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EXPONENTIAL EXPANSIVENESS AND COMPLETE ADMISSIBILITY FOR EVOLUTION FAMILIES

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Abstract. Connections between uniform exponential expansiveness and complete admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ are studied. A discrete version for a theorem due to Van Minh, Räbiger and Schnaubelt is presented. Equivalent characterizations of Perron type for uniform exponential expansiveness of evolution families in terms of complete admissibility are given.

Keywords: evolution family, uniform exponential expansiveness, complete admissibility

MSC 2000: 34E05, 34D05

0. Introduction

In the theory of evolution equations a central position is held by the "bounded input–bounded output" characterizations, or the so-called theorems of the Perron type. A new approach in this direction has been introduced by Henry in [5], where discrete dichotomy of a sequence $\{T_n\}_{n\in\mathbb{Z}}$ of bounded linear operators has been introduced and characterized in terms of existence and uniqueness of bounded solutions for $x_{n+1} = T_n x_n + f_n$ for every bounded sequence $(f_n)_{n\in\mathbb{Z}}$. In this context, he pointed out the connection between the discrete dichotomy and the exponential dichotomy of evolution families.

In the last few years, outstanding results concerning the asymptotic behaviour of evolution equations have been obtained using discrete-time methods (see [1]–[3], [6], [7], [14]). In [3], Chow and Leiva gave discrete and continuous characterizations for exponential dichotomy of linear skew-product semiflows. In [7], Latushkin and Schnaubelt expressed dichotomy in terms of the hyperbolicity of a family of weighted shift operators defined on $c_0(\mathbb{Z}, X)$. Sequence spaces over $\mathbb{N}$, in the study of exponential dichotomy, have been also considered by Ben-Artzi, Gohberg and Kaashoek in [1].

In [8] and [9], uniform exponential stability of periodic evolution families and linear skew-product semiflows, respectively, has been characterized by the presence of certain orbits in certain Banach sequence spaces.

A significant result concerning uniform exponential dichotomy of evolution families is the theorem of Perron type presented by Van Minh, Räbiger and Schnaubelt in [12]. They have shown that an evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ on a Banach space X, with the property that for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous, is uniformly exponentially dichotomic if and only if the pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ is admissible for $\mathscr{U}$ and the subspace $X_1 = \{x \in X \colon \lim\limits_{t\to\infty} U(t,0)x = 0\}$ is closed and complemented in X. An extension of this result to the nonuniform case has been treated in [11]. In [10], uniform exponential dichotomy of an evolution family has been expressed in terms of $(c_0(\mathbb{N}, X), c_{00}(\mathbb{N}, X))$-admissibility. As a consequence, using discrete methods, we have presented another proof for the theorem due to Van Minh, Räbiger and Schnaubelt.

The purpose of the present paper is to give discrete and continuous characterizations of Perron type for uniform exponential expansiveness. Our starting point is the theorem of Perron type contained in the paper of Van Minh, Räbiger and Schnaubelt (see [12]), where it is proved that the uniform exponential expansiveness of an evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ on a Banach space X, with the property that for every $x \in X$ the mapping $(t,s) \mapsto U(t,s)x$ is continuous, is equivalent to the complete admissibility of the pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ for $\mathscr{U}$. In what follows, we will present the connections between complete admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ for an evolution family on a Banach space X and its uniform exponential expansiveness. We shall prove that in certain conditions, the uniform exponential expansiveness of an evolution family is equivalent to the complete admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$. Thus, we will present a discrete variant for the theorem due to Van Minh, Räbiger and Schnaubelt, giving also a new proof for their result.

1. Preliminaries

Let X be a real or complex Banach space. The norm on X and on the space $\mathscr{B}(X)$ of all bounded linear operators on X will be denoted by $\|\cdot\|$.

Definition 1.1. A family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ of bounded linear operators is called an *evolution family* if the following conditions are satisfied:

(i) $U(t,t) = I$, the identity operator on X, for all $t \geqslant 0$;

(ii) $U(t,r)U(r,s) = U(t,s)$ for all $t \geqslant r \geqslant s \geqslant 0$;
(iii) there exist $M \geqslant 1$ and $\omega > 0$ such that

$$\|U(t,s)\| \leqslant M\mathrm{e}^{\omega(t-s)}, \quad \forall\, t \geqslant s \geqslant 0. \tag{1.1}$$

If, in addition, for every $x \in X$ and every t, $t_0 \geqslant 0$ the mapping $s \mapsto U(s,t_0)x$ is continuous on $[t_0,\infty)$ and the mapping $s \mapsto U(t,s)x$ is continuous on $[0,t]$, then $\mathscr{U}$ is called a *strongly continuous evolution family.*

Definition 1.2. An evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ is said to be *uniformly exponentially expansive* if $U(t,s)$ is invertible for all $t \geqslant s \geqslant 0$ and there are $N, \nu > 0$ such that

$$\|U(t,s)x\| \geqslant N\mathrm{e}^{\nu(t-s)}\|x\|, \quad \forall\, t \geqslant s \geqslant 0,\ \forall\, x \in X.$$

For every $n \in \mathbb{N}$ we consider

$$c_0(n,X) = \{s\colon \{k \in \mathbb{N}\colon\ k \geqslant n\} \to X \mid \lim_{k\to\infty} s(k) = 0\},$$

which is a Banach space with respect to the norm

$$\|s\|_{c_0(n,X)} = \sup_{k\geqslant n} \|s(k)\|.$$

Throughout the paper we shall denote $c_0(\mathbb{N},X) = c_0(0,X)$.

Definition 1.3. The pair $(c_0(\mathbb{N},X), c_0(\mathbb{N},X))$ is said to be *completely admissible* for an evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ if for every $n_0 \in \mathbb{N}$ and every $s \in c_0(n_0,X)$ there is a unique $\gamma_s \in c_0(n_0,X)$ such that

$$\gamma_s(m) = U(m,n)\gamma_s(n) + \sum_{j=n}^{m-1} U(m,j)s(j), \quad \forall\, m,n \in \mathbb{N},\ \ m > n \geqslant n_0. \tag{$\mathrm{E}_d^{n_0}$}$$

Remark 1.1. If the pair $(c_0(\mathbb{N},X), c_0(\mathbb{N},X))$ is completely admissible for the evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$, then we may consider the operator

$$\Gamma\colon\ c_0(\mathbb{N},X) \to c_0(\mathbb{N},X), \quad \Gamma(s) = \gamma_s.$$

It is easy to see that Γ is a closed linear operator on $c_0(\mathbb{N},X)$. Using the closed graph principle we obtain that Γ is bounded, so there exists $c > 0$ such that

$$\|\gamma_s\|_{c_0(\mathbb{N},X)} \leqslant c\|s\|_{c_0(\mathbb{N},X)}, \quad \forall\, s \in c_0(\mathbb{N},X).$$

In what follows we will denote by $C_0(\mathbb{R}_+, X)$ the Banach space of all continuous functions $u\colon \mathbb{R}_+ \to X$ with the property that $\lim_{t\to\infty} u(t) = 0$, which is a Banach space with respect to the norm

$$|||u||| = \sup_{t\geqslant 0} \|u(t)\|.$$

Simillarly, if $t_0 \geqslant 0$ then $C_0(t_0, X)$ denotes the space of all continuous functions $u\colon [t_0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} u(t) = 0$.

Definition 1.4. The pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ is said to be *completely admissible* for a strongly continuous evolution family $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ if for every $t_0 \geqslant 0$ and for every $u \in C_0(t_0, X)$ there exists a unique $f_u \in C_0(t_0, X)$ such that

$$(\mathrm{E}_c^{t_0}) \qquad f_u(t) = U(t,s)f_u(s) + \int_s^t U(t,\tau)u(\tau)\,\mathrm{d}\tau, \quad \forall\, t \geqslant s \geqslant t_0.$$

2. Main results

In this section we will establish the connections between complete admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ and uniform exponential expansiveness of an evolution family. We give necessary and sufficient conditions for uniform exponential expansiveness in terms of $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ complete admissibility and $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ complete admissibility. We present the manner how discrete methods can be used in order to characterize asymptotic properties.

We start with some necessary conditions given by the complete admissibility of the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ for an evolution family.

Theorem 2.1. *Let $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ be an evolution family on the Banach space X. If the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ is completely admissible for $\mathscr{U}$, then for every $n \in \mathbb{N}$ the operator $U(n,0)$ is invertible.*

Proof. *Injectivity.* Let $n_0 \in \mathbb{N}^*$ and $x \in X$ with $U(n_0,0)x = 0$. We observe that the sequences $\gamma_1, \gamma_2\colon \mathbb{N} \to X$, given by

$$\gamma_1(j) = 0 \quad \text{and} \quad \gamma_2(j) = U(j,0)x, \quad j \in \mathbb{N},$$

belong to $c_0(\mathbb{N}, X)$ and γ_1, γ_2 verify the equation (E_d^0) for $s = 0$. Since the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ is completely admissible for $\mathscr{U}$, it follows that $\gamma_1 = \gamma_2$. In particular, we obtain that $x = \gamma_2(0) = \gamma_1(0) = 0$. So, $U(n_0,0)$ is injective.

Surjectivity. Let $n_0 \in \mathbb{N}^*$ and $y \in X$. If

$$s\colon \mathbb{N} \to X, \quad s(n) = \begin{cases} -y, & n = n_0, \\ 0, & n \neq n_0, \end{cases}$$

and $\delta = -s$ we have

$$\delta(m) = U(m,n)\delta(n) + \sum_{j=n}^{m-1} U(m,j)s(j), \quad \forall\, m,n \in \mathbb{N}, \ \ m > n \geqslant n_0.$$

On the other hand, there exists $\gamma_s \in c_0(\mathbb{N}, X)$ such that

$$\gamma_s(m) = U(m,n)\gamma_s(n) + \sum_{j=n}^{m-1} U(m,j)s(j), \quad \forall\, m,n \in \mathbb{N}, \ \ m > n.$$

Since the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ is completely admissible for $\mathscr{U}$, it follows that $\gamma_s(n) = \delta(n)$ for all $n \geqslant n_0$. In particular, we obtain

$$y = \delta(n_0) = \gamma_s(n_0) = U(n_0, 0)\gamma_s(0) \in \operatorname{Range} U(n_0, 0),$$

so $U(n_0, 0)$ is surjective, which completes the proof. □

Theorem 2.2. *Let $\mathscr{U} = \{U(t,s)\}_{t \geqslant s \geqslant 0}$ be an evolution family on a Banach space X. If the pair $(c_0(\mathbb{N}, X), c_0(\mathbb{N}, X))$ is completely admissible for $\mathscr{U}$, then there are $N, \nu > 0$ such that*

$$\|U(m,n)x\| \geqslant N\mathrm{e}^{\nu(m-n)}\|x\|, \quad \forall\, m,n \in \mathbb{N}, \ \ m \geqslant n, \ \ \forall\, x \in X. \tag{2.1}$$

P r o o f. Let $x \neq 0$. From Theorem 2.1 we have that $U(n,0)x \neq 0$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, we consider sequences

$$s_n\colon \mathbb{N} \to X, \quad s_n(k) = \begin{cases} -\dfrac{U(k,0)x}{\|U(k,0)x\|}, & k \leqslant n, \\ 0, & k \geqslant n+1, \end{cases}$$

and

$$\gamma_n\colon \mathbb{N} \to X, \quad \gamma_n(k) = \begin{cases} \displaystyle\sum_{j=k}^{n} \frac{U(k,0)x}{\|U(j,0)x\|}, & k \leqslant n, \\ 0, & k \geqslant n+1. \end{cases}$$

Then $s_n, \gamma_n \in c_0(\mathbb{N}, X)$ and the pair (γ_n, s_n) verifies the discrete equation (E_d^0) for all $n \in \mathbb{N}$. So, for every $n \in \mathbb{N}$, γ_n is the solution of (E_d^0) corresponding to s_n. From Remark 1.1 it follows that there is $\nu \in (0,1)$ such that

$$2\nu \|\gamma_n\|_{c_0(\mathbb{N},X)} \leqslant \|s_n\|_{c_0(\mathbb{N},X)}, \quad \forall\, n \in \mathbb{N}.$$

Without loss of generality we may suppose that $2\nu \geqslant \mathrm{e}^{\nu} - 1$. Because $\|s_n\|_{c_0(\mathbb{N},X)} = 1$ for all $n \in \mathbb{N}$, we obtain

$$2\nu \sum_{j=k}^{n} \frac{1}{\|U(j,0)x\|} \leqslant \frac{1}{\|U(k,0)x\|}, \quad \forall\, n,k \in \mathbb{N}, \quad n \geqslant k,$$

so

$$2\nu \sum_{j=k}^{\infty} \frac{1}{\|U(j,0)x\|} \leqslant \frac{1}{\|U(k,0)x\|}, \quad \forall\, k \in \mathbb{N}. \tag{2.2}$$

We consider the sequence

$$\alpha\colon \mathbb{N} \to X, \quad \alpha(n) = \sum_{j=n}^{\infty} \frac{1}{\|U(j,0)x\|}.$$

From (2.2) it follows that

$$\frac{1}{\|U(n,0)x\|} \geqslant 2\nu\alpha(n+1) \geqslant (\mathrm{e}^{\nu} - 1)\alpha(n+1), \quad \forall\, n \in \mathbb{N},$$

so, $\alpha(n) \geqslant \mathrm{e}^{\nu}\alpha(n+1)$ for all $n \in \mathbb{N}$. Then for every $m > n$ we obtain

$$\frac{1}{\|U(m,0)x\|} \leqslant \alpha(m) \leqslant \mathrm{e}^{-\nu(m-n)}\alpha(n) \leqslant \frac{1}{2\nu}\mathrm{e}^{-\nu(m-n)}\frac{1}{\|U(n,0)x\|}.$$

Since $\nu \in (0,1)$ it follows that

$$\|U(m,0)x\| \geqslant \nu\mathrm{e}^{\nu(m-n)}\|U(n,0)x\|, \quad \forall\, m,n \in \mathbb{N}, \quad m \geqslant n, \quad \forall\, x \in X.$$

This last inequality and the invertibility of the operator $U(n,0)$ for all $n \in \mathbb{N}$ implies the conclusion. □

Theorem 2.3. *Let $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ be an evolution family on a Banach space X. Then $\mathscr{U}$ is uniformly exponentially expansive if and only if*

(i) *the pair $(c_0(\mathbb{N},X), c_0(\mathbb{N},X))$ is completely admissible for $\mathscr{U}$;*

(ii) *$U(t,s)$ is surjective for all $t\geqslant s\geqslant 0$.*

P r o o f. *Necessity.* From Definition 1.2, the assertion (ii) is obviously verified. Let $n_0\in\mathbb{N}$ and $s\in c_0(n_0,X)$. We consider the sequence

$$\gamma\colon \{n\in\mathbb{N}\colon\ n\geqslant n_0\}\to X,\quad \gamma(n) = -\sum_{j=n}^{\infty} U(j,n)^{-1}s(j).$$

Then $\gamma\in c_0(n_0,X)$ and the pair (γ,s) verifies the discrete equation $(\mathrm{E}_d^{n_0})$. To prove the uniqueness of γ, let $\tilde\gamma\in c_0(n_0,X)$ be such that the pair $(\tilde\gamma,s)$ verifies the discrete equation $(\mathrm{E}_d^{n_0})$. Then

$$\gamma(m)-\tilde\gamma(m) = U(m,n_0)(\gamma(n_0)-\tilde\gamma(n_0)),\quad \forall\, m\in\mathbb{N},\ \ m\geqslant n_0. \tag{2.3}$$

From (2.3), using the expansiveness of U and the fact that $\gamma,\tilde\gamma\in c_0(n_0,X)$, it follows that $\gamma(n_0)=\tilde\gamma(n_0)$ and so, $\gamma=\tilde\gamma$.

Sufficiency. First, we shall prove that for every $t\geqslant s\geqslant 0$, $U(t,s)$ is invertible. Let $t\geqslant 0$ and $n=[t]$. By Theorem 2.2 there are $N\in(0,1)$ and $\nu>0$ such that $\mathscr{U}$ satisfies relation (2.1). If M, ω are given by Definition 1.1, then it follows that

$$N\mathrm{e}^{\nu(n+1)}\|x\|\leqslant\|U(n+1,0)x\|\leqslant M\mathrm{e}^{\omega}\|U(t,0)x\|,\quad \forall\, x\in X.$$

Hence $U(t,0)$ is injective and using the hypothesis it is invertible for all $t\geqslant 0$. Using Definition 1.1 (ii) we deduce that $U(t,s)$ is invertible for all $t\geqslant s\geqslant 0$.

Now, we show that there is $K>0$ such that

$$\|U(n,s)x\|\geqslant K\|x\|,\quad \forall\, n\in\mathbb{N}^*,\ \ \forall\, s\in[n-1,n),\ \ \forall\, x\in X. \tag{2.4}$$

Let $x\in X$, $n\in\mathbb{N}^*$ and $s\in[n-1,n)$. Then

$$\|x\|\leqslant\|U(n,s)^{-1}\|\,\|U(n,s)x\|.$$

Since

$$U(n,s)^{-1} = U(s,n-1)U(n,n-1)^{-1},$$

we have

$$\|U(n,s)^{-1}\|\leqslant M\mathrm{e}^{\omega}\|U(n,n-1)^{-1}\|\leqslant\frac{M\mathrm{e}^{\omega}}{N\mathrm{e}^{\nu}}$$

and setting $K=(N\mathrm{e}^{\nu-\omega})/M$, we obtain the relation (2.4).

Let $x \in X$ and $t \geqslant s \geqslant 0$. If $k = [s] + 1$ we have two possible cases:

Case 1: $t \leqslant k$. Then an easy computation shows that

$$K\|x\| \leqslant \|U(k,s)x\| \leqslant M\mathrm{e}^{\omega}\|U(t,s)x\|,$$

which implies

$$\|U(t,s)x\| \geqslant \frac{K}{M}\mathrm{e}^{-(\omega+\nu)}\mathrm{e}^{\nu(t-s)}\|x\|.$$

Case 2: $t > k$. Then for $n = [t]$ we deduce that

$$\begin{aligned} KN\mathrm{e}^{-\nu}\mathrm{e}^{\nu(t-s)}\|x\| &\leqslant KN\mathrm{e}^{\nu(n+1-k)}\|x\| \leqslant \|U(n+1,k)U(k,s)x\| \\ &= \|U(n+1,s)x\| \leqslant M\mathrm{e}^{\omega}\|U(t,s)x\|. \end{aligned}$$

From the last two cases, for $L = KN\mathrm{e}^{-(\omega+\nu)}/M$, it follows that

$$\|U(t,s)x\| \geqslant L\mathrm{e}^{\nu(t-s)}\|x\|, \quad \forall\, t \geqslant s \geqslant 0, \ \forall\, x \in X.$$

□

The next theorem establishes the connection between discrete complete admissibility, continuous complete admissibility and uniform exponential expansiveness of a strongly continuous evolution family.

Theorem 2.4. *Let $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ be a strongly continuous evolution family on a Banach space X. Then the following assertions are equivalent:*

(i) *$\mathscr{U}$ is uniformly exponentially expansive;*

(ii) *the pair $(C_0(\mathbb{R}_+,X), C_0(\mathbb{R}_+,X))$ is completely admissible for $\mathscr{U}$;*

(iii) *the pair $(c_0(\mathbb{N},X), c_0(\mathbb{N},X))$ is completely admissible for $\mathscr{U}$ and $U(t,s)$ is surjective for all $t \geqslant s \geqslant 0$.*

Proof. (i) $\Longrightarrow$ (ii) Let $t_0 \geqslant 0$ and $u \in C_0(t_0,X)$. We define

$$f\colon [t_0,\infty) \to X, \quad f(t) = -\int_t^{\infty} U(s,t)^{-1}u(s)\,\mathrm{d}s.$$

Then $f \in C_0(t_0,X)$ and the pair (f,u) verifies the equation $(\mathrm{E}_c^{t_0})$. Using the same arguments as in the proof of necessity of Theorem 2.3 we obtain the uniqueness of f.

(ii) $\Longrightarrow$ (iii) To prove that $U(t,s)$ is surjective for all $t \geqslant s \geqslant 0$, it is sufficient to show that $U(t,0)$ is surjective for all $t \geqslant 0$.

Let $\alpha\colon \mathbb{R}_+ \to [0,2]$ be a continuous function with the support contained in $(0,1)$ such that $\int_0^1 \alpha(\tau)\,\mathrm{d}\tau = 1$. For $t_0 > 0$ and $x \in X$, we consider the functions

$$f\colon [t_0,\infty) \to X, \quad f(t) = U(t,t_0)x\int_t^\infty \alpha(\tau - t_0)\,\mathrm{d}\tau,$$
$$v\colon \mathbb{R}_+ \to X, \quad v(t) = \begin{cases} -\alpha(t-t_0)U(t,t_0)x, & t \geqslant t_0,\\ 0, & t < t_0. \end{cases}$$

An easy computation shows that

$$f(t) = U(t,s)f(s) + \int_s^t U(t,\tau)v(\tau)\,\mathrm{d}\tau, \quad \forall\, t \geqslant s \geqslant t_0.$$

But, by the hypothesis there is $g \in C_0(\mathbb{R}_+, X)$ such that

$$g(t) = U(t,s)g(s) + \int_s^t U(t,\tau)v(\tau)\,\mathrm{d}\tau, \quad \forall\, t \geqslant s \geqslant 0.$$

Using the uniqueness on $[t_0,\infty)$ we deduce that

$$x = f(t_0) = g(t_0) = U(t_0,0)g(0) \in \operatorname{Range} U(t_0,0),$$

so $U(t_0,0)$ is surjective.

Now, we prove that the pair $(c_0(\mathbb{N},X), c_0(\mathbb{N},X))$ is completely admissible for $\mathscr{U}$. For $n_0 \in \mathbb{N}$ and $s \in c_0(n_0,X)$ we define

$$w\colon [n_0,\infty) \to X, \quad w(t) = U(t,[t])s([t])\alpha(t-[t]).$$

Then w is continuous and if M, ω are given by Definition 1.1, we observe that

$$\|w(t)\| \leqslant 2M\mathrm{e}^{\omega}\|s([t])\|, \quad \forall\, t \geqslant n_0,$$

so $w \in C_0(n_0,X)$. By the hypothesis, there is a unique $h \in C_0(n_0,X)$ such that

$$h(t) = U(t,s)h(s) + \int_s^t U(t,\tau)w(\tau)\,\mathrm{d}\tau, \quad \forall\, t \geqslant s \geqslant n_0.$$

Let $m, n \in \mathbb{N}$, $m > n \geqslant n_0$. Then

$$\begin{aligned} h(m) &= U(m,n)h(n) + \int_n^m U(m,\tau)w(\tau)\,\mathrm{d}\tau \\ &= U(m,n)h(n) + \sum_{j=n}^{m-1} \int_j^{j+1} U(m,\tau)w(\tau)\,\mathrm{d}\tau \\ &= U(m,n)h(n) + \sum_{j=n}^{m-1} U(m,j)s(j) \int_j^{j+1} \alpha(\tau - j)\,\mathrm{d}\tau \\ &= U(m,n)h(n) + \sum_{j=n}^{m-1} U(m,j)s(j). \end{aligned}$$

Because $h \in C_0(n_0, X)$, we have $(h(n))_{n \geqslant n_0} \in c_0(n_0, X)$.

To prove the uniqueness, it is sufficient to show that if $s \in c_0(n_0, X)$ is such that

$$s(m) = U(m,n)s(n), \quad \forall\, m, n \in \mathbb{N}, \quad m \geqslant n \geqslant n_0, \tag{2.5}$$

then $s = 0$.

Indeed, let $s \in c_0(n_0, X)$ verify (2.5). We define

$$\varphi\colon\ [n_0, \infty) \to X, \quad \varphi(t) = U(t, [t])s([t]).$$

Using (2.5) it is easy to see that φ is continuous and

$$\varphi(t) = U(t,\tau)\varphi(\tau), \quad \forall\, t \geqslant \tau \geqslant n_0.$$

Since the pair $(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))$ is completely admissible for $\mathscr{U}$, it follows that $\varphi = 0$ and hence $s = 0$, which completes the proof.

(iii) $\Longrightarrow$ (i) It follows from Theorem 2.3. □

Remark 2.2. Using other methods, the equivalence (i) $\Longleftrightarrow$ (ii) has been proved by Van Minh, Räbiger and Schnaubelt in [12] for evolution families $\mathscr{U} = \{U(t,s)\}_{t \geqslant s \geqslant 0}$ which satisfy the additional assumption that for every $x \in X$, the mapping $(t,s) \mapsto U(t,s)x$ is continuous.

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