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MULTIPOINT BOUNDARY VALUE PROBLEMS FOR ODES. PART II

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Abstract. We apply the method of quasilinearization to multipoint boundary value problems for ordinary differential equations showing that the corresponding monotone iterations converge to the unique solution of our problem and this convergence is quadratic.

Keywords: quasilinearization, monotone iterations, quadratic convergence, multipoint boundary value problems

MSC 2000: 34A45, 34B99

Introduction

In this paper, we shall consider the following differential problem

$$\text{(1)} \qquad \begin{cases} x'(t) = h(t, x(t)), \ \ t \in J = [0, T], \ \ T > 0, \\ x(0) = -\alpha x(t_1) + \beta x(T) + k, \ \ 0 < t_1 < T, \end{cases}$$

where $h \in C(J \times \mathbb{R}, \mathbb{R})$, $\alpha, \beta, k \in \mathbb{R}$. We see that if $\alpha = k = 0$ and $\beta = 1$ or $\beta = -1$, then we have a periodic or anti-periodic boundary value problem, respectively.

It is well known ([2], [3], [8]–[12], [15]) that the method of quasilinearization offers an approach for obtaining approximate solutions to nonlinear differential problems. Recently it was generalized and extended under less restrictive assumptions.

The aim of this paper is to apply this method to general multipoint boundary value problems for ordinary differential equations. We construct monotone sequences as the solutions of the corresponding linear systems. Under the assumption that $f + \Delta$ is convex and $g + \Psi$ is concave for some convex function Δ and concave function Ψ (see assumption H_3(a)) with $h = f + g$, we prove that those sequences converge quadratically to the unique solution of our problem. This result is of both theoretical and computational interest. We must point out that our considerations lead us to prove some results for linear problems which are important in our investigations.

In Sections 1 and 2 we study differential problems with conditions of the form $x(0) = \omega_1 x(t_1) + \omega_2 x(T) + k$, $0 < t_1 < T$, for $\omega_1, \omega_2, k \in \mathbb{R}$ (see problems (1) and (13)). Our considerations crucially depend on the notion of weakly coupled lower and upper solutions connected with positive or negative values of ω_1 and ω_2. Section 3 contains the general case with a multipoint boundary condition of the form $x(0) = \sum_{i=1}^{r} a_i x(t_i)$, $a_i \in \mathbb{R}$.

This paper generalizes the results of [5] and [6] in which multipoint problems were studied for cases when all coefficients a_i are positive or all are negative. This paper also contains some results for periodic and anti-periodic boundary value problems obtained in [8]–[12], [16]–[17]. Note that if all coefficients $a_i = 0$, then we have an initial value problem.

Two and multipoint boundary value problems have been considered by several authors, see for example [4], [13], [14] and the references therein and in [1]. It is difficult to compare those results with the corresponding results of this paper because we use a different technique based on the theory of differential inequalities, see for the details [7] and [12].

Section 1

Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled lower and upper solutions of the problem (1) if

$$\begin{cases} u'(t) \leqslant h(t, u(t)), \ \ t \in J, \quad u(0) \leqslant -\alpha v(t_1) + \beta u(T) + k, \\ v'(t) \geqslant h(t, v(t)), \ \ t \in J, \quad v(0) \geqslant -\alpha u(t_1) + \beta v(T) + k. \end{cases}$$

Let $\Omega = \{(t, u) \in J \times \mathbb{R}\colon\ y_0(t) \leqslant u \leqslant z_0(t),\ t \in J\}$ be nonempty. The notation $h \in C^{0,2}(\Omega, \mathbb{R})$ means that $h, h_x, h_{xx} \in C(\Omega, \mathbb{R})$.

We introduce the following assumptions for later use.

(H_1) $h \in C(\Omega, \mathbb{R})$, $\alpha, \beta \geqslant 0$,

(H_2) y_0, $z_0 \in C^1(J, \mathbb{R})$ are weakly coupled lower and upper solutions of (1) and such that $y_0(t) \leqslant z_0(t)$, $t \in J$,

(H_3) $f, g, \Delta, \Psi \in C^{0,2}(\Omega, \mathbb{R})$ with $h = f + g$, and moreover
  - (a) $F_{xx}(t, u) \geqslant 0$, $\Delta_{xx}(t, u) \geqslant 0$, $G_{xx}(t, u) \leqslant 0$, $\Psi_{xx}(t, u) \leqslant 0$ on Ω for $F = f + \Delta$, $G = g + \Psi$,
  - (b) $\alpha K(t_1) + \beta K(T) < 1$ for $K(t) = \exp(\int_0^t L(s)\,\mathrm{d}s)$ with $L(s) = F_x(s, z_0(s)) + G_x(s, y_0(s)) - \Delta_x(t, y_0(s)) - \Psi_x(s, z_0(s))$.

Lemma 1. *Let $a, b \in C(J, \mathbb{R})$, $M, N \in C(J, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$. Assume that $k_1, k_2, \alpha, \alpha_1, \beta, \beta_1 \geqslant 0$ and the conditions*

(i) $1 - \beta A(T) > 0,\quad 1 - \beta_1 B(T) > 0,\quad \alpha_1 \alpha A(t_1) B(t_1) < [1 - \beta A(T)][1 - \beta_1 B(T)]$

are satisfied for

$$A(t) = \exp\Bigl(\int_0^t a(s)\,\mathrm{d}s\Bigr), \quad B(t) = \exp\Bigl(\int_0^t b(s)\,\mathrm{d}s\Bigr).$$

Let $p, q \in C^1(J, \mathbb{R})$ and

$$\begin{cases} p'(t) \leqslant a(t)p(t) + M(t), & t \in J, \\ p(0) \leqslant \alpha q(t_1) + \beta p(T) + k_1, \end{cases} \qquad \begin{cases} q'(t) \leqslant b(t)q(t) + N(t), & t \in J, \\ q(0) \leqslant \alpha_1 p(t_1) + \beta_1 q(T) + k_2. \end{cases}$$

Then

$$w(t) \leqslant \mathcal{C}(t)\mathcal{A}^{-1}\mathcal{B} + \mathcal{D}(t), \quad t \in J \tag{2}$$

with

$$w(t) = \begin{bmatrix} p(t) \\ q(t), \end{bmatrix} \quad \mathcal{C}(t) = \begin{bmatrix} A(t) & 0 \\ 0 & B(t) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 1 - \beta A(T) & -\alpha B(t_1) \\ -\alpha A(t_1) & 1 - \beta_1 B(T) \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} \alpha B(t_1) D(t_1) + \beta A(T) C(T) + k_1 \\ \alpha_1 A(t_1) C(t_1) + \beta B(T) D(T) + k_2 \end{bmatrix}, \quad \mathcal{D}(t) = \begin{bmatrix} A(t)C(t) \\ B(t)D(t) \end{bmatrix},$$

$$C(t) = \int_0^t M(s) \exp\Bigl(-\int_0^s a(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s, \quad D(t) = \int_0^t N(s) \exp\Bigl(-\int_0^s b(u)\,\mathrm{d}u\Bigr)\,\mathrm{d}s.$$

Moreover, if $M(t) = N(t) = 0$ on J and $k_1 = k_2 = 0$, then

$$p(t) \leqslant 0, \quad q(t) \leqslant 0 \ \text{ on } J. \tag{3}$$

Proof. Note that

$$\begin{cases} p(t) \leqslant A(t)[p(0) + C(t)], & t \in J, \\ q(t) \leqslant B(t)[q(0) + D(t)], & t \in J, \end{cases}$$

so

$$w(t) \leqslant \mathcal{C}(t) w(0) + \mathcal{D}(t), \quad t \in J. \tag{4}$$

Moreover, using the boundary conditions we get

$$\begin{cases} p(0) \leqslant \alpha B(t_1)[q(0) + D(t_1)] + \beta A(T)[p(0) + C(T)] + k_1, \\ q(0) \leqslant \alpha_1 A(t_1)[p(0) + C(t_1)] + \beta_1 B(T)[q(0) + D(T)] + k_2, \end{cases}$$

or $\mathcal{A}w(0) \leqslant \mathcal{B}$. Hence $w(0) \leqslant \mathcal{A}^{-1}\mathcal{B}$ because $\mathcal{A}^{-1}$ exists and is positive (so its entries are nonnegative), by assumption (i). Combining this with (4) we have (2).

Note that if $M(t) = N(t) = 0$ on J and $k_1 = k_2 = 0$, then $C(t) = D(t) = 0$ on J, so $\mathcal{B} = [0 \quad 0]^T$, $\mathcal{D}(t) = [0 \quad 0]^T$, $t \in J$. In this case, the inequality (2) yields $p(t) \leqslant 0$, $q(t) \leqslant 0$ on J.

This completes the proof. $\square$

Remark 1. If $\alpha = \alpha_1$, $\beta = \beta_1$ and $a(t) = b(t)$ on J, then $A(t) = B(t)$ on J. Consequently, the condition (i) takes the form

$$\alpha A(t_1) + \beta A(T) < 1. \tag{5}$$

Note that in this case we have also

$$p(t) + q(t) \leqslant \frac{A(t)}{m}[\alpha\gamma(t_1) + \beta\gamma(T) + k_1 + k_2] + \gamma(t), \quad t \in J \tag{6}$$

with $m = 1 - \alpha A(t_1) - \beta A(T)$, $\gamma(t) = A(t)[C(t) + D(t)]$, $t \in J$.

Similarly as Lemma 1 we can prove

Lemma 2. *Assume that $a \in C(J, \mathbb{R})$, $M \in C(J, \mathbb{R}_+)$, $\alpha, \beta, k_1 \geqslant 0$ and*

$$m \equiv 1 - \alpha A(t_1) - \beta A(T) > 0 \quad \text{for } A(t) = \mathrm{e}^{\int_0^t a(s)\,\mathrm{d}s}.$$

Let $p \in C^1(J, \mathbb{R})$, and

$$\begin{cases} p'(t) \leqslant a(t)p(t) + M(t), \quad t \in J, \\ p(0) \leqslant \alpha p(t_1) + \beta p(T) + k_1. \end{cases}$$

Then

$$p(t) \leqslant A(t)\Big\{\frac{1}{m}[\alpha A(t_1)C(t_1) + \beta A(T)C(T) + k_1] + C(t)\Big\}$$

with C defined as in Lemma 1.

Moreover, if $k_1 = 0$ and $M(t) = 0$ on J, then $p(t) \leqslant 0$ on J.

Lemma 3. *Assume that $a, b, M, N \in C(J, \mathbb{R})$, and*

$$[1 - \beta A(T)][1 - \beta_1 B(T)] - \alpha\alpha_1 A(t_1) B(t_1) \neq 0 \tag{7}$$

for A and B defined as in Lemma 1.

Then the system

$$\begin{cases} y'(t) = a(t)y(t) + M(t), \ \ t \in J, \quad y(0) = -\alpha z(t_1) + \beta y(T) + k_1, \\ z'(t) = b(t)z(t) + N(t), \ \ t \in J, \quad z(0) = -\alpha_1 y(t_1) + \beta_1 z(T) + k_2 \end{cases} \tag{8}$$

has a unique solution (y, z).

Proof. Indeed,

$$\begin{cases} y(t) = A(t)[y(0) + C(t)], \ \ t \in J, \\ z(t) = B(t)[z(0) + D(t)], \ \ t \in J \end{cases}$$

with C and D defined as in Lemma 1. Using the boundary conditions we obtain

$$\mathcal{P} \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -\alpha B(t_1) D(t_1) + \beta A(T) C(T) + k_1 \\ -\alpha_1 A(t_1) C(t_1) + \beta_1 B(T) D(T) + k_2 \end{bmatrix}$$

with the matrix $\mathcal{P}$ defined by

$$\mathcal{P} = \begin{bmatrix} 1 - \beta A(T) & \alpha B(t_1) \\ \alpha_1 A(t_1) & 1 - \beta_1 B(T) \end{bmatrix}.$$

By (7), the matrix $\mathcal{P}$ is invertible which proves that the problem (8) has a unique solution (y, z).

This completes the proof. □

Lemma 4. *Let the assumption* H_1 *hold. Assume that $h_x \in C(\Omega, \mathbb{R})$, and*

$$\alpha \exp\Big(\int_0^{t_1} h_x(s, u(s))\,\mathrm{d}s\Big) + \beta \exp\Big(\int_0^{T} h_x(s, u(s))\,\mathrm{d}s\Big) \neq 1 \quad \text{for } (s, u) \in \Omega. \tag{9}$$

Let $y, z \in C^1(J, \mathbb{R})$ and

$$\begin{cases} y'(t) = h(t, y), \ \ t \in J, \quad y(0) = -\alpha z(t_1) + \beta y(T) + k, \\ z'(t) = h(t, z), \ \ t \in J, \quad z(0) = -\alpha y(t_1) + \beta z(T) + k. \end{cases}$$

Then y and z are solutions of the problem (1).

P r o o f. Put $p = y - z$. Then $p(0) = \alpha p(t_1) + \beta p(T)$ and

$$p'(t) = h(t, y) - h(t, z) = h_x(t, \xi(t))p(t), \quad t \in J,$$

where ξ is between y and z. Hence

$$p(t) = d(t)p(0), \ \ t \in J \quad \text{with } \ d(t) = \exp\biggl(\int_0^t h_x(s, \xi(s)) \, \mathrm{d}s\biggr).$$

Now, the boundary condition yields $p(0)[1 - \alpha d(t_1) - \beta d(T)] = 0$. By the condition (9), $p(0) = 0$, and hence $p(t) = 0$ on J. This means that $y(t) = z(t)$ on J, so y and z satisfy the equations

$$\begin{cases} y'(t) = h(t, y), \ \ t \in J, \quad y(0) = -\alpha y(t_1) + \beta y(T) + k, \\ z'(t) = h(t, z), \ \ t \in J, \quad z(0) = -\alpha z(t_1) + \beta z(T) + k. \end{cases}$$

This proves that y and z are solutions of the problem (1).

This ends the proof. □

Lemma 5. *Let the assumption* H_1 *hold. Assume that* $h_x \in C(\Omega, \mathbb{R})$*, and*

$$(10) \quad -\alpha \exp\biggl(\int_0^{t_1} h_x(s, u(s)) \, \mathrm{d}s\biggr) + \beta \exp\biggl(\int_0^{T} h_x(s, u(s)) \, \mathrm{d}s\biggr) \neq 1 \quad \textit{for} \ \ (s, u) \in \Omega.$$

Then the problem (1) *has at most one solution.*

P r o o f. Assume that problem (1) has two distinct solutions x and y on the segment $[y_0, z_0]$. Put $p = x - y$, so $p(0) = -\alpha p(t_1) + \beta p(T)$. The mean value theorem yields

$$p'(t) = h(t, x) - h(t, y) = h_x(t, \xi(t))p(t), \quad t \in J,$$

where ξ is between x and y. Hence $p(t) = c(t)p(0)$, $t \in J$ with $c(t) = \mathrm{e}^{\int_0^t h_x(s, \xi(s)) \, \mathrm{d}s}$. Moreover, by the boundary condition, $p(0)[1 + \alpha c(t_1) - \beta c(T)] = 0$. By (10), it follows that $p(0) = 0$ showing that $p(t) = 0$ on J. This means that $x(t) = y(t)$ on J.

This completes the proof. □

Lemma 6. *Assume that the assumptions* H_1, H_3 *are satisfied. Let* $u, v \in C^1(J, \mathbb{R})$ *be weakly coupled lower and upper solutions of* (1) *such that* $y_0(t) \leqslant u(t) \leqslant v(t) \leqslant z_0(t)$ *on* J. *Let*

$$\left\{\begin{aligned} y'(t) &= h(t,u) + W(t,u,v)[y(t) - u(t)], \ \ t \in J, \\ y(0) &= \ -\alpha z(t_1) + \beta y(T) + k, \\ z'(t) &= h(t,v) + W(t,u,v)[z(t) - v(t)], \ \ t \in J, \\ z(0) &= \ -\alpha y(t_1) + \beta z(T) + k \end{aligned}\right. \tag{11}$$

with $W(t,u,v) = F_x(t,u) + G_x(t,v) - \Delta_x(t,v) - \Psi_x(t,u)$.

Then

$$u(t) \leqslant y(t) \leqslant z(t) \leqslant v(t), \quad t \in J, \tag{12}$$

and moreover, y, z *are weakly coupled lower and upper solutions of* (1).

Proof. By the assumption H_3, we get $W(t,u,v) \leqslant L(t)$. This, the assumption H_3(b) and Lemma 3 prove that the system (11) has a unique solution (y, z).

Now we are going to show that (12) holds. Put $p = u - y$, $q = z - v$, so $p(0) \leqslant \alpha q(t_1) + \beta p(T)$, $q(0) \leqslant \alpha p(t_1) + \beta q(T)$. Moreover,

$$\left\{\begin{aligned} p'(t) &\leqslant h(t,u) - h(t,u) - W(t,u,v)[y(t) - u(t)] = W(t,u,v)p(t), \ \ t \in J, \\ q'(t) &\leqslant h(t,v) + W(t,u,v)[z(t) - v(t)] - h(t,v) = W(t,u,v)q(t), \ \ t \in J. \end{aligned}\right.$$

This and Lemma 1 give $p(t) \leqslant 0$, $q(t) \leqslant 0$ on J showing that $u(t) \leqslant y(t), z(t) \leqslant v(t)$ on J. Now, let $p = y - z$. Hence $p(0) = \alpha p(t_1) + \beta p(T)$. Furthermore, by the mean value theorem and the assumption H_3(a) we obtain

$$\begin{aligned} p'(t) &= h(t,u) - h(t,v) + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &= h_x(t,\xi(t))[u(t) - v(t)] + W(t,u,v)[y(t) - u(t) - z(t) + v(t)] \\ &= [F_x(t,\xi(t)) + G_x(t,\xi(t)) - \Delta_x(t,\xi(t)) - \Psi_x(t,\xi(t)) - W(t,u,v)][u(t) - v(t)] \\ &\quad + W(t,u,v)p(t) \leqslant W(t,u,v)p(t), \quad t \in J, \end{aligned}$$

where $u(t) < \xi(t) < v(t)$, $t \in J$. This and Lemma 2 prove that $y(t) \leqslant z(t)$ on J which means that (12) holds.

Now, the mean value theorem and the assumption H_3(a) yield

$$\begin{aligned} y'(t) &= h(t,u) + W(t,u,v)[y(t) - u(t)] - h(t,y) + h(t,y) \\ &= h(t,y) + h_x(t,\xi_1(t))[u(t) - y(t)] + W(t,u,v)[y(t) - u(t)] \leqslant h(t,y), \ \ t \in J, \\ z'(t) &= h(t,v) + W(t,u,v)[z(t) - v(t)] - h(t,z) + h(t,z) \\ &= h(t,z) + h_x(t,\xi_2(t))[v(t) - z(t)] + W(t,u,v)[z(t) - v(t)] \geqslant h(t,z), \ \ t \in J. \end{aligned}$$

It follows that y and z are weakly coupled lower and upper solutions of (1).

This completes the proof. □

Theorem 1. *Let the assumptions* H_1, H_2 *and* H_3 *hold.*

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ *which converge monotonically and uniformly to the unique solution of the problem* (1) *and the convergence is quadratic.*

Proof. Let

$$\begin{cases} y'_{n+1}(t) = h(t, y_n) + W(t, y_n, z_n)[y_{n+1}(t) - y_n(t)], & t \in J, \\ y_{n+1}(0) = -\alpha z_{n+1}(t_1) + \beta y_{n+1}(T) + k, \end{cases}$$

$$\begin{cases} z'_{n+1}(t) = h(t, z_n) + W(t, y_n, z_n)[z_{n+1}(t) - z_n(t)], & t \in J, \\ z_{n+1}(0) = -\alpha y_{n+1}(t_1) + \beta z_{n+1}(T) + k, \end{cases}$$

where W is defined as in Lemma 6. By Lemma 3, y_1 and z_1 are well defined. Moreover, Lemma 6 yields the relation

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J.$$

Also, by Lemma 6, y_1 and z_1 are weakly coupled lower and upper solutions of (1). Now, using induction argument, we can prove that for all n and $t \in J$,

$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_{n-1}(t) \leqslant y_n(t) \leqslant z_n(t) \leqslant z_{n-1}(t) \leqslant \ldots \leqslant z_1(t) \leqslant z_0(t).$$

Employing a standard argument (see [7]), it is easy to conclude that the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the limit functions y and z, respectively, where y and z satisfy the equations

$$\begin{cases} y'(t) = h(t, y), \ t \in J, \\ y(0) = -\alpha z(t_1) + \beta y(T) + k, \\ z'(t) = h(t, z), \ t \in J, \\ z(0) = -\alpha y(t_1) + \beta z(T) + k. \end{cases}$$

By Lemma 4, y and z are solutions of (1). This and Lemma 5 show that problem (1) has a unique solution x, so $y = z = x$.

The proof will be completed if we show that the convergence of $\{y_n\}$ and $\{z_n\}$ to x is quadratic. Let $p_{n+1} = x - y_{n+1} \geqslant 0$, $q_{n+1} = z_{n+1} - x \geqslant 0$. Note that

$p_{n+1}(0) = \alpha q_{n+1}(t_1) + \beta p_{n+1}(T)$, $q_{n+1}(0) = \alpha p_{n+1}(t_1) + \beta q_{n+1}(T)$. Hence, by the mean value theorem and the assumption H_3(a), we obtain

$$\begin{aligned}
p'_{n+1}(t) &= h(t,x) - h(t,y_n) - W(t,y_n,z_n)[y_{n+1}(t) - y_n(t)] \\
&= h_x(t,\xi_1)p_n(t) - W(t,y_n,z_n)[p_n(t) - p_{n+1}(t)] \\
&\leqslant [F_x(t,x) - F_x(t,y_n) + G_x(t,y_n) - G_x(t,z_n) + \Delta_x(t,z_n) - \Delta_x(t,y_n) \\
&\quad + \Psi_x(t,y_n) - \Psi_x(t,x)]p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\
&= \{F_{xx}(t,\xi_2)p_n(t) - G_{xx}(t,\xi_3)[p_n(t) + q_n(t)] + \Delta_{xx}(t,\xi_4)[q_n(t) + p_n(t)] \\
&\quad - \Psi_{xx}(t,\xi_5(t))p_n(t)\}p_n(t) + W(t,y_n,z_n)p_{n+1}(t) \\
&\leqslant \{(A_1 + A_4)p_n(t) + (A_2 + A_3)[p_n(t) + q_n(t)]\}p_n(t) + L(t)p_{n+1}(t), \\
&\leqslant L(t)p_{n+1}(t) + D_1,
\end{aligned}$$

where $y_n(t) < \xi_1(t), \xi_2(t), \xi_5(t) < x(t)$, $y_n(t) < \xi_3(t), \xi_4(t) < z_n(t)$ with L defined as in the assumption H_3(b), and

$$|F_{xx}(t,u)| \leqslant A_1, \ \ |G_{xx}(t,u)| \leqslant A_2, \ \ |\Delta_{xx}(t,u)| \leqslant A_3, \ \ |\Psi_{xx}(t,u)| \leqslant A_4 \quad \text{on } \Omega,$$
$$D_1 = \max_{t\in J}[K_1 p_n^2(t) + K_2 q_n^2(t)], \ \ K_1 = A_1 + A_4 + \frac{3}{2}(A_2 + A_3), \ \ K_2 = \frac{1}{2}(A_2 + A_3).$$

In a similar way, we obtain

$$q'_{n+1}(t) \leqslant L(t)q_{n+1}(t) + D_2 \quad \text{with } D_2 = \max_{t\in J}[K_3 p_n^2(t) + K_4 q_n^2(t)],$$

where

$$K_3 = \frac{1}{2}(A_1 + A_4), \qquad K_4 = A_2 + A_3 + \frac{3}{2}(A_1 + A_4).$$

Put $w = p_{n+1} + q_{n+1}$, so $w(0) = \alpha w(t_1) + \beta w(T)$, and

$$w'(t) \leqslant L(t)w(t) + D, \quad t \in J$$

with $D = D_1 + D_2$. Consequently, by Lemma 2, we have

$$w(t) \leqslant K(t)\Big\{\frac{1}{m}[\alpha K(t_1)C(t_1) + \beta K(T)C(T)] + C(t)\Big\}D, \quad t \in J$$

with K defined as in the assumption H_3(a) and

$$C(t) = \int_0^t \mathrm{e}^{-\int_0^s L(r)\,\mathrm{d}r}\,\mathrm{d}s, \quad m = 1 - \alpha K(t_1) - \beta K(T).$$

Put

$$\gamma = \max_{t\in J} K(t)\Big\{\frac{1}{m}[\alpha K(t_1)C(t_1) + \beta K(T)C(T)] + C(t)\Big\}.$$

Then

$$\max_{t\in J} w(t) \leqslant (K_1 + K_3)\gamma\big[\max_{t\in J} p_n^2(t) + \max_{t\in J} q_n^2(t)\big].$$

Since $\max_{t\in J} p_{n+1}(t) \leqslant \max_{t\in J} w(t)$ and $\max_{t\in J} q_{n+1}(t) \leqslant \max_{t\in J} w(t)$ we get the desired quadratic convergence.

The proof is therefore complete. □

Remark 2. Theorem 1 contains some results of [5] (when $\alpha = 0$), [8], [12] (when $\alpha = \beta = 0$), [10], [12] (when $\alpha = k = 0$, $\beta = 1$).

Section 2

Now, we shall consider the following differential problem

$$(13) \qquad \begin{cases} x'(t) = h(t, x(t)), & t \in J = [0, T],\ \ T > 0, \\ x(0) = \alpha x(t_1) - \beta x(T) + k, & 0 < t_1 < T, \end{cases}$$

where $h \in C(J \times \mathbb{R}, \mathbb{R}),\ k \in \mathbb{R},\ \alpha, \beta \geqslant 0$.

Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled lower and upper solutions of the problem (13) if

$$\begin{cases} u'(t) \leqslant h(t, u(t)),\ \ t \in J, \quad u(0) \leqslant \alpha u(t_1) - \beta v(T) + k, \\ v'(t) \geqslant h(t, v(t)),\ \ t \in J, \quad v(0) \geqslant \alpha v(t_1) - \beta u(T) + k. \end{cases}$$

Theorem 2. *Let the assumptions* H_1 *and* H_3 *hold. Let* y_0, z_0 *be weakly coupled lower and upper solutions of the problem* (13) *and such that* $y_0(t) \leqslant z_0(t)$, $t \in J$. *Then there exist monotone sequences* $\{y_n\}, \{z_n\}$ *which converge monotonically and uniformly to the unique solution of the problem* (13) *and the convergence is quadratic.*

Proof. Let

$$\begin{cases} y'_{n+1}(t) = h(t, y_n) + W(t, y_n, z_n)[y_{n+1}(t) - y_n(t)],\ \ t \in J, \\ y_{n+1}(0) = \alpha y_{n+1}(t_1) - \beta z_{n+1}(T) + k \end{cases}$$

$$\begin{cases} z'_{n+1}(t) = h(t, z_n) + W(t, y_n, z_n)[z_{n+1}(t) - z_n(t)],\ \ t \in J, \\ z_{n+1}(0) = \alpha z_{n+1}(t_1) - \beta y_{n+1}(T) + k, \end{cases}$$

where W is defined as in Lemma 6. The proof is similar to the proof of Theorem 1 and therefore is omitted. □

Remark 3. Theorem 2 contains some results of [5] (when $\alpha = 0$), [12], [16], [17] (when $\alpha = k = 0$, $\beta = 1$).

Section 3

Let us consider a generalization of the problems (1) and (13), namely

$$(14)\qquad \begin{cases} x'(t) = h(t,x), \quad t \in J, \\ x(0) = \sum\limits_{i=1}^{r} a_i x(t_i) + k, \quad 0 < t_1 < t_2 < \ldots < t_{r-1} < t_r = T, \end{cases}$$

where $h \in C(J \times \mathbb{R})$, $k \in \mathbb{R}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, r$. Functions $u, v \in C^1(J, \mathbb{R})$ are called weakly coupled lower and upper solutions of the problem (14) if

$$\begin{cases} u'(t) \leqslant h(t,u), \quad t \in J, \qquad u(0) \leqslant \sum\limits_{i=1}^{r} a_i \zeta(t_i, u, v) + k, \\ v'(t) \geqslant h(t,v), \quad t \in J, \qquad v(0) \geqslant \sum\limits_{i=1}^{r} a_i \eta(t_i, u, v) + k, \end{cases}$$

where

$$\zeta(t_i, u, v) = \begin{cases} u(t_i) & \text{if } a_i > 0, \\ v(t_i) & \text{if } a_i < 0, \end{cases} \qquad \eta(t_i, u, v) = \begin{cases} v(t_i) & \text{if } a_i > 0, \\ u(t_i) & \text{if } a_i < 0. \end{cases}$$

The proof of the next theorem is similar to the proofs of Theorems 1 and 2 and therefore is omitted.

Theorem 3. *Let the assumption* $\mathrm{H}_3(\mathrm{a})$ *hold. Assume that the condition*

$$\sum_{i=1}^{r} |a_i| K(t_i) < 1 \quad \text{for } K(t) = \mathrm{e}^{\int_0^t L(s)\,\mathrm{d}s}$$

is satisfied with L defined as in the assumption $\mathrm{H}_3(\mathrm{b})$. *Let $y_0, z_0 \in C^1(J, \mathbb{R})$ be weakly coupled lower and upper solutions of the problem* (14) *such that $y_0(t) \leqslant z_0(t)$, $t \in J$.*

Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ which converge monotonically and uniformly to the unique solution of the problem (14) *and the convergence is quadratic.*

Remark 4. Note that Theorems 1 and 2 are special cases of Theorem 3. Moreover, Theorem 3 contains some results of [6] when all the coefficients a_i of (14) are positive or all are negative.

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