i \in \{0, 1, \dots, k - 1\}$, let $\{\alpha_{i,j}\}_{j \in \mathbb{Z}} \subset (a_i, a_{i+1})$ be an increasing sequence such that $\lim_{j \rightarrow -\infty} \alpha_{i,j} = a_i$ and $\lim_{j \rightarrow \infty} \alpha_{i,j} = a_{i+1}$.

Fixed $\varepsilon > 0$, $i \in \{0, 1, \dots, k - 1\}$ and $j \in \mathbb{Z}$, by Henstock's lemma there exists a gauge $\delta_{i,j}$ on $(\alpha_{i,j}, \alpha_{i,j+1})$ such that

$$(3) \quad \sum_{(A,x) \in P_{i,j}} \left| f(x)|A| - \int_A f \right| < \frac{\varepsilon}{k2^{|j|+3}}$$

for each $\delta_{i,j}$ -fine McShane's partition $P_{i,j}$ of $(\alpha_{i,j}, \alpha_{i,j+1})$.

For $x \neq a_i$, $0 \leq i \leq k$ we set

$$\delta(x) = \begin{cases} \min\{\delta_{i,j}(x), x - \alpha_{i,j}, \alpha_{i,j+1} - x\}, & \text{if } \alpha_{i,j} < x < \alpha_{i,j+1}; \\ \min\{\delta_{i,j-1}(x), \delta_{i,j}(x)\}, & \text{if } x = \alpha_{i,j}. \end{cases}$$

Moreover, for $x = a_i$, $0 \leq i \leq k$, we define $\delta(a_i) > 0$ such that

$$(4) \quad \delta(a_i)|f(a_i)| < \frac{\varepsilon}{8(k+1)},$$

and

$$(5) \quad \left| \int_{a_i}^{\alpha} f \right| < \frac{\varepsilon}{8(k+1)}$$

for each $\alpha \in [a, b]$ with $|\alpha - a_i| < \delta(a_i)$.

Let $\{(A_h, x_h)\}_{h=1}^p$ be a δ -fine McShane's partition of $[a, b]$ satisfying the K-H condition on $\{a_0, \dots, a_k\}$. Then, by (4) and (5) we have

$$\begin{aligned} \sum_{x_h \in \{a_0, \dots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| &\leq \sum_{x_h \in \{a_0, \dots, a_k\}} \left(|f(x_h)||A_h| + \left| \int_{A_h} f \right| \right) \\ &\leq \sum_{x_h \in \{a_0, \dots, a_k\}} \left(2 \cdot \frac{\varepsilon}{8(k+1)} + 2 \cdot \frac{\varepsilon}{8(k+1)} \right) \\ &\leq (k+1) \cdot \left(\frac{\varepsilon}{4(k+1)} + \frac{\varepsilon}{4(k+1)} \right) = \frac{\varepsilon}{2}. \end{aligned}$$

So, by (3) we get

$$\begin{aligned} \left| \sum_{h=1}^p f(x_h)|A_h| - \int_a^b f \right| &\leq \sum_{x_h \in \{a_0, \dots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \sum_{x_h \in (\alpha_{i,j}, \alpha_{i,j+1})} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \frac{\varepsilon}{k2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore condition (LI) is satisfied.

Vice versa, assume that there exist $a = a_0 < a_1 \dots < a_k = b$, and a real number I satisfying condition (LI). Then f is Kurzweil-Henstock integrable on $[a, b]$ with $I = \int_a^b f$. Moreover, by an easy adaptation of Henstock's lemma (see [4, Lemma 9.11]) we have

$$\sum_{h=1}^p \left| f(x_h)|A_h| - \int_{A_h} f \right| < 2\varepsilon,$$

for each δ -fine McShane's partition $\{(A_h, x_h)\}_{h=1}^p$ of $[a, b]$ satisfying the K-H condition on E . This implies that f is McShane integrable (hence Lebesgue integrable) on each compact subinterval of (a_i, a_{i+1}) , $i = 0, 1, \dots, k-1$ and, by the continuity

of the Kurzweil-Henstock integral, the limits

$$\lim_{\substack{\alpha \searrow a_i \\ \beta \nearrow a_{i+1}}} \int_{\alpha}^{\beta} f$$

exist finite, for $i = 0, 1, \dots, k - 1$.

In conclusion f is Lebesgue improper integrable on $[a, b]$. □

4. THE RIEMANN IMPROPER INTEGRAL

In this section we prove

Theorem 2. *A function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann improper integrable on $[a, b]$ if and only if there exists a finite set $E \subset [a, b]$ and a real number I such that*

(RI) *for each $\varepsilon > 0$ there exists a gauge δ so that*

(1) *δ is continuous on $[a, b] \setminus E$;*

(2) *$\left| \sum_P f - I \right| < \varepsilon$, for each δ -fine McShane's partition P of $[a, b]$ satisfying the K-H condition on E .*

The proof is based on the following lemma:

Lemma 1. *If $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$, then for each $\varepsilon > 0$ there exists a constant $\eta > 0$ such that*

$$\sum_{h=1}^p \left| f(x_h) |A_h| - \int_{A_h} f \right| < \varepsilon,$$

for each η -fine McShane's partition $\{(A_h, x_h)\}_{h=1}^p$ of $[a, b]$.

Proof. By the definition of Riemann integral, to each $\varepsilon > 0$ there exists a constant $\eta_1 > 0$ such that

$$\left| \sum_{h=1}^p f(y_h) |A_h| - \int_a^b f \right| < \frac{\varepsilon}{8},$$

for each η_1 -fine Kurzweil-Henstock's partition $\{(A_h, y_h)\}_{h=1}^p$ of $[a, b]$. Then, by an easy adaptation of Henstock's lemma, we have

$$(6) \quad \sum_{h=1}^p \left| f(y_h) |A_h| - \int_{A_h} f \right| < \frac{\varepsilon}{4},$$

for each η_1 -fine Kurzweil-Henstock's partition $\{(A_h, y_h)\}_{h=1}^p$ of $[a, b]$. Now, for each natural n , let

$$E_n = \left\{ x \in [a, b] : \omega\left(f, \left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right) \geq \frac{\varepsilon}{4(b-a)} \right\},$$

where $\omega(f, A)$ stands for the oscillation of f on the set A .

It is easy to see that $\overline{E}_n \subset E_{n+1}$. Moreover, since the set of points of discontinuity of a Riemann integrable function is a Lebesgue null set, we have $\left| \bigcap_{n=1}^{\infty} E_n \right| = 0$. Hence $|\overline{E}_n| \rightarrow 0$. Let $M = \sup\{|f(x)| : x \in [a, b]\}$, and let n_0 be such that $|\overline{E}_{n_0}| < \varepsilon/(4M)$.

Now, let $\{(\alpha_i, \beta_i)\}_{i=1}^{\infty}$ be a covering of \overline{E}_{n_0} with

$$(7) \quad \sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \frac{\varepsilon}{4M}.$$

Then, there is i_0 such that $\overline{E}_{n_0} \subset \bigcup_{i=1}^{i_0} (\alpha_i, \beta_i)$. Set

$$\eta_2 = \inf \left\{ |x - y| : x \in \overline{E}_{n_0} \text{ and } y \notin \bigcup_{i=1}^{i_0} (\alpha_i, \beta_i) \right\},$$

and

$$\eta = \min\{\eta_1, \eta_2, 1/n_0\}.$$

Let $\{(A_h, x_h)\}_{h=1}^p$ be an η -fine McShane's partition of $[a, b]$, and let $y_h \in A_h$ for $h = 1, \dots, p$. Then (6) holds. Moreover, by definitions of E_{n_0} and η , for $h = 1, \dots, p$ we have

$$|f(x_h) - f(y_h)| < \frac{\varepsilon}{4(b-a)}, \text{ if } x_h \notin E_{n_0},$$

and

$$A_n \subset \bigcup_{i=1}^{i_0} (\alpha_i, \beta_i), \text{ if } x_h \in E_{n_0}.$$

Thus, by (7) we get

$$\sum_{x_h \in E_{n_0}} |A_h| < \frac{\varepsilon}{4M}.$$

In conclusion

$$\begin{aligned}
& \sum_{h=1}^p \left| f(x_h) |A_h| - \int_{A_h} f \right| \\
& \leq \sum_{h=1}^p |f(x_h) - f(y_h)| |A_h| + \sum_{h=1}^p \left| f(y_h) |A_h| - \int_{A_h} f \right| \\
& \leq \sum_{h=1}^p |f(x_h) - f(y_h)| |A_h| + \frac{\varepsilon}{4} \\
& = \sum_{x_h \in E_{n_0}} |f(x_h) - f(y_h)| |A_h| + \sum_{x_h \notin E_{n_0}} |f(x_h) - f(y_h)| |A_h| + \frac{\varepsilon}{4} \\
& < 2M \sum_{x_h \in E_{n_0}} |A_h| + \frac{\varepsilon}{4(b-a)} \sum_{x_h \notin E_{n_0}} |A_h| + \frac{\varepsilon}{4} \\
& < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

□

Corollary 1. *A function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$, if and only if there exists a real number I satisfying the following condition: for each $\varepsilon > 0$ there exists a positive constant η such that $\left| \sum_P f - I \right| < \varepsilon$, for each η -fine McShane's partition $P = \{(A_h, x_h)\}_{h=1}^p$ of $[a, b]$.*

Proof of Theorem 2. Assume that f is Riemann improper integrable on $[a, b]$. Then there exist $a = a_0 < a_1 < \dots < a_k = b$ such that f is Riemann integrable on each interval $[\alpha, \beta] \subset (a_i, a_{i+1})$, and the limits (2) exist finite, for $i = 0, 1, \dots, k-1$.

Given $i \in \{0, 1, \dots, k-1\}$, let $\{\alpha_{i,j}\}_{j \in \mathbb{Z}} \subset (a_i, b_i)$ be an increasing sequence such that $\lim_{j \rightarrow -\infty} \alpha_{i,j} = a_i$ and $\lim_{j \rightarrow \infty} \alpha_{i,j} = a_{i+1}$.

For $i \in \{0, 1, \dots, k-1\}$, $j \in \mathbb{Z}$, and $\varepsilon > 0$, by Lemma 1 there exist $0 < \tau_{i,j} < \min\{\alpha_{i,j+1} - \alpha_{i,j}, \alpha_{i,j+2} - \alpha_{i,j+1}, \alpha_{i,j+3} - \alpha_{i,j+2}\}$ such that condition (3) is satisfied for each $\tau_{i,j}$ -fine McShane's partition $P_{i,j}$ of $[\alpha_{i,j}, \alpha_{i,j+3}]$. Define $\delta_{i,j} \leq \min\{\tau_{i,j-2}, \tau_{i,j-1}, \tau_{i,j}\}$ such that $\delta_{i,j} \leq \delta_{i,j+1}$ for $j < 0$, $\delta_{i,-1} = \delta_{i,0}$ and $\delta_{i,j} \geq \delta_{i,j+1}$, for $j \geq 0$. Moreover, for $i = 0, 1, \dots, k$ take $\delta(a_i) > 0$ such that conditions (4) and (5) are satisfied. Finally, for $i \in \{0, 1, \dots, k-1\}$ and $x \in (a_i, a_{i+1})$ define

$$\delta(x) = \begin{cases} \delta_{i,j-1} + \frac{x - \alpha_{i,j}}{\alpha_{i,j+1} - \alpha_{i,j}} (\delta_{i,j} - \delta_{i,j-1}), & \text{if } x \in [\alpha_{i,j}, \alpha_{i,j+1}) \text{ and } j < 0; \\ \delta_{i,j} + \frac{x - \alpha_{i,j}}{\alpha_{i,j+1} - \alpha_{i,j}} (\delta_{i,j+1} - \delta_{i,j}), & \text{if } x \in [\alpha_{i,j}, \alpha_{i,j+1}) \text{ and } j \geq 0. \end{cases}$$

Note that δ is continuous on $[a, b] \setminus \bigcup_{i=1}^p \{a_i\}$ and $\delta(x) \leq \delta_{i,j}$ for $x \in [\alpha_{i,j}, \alpha_{i,j+1})$, $i = 0, 1, \dots, k-1, j \in \mathbb{Z}$.

To verify condition (RI)₂, let $\{(A_h, x_h)\}_{h=1}^p$ be a δ -fine McShane's partition of $[a, b]$ satisfying the K-H condition on $\{a_0, \dots, a_k\}$. Then, by (4) and (5) we have

$$(8) \quad \sum_{x_h \in \{a_0, \dots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \leq \frac{\varepsilon}{2}.$$

Now, whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j \geq 0$, then

$$\begin{aligned} A_h &\subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j}, \alpha_{i,j+1} + \delta_{i,j}) \\ &\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}], \end{aligned}$$

and whenever $x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]$ with $j < 0$, then

$$\begin{aligned} A_h &\subset (x_h - \delta(x_h), x_h + \delta(x_h)) \subset (\alpha_{i,j} - \delta_{i,j+1}, \alpha_{i,j+1} + \delta_{i,j+1}) \\ &\subset (\alpha_{i,j} - \tau_{i,j-1}, \alpha_{i,j+1} + \tau_{i,j-1}) \subset [\alpha_{i,j-1}, \alpha_{i,j+2}]. \end{aligned}$$

Therefore the family $\{(A_h, x_h) : x_h \in [\alpha_{i,j}, \alpha_{i,j+1}]\}$ is a $\tau_{i,j-1}$ -fine McShane's partial partition in $[\alpha_{i,j-1}, \alpha_{i,j+2}]$. Thus, by (3) and (8) we have

$$\begin{aligned} \left| \sum_{h=1}^p f(x_h)|A_h| - \int_a^b f \right| &\leq \sum_{x_h \in \{a_0, \dots, a_k\}} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \sum_{x_h \in (\alpha_{i,j}, \alpha_{i,j+1})} \left| f(x_h)|A_h| - \int_{A_h} f \right| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=0}^{k-1} \sum_{j=-\infty}^{+\infty} \frac{\varepsilon}{k2^{|j|+3}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This completes the proof of condition (RI)₂.

Vice versa, assume that there exist $a = a_0 < a_1 < \dots < a_k = b$, and a real number I satisfying condition (RI). Since condition (RI) implies condition (LI), then f is Lebesgue improper integrable on $[a, b]$, by Theorem 1. Thus we have only to prove that f is Riemann integrable on each compact subinterval of (a_i, a_{i+1}) , for $i = 0, 1, \dots, k-1$. Let $[\alpha, \beta] \subset (a_i, a_{i+1})$, $i = 0, 1, \dots, k-1$. By (RI)₂ and by an easy adaptation of Henstock's lemma, we have $\left| \sum_P f - I \right| < 2\varepsilon$, for each δ -fine McShane's partition P of $[\alpha, \beta]$. Now, by continuity of δ on $[\alpha, \beta]$, we have $\eta = \min\{\delta(x) : x \in [\alpha, \beta]\} > 0$. Therefore $\left| \sum_P f - I \right| < 2\varepsilon$, for each η -fine McShane's (hence Kurzweil-Henstock's) partition P of $[\alpha, \beta]$. Thus f is Riemann integrable on $[\alpha, \beta]$, and the proof is complete. \square

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