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NATURAL OPERATORS LIFTING VECTOR FIELDS TO BUNDLES OF WEIL CONTACT ELEMENTS

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Abstract. Let A be a Weil algebra. The bijection between all natural operators lifting vector fields from m-manifolds to the bundle functor K^A of Weil contact elements and the subalgebra of fixed elements SA of the Weil algebra A is determined and the bijection between all natural affinors on K^A and SA is deduced. Furthermore, the rigidity of the functor K^A is proved. Requisite results about the structure of SA are obtained by a purely algebraic approach, namely the existence of nontrivial SA is discussed.

Keywords: Weil algebra, Weil bundle, contact element, natural operator

MSC 2000: 58A32, 12D05, 58A20, 53A55

Introduction

A Weil algebra A is a local commutative $\mathbb{R}$-algebra with identity, the nilpotent ideal $\mathfrak{n}$ of which has a finite dimension as a vector space and $A/\mathfrak{n} = \mathbb{R}$. We call the *order* of A the minimum $\operatorname{ord}(A)$ of the integers r satisfying $\mathfrak{n}^{r+1} = 0$ and the *width* of A $w(A) = \dim(\mathfrak{n}/\mathfrak{n}^2)$.

One can assume that Weil algebras are finite dimensional factor $\mathbb{R}$-algebras of the algebra $\mathbb{R}[t^1, \ldots, t^k]$ of real polynomials in several indeterminates. That is, a Weil algebra A has the form $\mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$, where $\mathfrak{m}^{r+1} \subset \mathfrak{i} \subset \mathfrak{m}$ for some r, $\mathfrak{m} = \langle t^1, \ldots, t^k \rangle$ being the maximal ideal of $\mathbb{R}[t^1, \ldots, t^k]$ ($\mathfrak{i}$ with this property is called the *Weil ideal*). We consider only the case $w(A) \geqslant 1$ and the minimal number of indeterminates, i.e. $k = w(A)$ (then $\mathfrak{i} \subset \mathfrak{m}^2$). Of course, such an expression of the Weil algebra is not unique. Really, $\mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i} \cong \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{j}$ if and only if there is $G \in \operatorname{Aut} \mathbb{R}[t^1 \ldots, t^k]$, $G(\mathfrak{i}) = \mathfrak{j}$.

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Alternatively, one can assume that Weil algebras are finite dimensional factor $\mathbb{R}$-algebras of the algebra of germs $\mathscr{E}_k = C_0^\infty(\mathbb{R}^k, \mathbb{R})$, see [7, Proposition 35.5]. The fact that ideals in $\mathscr{E}_k$ can be generated by some polynomials induces the corresponding ideal $\underline{\mathfrak{i}}$ in $\mathscr{E}_k$ for every Weil ideal $\mathfrak{i}$ in $\mathbb{R}[t^1, \dots, t^k]$.

Let $A = \mathscr{E}_k/\underline{\mathfrak{i}}$ be a Weil algebra and M an m-manifold. Two maps $g, h\colon \mathbb{R}^k \to M$, $g(0) = h(0) = x$ are said to be *A-equivalent* if $\alpha \circ g - \alpha \circ h \in \underline{\mathfrak{i}}$ for every germ α of a smooth function on M at x. Such an equivalence class will be denoted by $j^A g$ and called an *A-velocity* on M. The point $x = g(0)$ is said to be the *target* of $j^A g$. Denote by $T^A M$ the set of all A-velocities on M and by $T_x^A M$ the set of all A-velocities on M with the target x. T^A is a bundle functor on the category of all manifolds, see [7], and $T^A M$ is called the *Weil bundle*.

The theory of Weil bundles is a powerful tool for many problems in differential geometry. The important problem how a vector field on an m-manifold M can induce canonically a vector field on $T^A M$ has been solved completely by I. Kolář in [6] with the aid of the concept of natural operators. We remark that the best known example of a Weil bundle is the bundle $T_k^r M$ of k-dimensional velocities of order r on M, in particular, for $r = k = 1$ the tangent bundle on M.

Let $\operatorname{reg} T^A M \subset T^A M$ be the open subbundle of so-called *regular* A-velocities on M, i.e. if $A = \mathscr{E}_k/\underline{\mathfrak{i}}$, then $j^A g \in \operatorname{reg} T^A M \subset T^A M$ if and only if $g\colon \mathbb{R}^k \to M$ is of rank k at 0. The *contact element of type* A on M determined by $X \in \operatorname{reg} T^A M$ is the equivalence class $\operatorname{Aut} A_M(X) := \{\varphi(X);\ \varphi \in \operatorname{Aut} A\}$, see [5]. We denote by $K^A M$ the set of all contact elements of type A on M. Quite recently, R. Alonso proved in [1] that $K^A M$ has a differentiable manifold structure and $\operatorname{reg} T^A M \to K^A M$ is a principal fiber bundle with structure group $\operatorname{Aut} A$. $K^A M$ is a generalization of higher order contact elements bundle $K_k^r M = \operatorname{reg} T_k^r M / G_k^r$ introduced by C. Ehresmann in [3].

In this paper, we study the problem how a vector field on an m-manifold M can induce canonically a vector field on $K^A M$. This problem is reflected in the concept of natural operators $\mathscr{A}\colon T_{|\mathscr{M}f_m} \rightsquigarrow TK^A$ in the sense of [7]. For $m \geqslant w(A) + 2$ we construct explicitly a bijection between all natural operators $\mathscr{A}\colon T_{|\mathscr{M}f_m} \rightsquigarrow TK^A$ and the subalgebra $SA = \{a \in A;\ \varphi(a) = a \text{ for all } \varphi \in \operatorname{Aut} A\}$ of fixed elements of a Weil algebra A. This main result of the paper is stated in Section 2. In addition, the classification of natural affinors on K^A is established and a rigidity theorem for K^A is presented also in Section 2. Section 1 gives a purely algebraic description of A and can be read independently.

All manifolds and maps are assumed to be of class C^∞.

1. On the subalgebra of fixed elements of a Weil algebra

1.1. Homogeneous Weil algebras.

We recall some known algebraic facts and formulate the definition of a homogeneous Weil algebra. First of all, the algebra $\mathbb{R}[t^1, \ldots, t^k]$ is Noetherian. Thus every ideal $\mathfrak{i}$ in $\mathbb{R}[t^1, \ldots, t^k]$ has a finite set of generators.

Every element $P \in \mathbb{R}[t^1, \ldots, t^k]$ can be written in the form of a finite sum $P = P_0 + P_1 + \ldots + P_j + \ldots$, where P_j is either zero or a homogeneous polynomial of degree j. P_j is called the *homogeneous component of degree* j of P. An ideal $\mathfrak{i}$ in $\mathbb{R}[t^1, \ldots, t^k]$ is said to be *homogeneous* if the relation $P \in \mathfrak{i}$ implies that all homogeneous components of P are in $\mathfrak{i}$. An ideal $\mathfrak{i}$ in $\mathbb{R}[t^1, \ldots, t^k]$ is homogeneous if and only if $\mathfrak{i}$ possesses homogeneous generators, see [15, Theorem VII.2.7]. In general, $G \in \operatorname{Aut} \mathbb{R}[t^1, \ldots, t^k]$ does not preserve the homogeneity of ideals, see Example (ix). (Nevertheless, linear automorphisms preserve the homogeneity of ideals.)

Let A be a Weil algebra. If there is an expression of A as $A \cong \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$, where i is a homogeneous Weil ideal, we call A a *homogeneous Weil algebra.*

Examples.

(i) For $k = 1$, every Weil algebra $A = \mathbb{R}[t]/\mathfrak{i}$ is homogeneous. In this case, $\mathfrak{i}$ is a principal ideal and a monomial of the lowest degree in $\mathfrak{i}$ can be taken as its generator.

(ii) $\mathbb{D}^r_k$ are homogeneous, $\mathbb{D}^r_k$ being the Weil algebras of functors of k-dimensional velocities of order r. Indeed, $\mathbb{D}^r_k = \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{m}^{r+1}$ and a power of the maximal ideal $\mathfrak{m}$ is generated by homogeneous polynomials.

(iii) $\tilde{\mathbb{D}}^r_k$ are homogeneous, $\tilde{\mathbb{D}}^r_k$ being the Weil algebras of functors of nonholonomic k-dimensional velocities of order r. Of course, we can realize $\tilde{\mathbb{D}}^r_k$ as the factor algebra of $\mathbb{D}^r_{rk}$ in the following way $\tilde{\mathbb{D}}^r_k \cong \mathbb{R}[t^1_1, \ldots, t^k_r]/\langle\langle t^1_1, \ldots, t^k_1\rangle^2, \ldots, \langle t^1_r, \ldots, t^k_r\rangle^2\rangle$ and the ideal has homogeneous generators. (Let us notice that $\tilde{\mathbb{D}}^r_k \cong \mathbb{D}^1_k \otimes \ldots \otimes \mathbb{D}^1_k$ and the use of example (vii) is possible, too.)

(iv) $\overline{\mathbb{D}}^r_k$ are homogeneous, $\overline{\mathbb{D}}^r_k$ being the Weil algebras of functors of semiholonomic k-dimensional velocities of order r. The proof is rather long, see [9].

(v) The first author introduced Weil algebras $\overset{\omega}{\mathbb{D}}{}^r_k$ of functors of ω-holonomic k-dimensional velocities of order r, which include nonholonomic and semiholonomic velocities as special cases. They are homogeneous, see also [9].

(vi) $\mathbb{Q}^r_k$ are homogeneous, $\mathbb{Q}^r_k$ being the Weil algebras of functors of k-dimensional quasivelocities of order r. For the proof, it suffices to take the expression of $\mathbb{Q}^r_k$ in the form $\mathbb{Q}^r_k = \mathbb{D}^r_{k(2^r-1)}/\mathfrak{i}$, where the ideal $\mathfrak{i}$ has homogeneous generators described in [13, Proposition 5].

(vii) If A and B are homogeneous Weil algebras, then $A \otimes B$ is homogeneous. Indeed, if $A = \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$ and $B = \mathbb{R}[t^{k+1}, \ldots, t^{k+l}]/\mathfrak{j}$, then $A \otimes B \cong R[t^1, \ldots, t^{k+l}]/\langle \mathfrak{i}, \mathfrak{j} \rangle$ where $\langle \mathfrak{i}, \mathfrak{j} \rangle$ is the least ideal in $R[t^1, \ldots, t^{k+l}]$ which contains $\mathfrak{i}$ and $\mathfrak{j}$ and its generators are homogeneous ditto generators $\mathfrak{i}$ and $\mathfrak{j}$.

(viii) If A is a homogeneous Weil algebra and $\mathfrak{n}$ the ideal of all its nilpotent elements, then q-th underlying Weil algebras $A_q = A/\mathfrak{n}^{q+1}$, [5], are homogeneous for all $q = 1, \ldots, r-1$, as for $A = \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$ we have $A_q \cong \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i} + \mathfrak{m}^{q+1}$.

(ix) Let $A = \mathbb{R}[s,t]/\langle s^2 + 2st^2 + t^4 \rangle + \mathfrak{m}^5$. We demonstrate that A is homogeneous. First, we prove the nonhomogeneity of $\mathfrak{i} = \langle s^2 + 2st^2 + t^4 \rangle + \mathfrak{m}^5$. As $s^2 + 2st^2 + t^4 \in \mathfrak{i}$, we assume $s^2 \in \mathfrak{i}$. Then $s^2 \in PQ + \mathfrak{m}^4$, where $P = P(s,t)$ is some polynomial in s, t and $Q = s^2 + 2st^2$. Hence $s^2 = (k_1 + k_2 s + k_3 t + k_4 s^2 + \ldots)(s^2 + 2st^2) + \ldots = k_1 s^2 + 2k_1 st^2 + \ldots$. Thus $k_1 = 1$ and $2k_1 = 0$. This is a contradiction, so $s^2 \notin \mathfrak{i}$ and $\mathfrak{i}$ is nonhomogeneous. We take $G \in \operatorname{Aut} \mathbb{R}[t^1, \ldots, t^k]$ in this way: $\bar{s} = s + t^2$, $\bar{t} = t$. Then $G(i) = \langle \bar{s}^2 \rangle + \mathfrak{m}^5$ and this is a homogeneous ideal in $\mathbb{R}[\bar{s}, \bar{t}]$. Hence A is homogeneous.

If $H\colon A \to B$ is a homomorphism of $\mathbb{R}$-algebras, then H induces the induced homomorphism $\overline{H}\colon A/\mathfrak{i} \to B/\mathfrak{j}$ if and only if $H(\mathfrak{i}) \subset \mathfrak{j}$. Let $\tau \in \mathbb{R}$. It is evident that for a homogeneous Weil ideal $\mathfrak{i}$, the homomorphism $H_\tau\colon \mathbb{R}[t^1, \ldots, t^k] \to \mathbb{R}[t^1, \ldots, t^k]$, $H_\tau\colon P(t^1, \ldots, t^k) \mapsto P(\tau t^1, \ldots, \tau t^k)$, induces the homomorphism $\overline{H}_\tau\colon \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i} \to \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$, and $\overline{H}_\tau$ is an element of $\operatorname{Aut} A$ for $\tau \neq 0$, $A = \mathbb{R}[t^1, \ldots, t^k]/\mathfrak{i}$.

Let $SA = \{a \in A;\ \varphi(a) = a \text{ for all } \varphi \in \operatorname{Aut} A\}$ be the subalgebra of fixed elements of a Weil algebra A. We find easily the following assertion.

Proposition 1. *If A is a homogeneous Weil algebra, then SA is the trivial subalgebra $\mathbb{R} \cdot 1$.*

Proof. We take an arbitrary $\tau \in \mathbb{R} - \{-1, 0, 1\}$. Then only constants possess the property $\overline{H}_\tau(a) = a$. □

1.2. Nonhomogeneous Weil algebras.

Example *of a nonhomogeneous Weil algebra with trivial subalgebra of fixed elements.*

Let $A = \mathbb{R}[s,t]/\langle s^2 + t^3 \rangle + \mathfrak{m}^4$. We demonstrate that A is nonhomogeneous and $SA = \mathbb{R} \cdot 1$.

In the first instance, we presume the homogeneity of A. This means that there is $G \in \operatorname{Aut} \mathbb{R}[s,t]$ such that $G(\mathfrak{i}) = \mathfrak{j}$, where $\mathfrak{i} = \langle s^2 + t^3 \rangle + \mathfrak{m}^4$ and $\mathfrak{j}$ is generated by homogeneous polynomials $P_1, \ldots, P_L$. We can assume that the matrix of the linear part of G is the identity matrix. (If not, we compose G with a linear automorphism.)

Since $G^{-1}(\mathfrak{m}^3) \subset \mathfrak{m}^3$ and $s^2 + t^3 \in \mathfrak{i} - \mathfrak{m}^3$, $\mathfrak{j} \not\subset \mathfrak{m}^3$. Thus there is a homogeneous generator of $\mathfrak{j}$ with degree 2 and we can suppose that it is P_1. We have $G^{-1}(P_1) \in P_1 + \mathfrak{m}^3$ and $G^{-1}(P_1) \in QR + \mathfrak{m}^4$, where $Q = Q(s,t)$ is some polynomial in s, t and $R = s^2 + t^3$. It follows that $P_1 = as^2$. Thus, we assume $P_1 = s^2$ hereafter. We have $G^{-1}(P_1) \in (s + \mathfrak{m}^2)^2$, i.e. $G^{-1}(P_1) = s^2 + k_1 s^3 + k_2 s^2 t + k_3 s^4 + \ldots$. We have also $G^{-1}(P_1) \in QR + \mathfrak{m}^4$ as above, i.e. $G^{-1}(P_1) = (l_1 + l_2 s + l_3 t + l_4 s^2 + \ldots)(s^2 + t^3) + \ldots = l_1 s^2 + l_1 t^3 + \ldots$. Thus $l_1 = 1$ and $l_1 = 0$. This is a contradiction, hence A is nonhomogeneous.

The elements of A have the form

$$k_1 + k_2 s + k_3 t + k_4 s^2 + k_5 st + k_6 t^2 + k_7 st^2$$

with all monomials of the fourth or higher order vanishing, in addition to s^3, $s^2 t$ and $s^2 + t^3$. We shall describe the automorphisms of A. The starting point for their identification is the form

$$\begin{aligned} \bar{s} &= As + Bt + Cs^2 + Dst + Et^2 + Fst^2, \\ \bar{t} &= Gs + Ht + Is^2 + Jst + Kt^2 + Lst^2. \end{aligned} \tag{1}$$

The matrix $\begin{pmatrix} A & B \\ G & H \end{pmatrix}$ of the linear part of an automorphism must be regular. We must now satisfy the conditions $\bar{s}^3 = 0$, $\bar{s}^2\bar{t} = 0$, and $\bar{s}^2 + \bar{t}^3 = 0$. The condition $\bar{s}^3 = 0$ gives $3AB^2 st^2 + B^3 t^3 = 0$. It follows that $B = 0$. Then $\bar{s}^2\bar{t} = 0$ gives no new nontrivial relation. For the condition $\bar{s}^2 + \bar{t}^3 = 0$, we obtain $A^2 s^2 + (2AE + 3GH^2)st^2 + H^3 t^3 = 0$ and it follows that $A^2 = H^3$ and $2AE + 3GH^2 = 0$. It is impossible that $A = H = 0$, hence $A = \tau^3$, $H = \tau^2$ for some $\tau \neq 0$ and $G = -\frac{2}{3}\tau E$.

Hence the automorphisms have the following form

$$\begin{aligned} \bar{s} &= \tau^3 s + Cs^2 + Dst + Et^2 + Fst^2, \\ \bar{t} &= -\frac{2}{3\tau} Es + \tau^2 t + Is^2 + Jst + Kt^2 + Lst^2. \end{aligned} \tag{1A}$$

We choose the automorphism φ

$$\begin{aligned} \bar{s} &= 8s, \\ \bar{t} &= 4t \end{aligned}$$

and it is not difficult to find that only constants possess the property $\varphi(a) = a$.

Now, it is not surprising that the following upgrade of Proposition 1 is possible by a relatively slight generalization. Let $\tau_1, \ldots, \tau_k \in \mathbb{R}$. We take as the homomorphism $H_{\tau_1,\ldots,\tau_k} \colon \mathbb{R}[t^1, \ldots, t^k] \to \mathbb{R}[t^1, \ldots, t^k]$, $H_{\tau_1,\ldots,\tau_k} \colon P(t^1, \ldots, t^k) \mapsto P(\tau_1 t^1, \ldots, \tau_k t^k)$.

Proposition 2. *If $A = \mathbb{R}[t^1, \dots, t^k]/\mathfrak{i}$ is a Weil algebra with $w(A) = k$ and if there exist some $\tau_1, \dots, \tau_k \in \mathbb{R} - [-1, 1]$ (or $\tau_1, \dots, \tau_k \in (-1, 1) - \{0\}$) such that $H_{\tau_1,\dots,\tau_k}(\mathfrak{i}) \subset \mathfrak{i}$, then SA is the trivial subalgebra $\mathbb{R} \cdot 1$.*

Proof. The idea is the same as in the proof of Proposition 1. □

Exercise 1. We leave it to the reader to prove that $A = \mathbb{R}[s, t]/\langle s^2 + t^3, s^3 + t^4\rangle + \mathfrak{m}^5$ is an example of a Weil algebra with these properties:

(α) A is nonhomogeneous,

(β) there are no $\tau_1, \tau_2 \in \mathbb{R} - [-1, 1]$ (or $\tau_1, \tau_2 \in (-1, 1) - \{0\}$) such that $H_{\tau_1,\tau_2}(\langle s^2 + t^3, s^3 + t^4\rangle + \mathfrak{m}^5) \subset \langle s^2 + t^3, s^3 + t^4\rangle + \mathfrak{m}^5$,

(γ) A has trivial subalgebra of fixed elements.

Example *of a nonhomogeneous Weil algebra with a nontrivial subalgebra of fixed elements.*

Let $A = \mathbb{R}[s, t]/\langle st^2 + s^4, s^2t + t^5\rangle + \mathfrak{m}^6$. We demonstrate that $SA \supsetneqq \mathbb{R} \cdot 1$. (Then the nonhomogeneity of A is a consequence of this fact.) The elements of A have the form

$$k_1 + k_2 s + k_3 t + k_4 s^2 + k_5 st + k_6 t^2 + k_7 s^3 + k_8 s^2 t + k_9 st^2 + k_{10} t^3 + k_{11} t^4$$

with all monomials of the sixth or higher order vanishing, in addition to s^5, s^3t, s^2t^2, st^3, $st^2 + s^4$ and $s^2t + t^5$. We shall describe the automorphisms of A. The starting point for their identification is the form

$$\begin{aligned} \bar{s} &= As + Bt + Cs^2 + Dst + Et^2 + Fs^3 + Gs^2t + Hst^2 + It^3 + Jt^4, \\ \bar{t} &= Ks + Lt + Ms^2 + Nst + Ot^2 + Ps^3 + Qs^2t + Rst^2 + St^3 + Tt^4. \end{aligned} \tag{2}$$

The matrix $\begin{pmatrix} A & B \\ K & L \end{pmatrix}$ of the linear part of an automorphism must be regular. We must now satisfy the conditions $\bar{s}^5 = 0$, $\bar{s}^3\bar{t} = 0$, $\bar{s}^2\bar{t}^2 = 0$, $\bar{s}\bar{t}^3 = 0$, $\bar{s}\bar{t}^2 + \bar{s}^4 = 0$, and $\bar{s}^2\bar{t} + \bar{t}^5 = 0$. The condition $\bar{s}^5 = 0$ gives $B^5t^5 = 0$. It follows that $B = 0$. The condition $\bar{s}^3\bar{t} = 0$ gives $A^3Ks^4 = 0$. It follows that $K = 0$. Then $\bar{s}^2\bar{t}^2 = 0$ gives no new nontrivial relation. The condition $\bar{s}\bar{t}^3 = 0$ gives $EL^3t^5 = 0$. It follows that $E = 0$. For the condition $\bar{s}\bar{t}^2 + \bar{s}^4 = 0$ we obtain $AL^2st^2 + IL^2t^5 + A^4s^4 = 0$ and it follows that $L^2 = A^3$ and $I = 0$. Finally, for the condition $\bar{s}^2\bar{t} + \bar{t}^5 = 0$, we obtain $A^2Ls^2t + A^2Ms^4 + L^5t^5 = 0$ and it follows that $A^2 = L^4$ and $M = 0$. The conditions $L^2 = A^3$ and $A^2 = L^4$ give $A = 1$ and $L = \pm 1$.

Hence the automorphisms have the following form

$$\begin{aligned} \bar{s} &= s + Cs^2 + Dst + Fs^3 + Gs^2t + Hst^2 + Jt^4, \\ \bar{t} &= \pm t + Nst + Ot^2 + Ps^3 + Qs^2t + Rst^2 + St^3 + Tt^4. \end{aligned} \tag{2A}$$

Consequently, we solve the equation

$$\begin{aligned}&k_1 + k_2\bar{s} + k_3\bar{t} + k_4\bar{s}^2 + k_5\bar{s}\bar{t} + k_6\bar{t}^2 + k_7\bar{s}^3 + k_8\bar{s}^2\bar{t} + k_9\bar{s}\bar{t}^2 + k_{10}\bar{t}^3 + k_{11}\bar{t}^4\\ &= k_1 + k_2 s + k_3 t + k_4 s^2 + k_5 st + k_6 t^2 + k_7 s^3 + k_8 s^2 t + k_9 st^2 + k_{10} t^3 + k_{11} t^4\end{aligned}$$

for k_i, $i = 1, \dots, 11$, using (2A). We obtain

$$\begin{aligned}&k_1 + k_2(s + Cs^2 + Dst + Fs^3 + Gs^2t + Hst^2 + Jt^4)\\ &\quad + k_3(\pm t + Nst + Ot^4 + Ps^3 + Qs^2t + Rst^2 + St^3 + Tt^4)\\ &\quad + k_4(s^2 + 2Cs^3 + 2Ds^2t + 2Fs^4 + C^2s^4)\\ &\quad + k_5(\pm st + Ns^2t + Ost^2 + Ps^4 \pm Cs^2t \pm Dst^2 \pm Jt^5)\\ &\quad + k_6(t^2 \pm 2Nst^2 \pm 2Ot^3 \pm 2St^4 \pm 2Tt^5 + O^2t^4 + 2OSt^5)\\ &\quad + k_7(s^3 + 3Cs^4) + k_8(\pm s^2t) + k_9 st^2\\ &\quad + k_{10}(\pm t^3 + 3Ot^4 + 3St^5 \pm 3O^2t^5) + k_{11}(t^4 \pm 4Ot^5)\\ &= k_1 + k_2 s + k_3 t + k_4 s^2 + k_5 st + k_6 t^2 + k_7 s^3 + k_8 s^2 t + k_9 st^2 + k_{10} t^3 + k_{11} t^4\end{aligned}$$

Comparing the coefficients standing at powers of s and t, we find that $k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = k_8 = k_{10} = k_{11} = 0$ and k_1, k_9 are arbitrary real coefficients. This means that

$$SA = \{k_1 + k_9 st^2;\ k_1, k_9 \in \mathbb{R}\} \tag{3}$$

and we have obtained the description of the subalgebra of fixed elements. Naturally, SA is nontrivial, i.e. $SA \supsetneqq \mathbb{R} \cdot 1$. This proves our claim.

Proposition 3. *There are Weil algebras with nontrivial subalgebras of fixed elements.*

2. The classification theorems

2.1. (a)**-lifts and** $\langle a\rangle$**-lifts. Affinors** $\mathrm{af}(a)$ **and** $\mathrm{Af}(a)$**.**

Let $X\colon M \rightsquigarrow TM$ be a vector field on an m-manifold M. Given a natural bundle F over m-manifolds, one general operator $T \to TF$ is the *flow operator* $\mathscr{F}$, which is defined by

$$\mathscr{F}_M(X) := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_0 F(Fl_s^X),$$

where Fl_s^X means the flow of a vector field X. The vector field $\mathscr{F}_M(X)$ on FM is called the *complete lift* of X to FM.

Let A be a Weil algebra and $a \in A$. Then a determines the following action on $TT^A\mathbb{R}^m\colon\ (p_1,\dots,p_m,v_1,\dots,v_m) \mapsto (p_1,\dots,p_m,av_1,\dots,av_m)$. This implies that the action of any $a \in A$ on TT^AM is a natural affinor $\mathrm{af}_M(a)\colon TT^AM \to TT^AM$, see [2], [4]. The vector field $X^{(a)}$ on T^AM defined as

$$X^{(a)} := \mathrm{af}_M(a) \circ \mathscr{T}^A_M(X)$$

is called the (a)-*lift* of X to T^AM. This lift was introduced by I. Kolář in [6], cf. also [4]. Immediately, $X^{(1)}$ is the complete lift.

So, let $a \in SA$. $\pi\colon \operatorname{reg} T^AM \to K^AM$ is a principal fiber bundle with structure group $\operatorname{Aut} A$. Let $u \in TK^AM$. Choose $v \in T(\operatorname{reg} T^AM)$ with $T\pi(v) = u$ and put

$$\mathrm{Af}_M(a)(u) := T\pi(\mathrm{af}_M(a)(v)).$$

We prove that our definition is correct. Let $w \in T(\operatorname{reg} T^AM)$ be another vector with $T\pi(w) = u$. Let $w_t, v_t \in \operatorname{reg} T^AM$ be the curves representing w and v, respectively. Since π is a submersion, we can assume $\pi(w_t) = \pi(v_t)$. Then there exists a smoothly parametrized family $\varphi_t \in \mathrm{Aut}(A)$ such that $w_t = \varphi_t(v_t)$. We define a vector field Y on T^AM by $Y_y = \mathrm{af}_M(a)(\mathrm{d}/\mathrm{d}t\big|_0\varphi_t(y))$, where $y \in T^AM$. Then Y is an absolute vector field on T^AM and the flow $F_s = F1^Y_s$ of Y belongs to $\mathrm{Aut}(A)$. Thus, $T\pi(\mathrm{af}_M(a)(\mathrm{d}/\mathrm{d}t\big|_0\varphi_t(v_0))) = T\pi(Y_{v_0}) = T\pi(\mathrm{d}/\mathrm{d}s\big|_0 F_s(v_0)) = \mathrm{d}/\mathrm{d}s\big|_0(\pi \circ F_s(v_0)) = 0$ as $\pi \circ F_s = \pi$ and $T\pi(\mathrm{af}_M(a)(w)) = T\pi(\mathrm{af}_M(a)(\mathrm{d}/\mathrm{d}t\big|_0\varphi_t(v_t))) = T\pi(\mathrm{af}_M(a)(T\varphi_0(v))) + T\pi(\mathrm{af}_M(a)(\mathrm{d}/\mathrm{d}t\big|_0\varphi_t(v_0))) = T\pi(T\varphi_0 \circ \mathrm{af}_M(\varphi_0^{-1}(a))(v)) = T\pi(\mathrm{af}_M(a)(v))$ as $T\varphi_0 \circ \mathrm{af}_M(\varphi_0^{-1}(a)) \circ T\varphi_0^{-1} = \mathrm{af}_M(a)$, $\varphi_0^{-1}(a) = a$ and $\pi \circ \varphi_0^{-1} = \pi$. Hence the definition is correct.

The family $\mathrm{Af}(a) = \{\mathrm{Af}_M(a)\}$ is a natural affinor on K^A depending linearly on $a \in SA$. If $a = 1$, $\mathrm{Af}(1)$ is the identity natural affinor on K^A and $\mathrm{Af}_M(1)$ is the identity map on TK^AM.

The vector field $X^{\langle a\rangle}$ on K^AM defined as

$$X^{\langle a\rangle} := \mathrm{Af}_M(a) \circ \mathscr{K}^A_M(X)$$

is called the $\langle a\rangle$-*lift* of X to K^AM. The correspondence $\mathscr{A}^{\langle a\rangle}\colon T_{|\mathscr{M}f_m} \to TK^A$, $X \to X^{\langle a\rangle}$ is a linear natural operator depending linearly on $a \in SA$. If $a = 1$, $\mathscr{A}^{\langle a\rangle}$ is the flow operator $\mathscr{K}^A$ and $X^{\langle 1\rangle}$ is the complete lift.

Exercise 2. Verify that another equivalent way how to define correctly $X^{\langle a\rangle}$ for $a \in SA$ is the following. Let $u \in K^AM$. Choose $v \in \operatorname{reg} T^AM$ with $\pi(v) = u$ and put $X^{\langle a\rangle}_{|u} := T\pi(X^{(a)}_{|v})$.

2.2. Liftings of vector fields to K^A.

The first main result of this paper is the following classification theorem.

Theorem 1. *Let A be a Weil algebra, $m \geqslant w(A)+2$. Then for every natural operator $\mathscr{A}\colon T_{|\mathscr{M}fm} \rightsquigarrow TK^A$ there exists uniquely determined $a \in SA$ such that $\mathscr{A} = \mathrm{Af}(a) \circ \mathscr{K}^A$.*

P r o o f. *Step 1. The choice of σ.*

We denote by $t^1,\dots,t^k$ and $x^1,\dots,x^m$ the coordinates on $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, $k = w(A)$. Since $m \geqslant k+2$, we have the embedding $\tilde\sigma\colon \mathbb{R}^k \to \mathbb{R}^m$, $\tilde\sigma(t^1,\dots,t^k) = (0,t^1,\dots,t^k,0,\dots,0)$. Then $j^A\tilde\sigma$ has 0 as the target and it is regular, i.e. $j^A\tilde\sigma \in \operatorname{reg} T^A_0\mathbb{R}^m$. It follows $\sigma = i(j^A\tilde\sigma) \in K^A_0\mathbb{R}^m$.

Step 2. $\mathscr{A}$ is determined by $\mathscr{A}(\partial/\partial x^1)_{|\sigma}$.

Consider a natural operator $\mathscr{A}\colon T_{|\mathscr{M}fm} \rightsquigarrow TK^A$. We prove that $\mathscr{A}$ is uniquely determined by $\mathscr{A}(\partial/\partial x^1)_{|\sigma}$. Every vector field X with non-zero value at x can be expressed in a suitable local coordinate system centered at x as the constant vector field $\partial/\partial x^1$. In addition, the well-known fact following from the theory of natural operators is that $\mathscr{A}$ is uniquely determined by $\mathscr{A}(\partial/\partial x^1)_{|K^A_0\mathbb{R}^m}$. We need to show that the orbit through $\sigma \in K^A_0\mathbb{R}^m$ with respect to the diffeomorphisms $\mathbb{R}^m \to \mathbb{R}^m$ preserving $\mathrm{germ}_0(\partial/\partial x^1)$ forms a dense subset in $K^A_0\mathbb{R}^m$. We consider an arbitrary map $\gamma\colon \mathbb{R}^k \to \mathbb{R}^m$, $\gamma(t^1,\dots,t^k) = (\gamma^1(t),\dots,\gamma^m(t))$ such that $\gamma(0)=0$ and the map $p\circ\gamma\colon \mathbb{R}^k \to \mathbb{R}^{m-1}$ is of rank k at 0 (where $p\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$, $p(x^1,\dots,x^m) = (x^2,\dots,x^m)$, is the canonical projection). Since all $\pi(j^A\gamma)$ with such a γ form a dense subset in $K^A_0\mathbb{R}^m$, it is sufficient to verify that $\pi(j^A\gamma)$ is in the mentioned orbit. We deduce this as follows. Since $k \geqslant 1$ and $m \geqslant k+1$, we have a diffeomorphism $\varphi\colon \mathbb{R}^m \to \mathbb{R}^m$, $\varphi(x^1,\dots,x^m) = (x^1+\gamma^1(x^2,\dots,x^{k+1}),x^2,\dots,x^m)$. Evidently, φ preserves $\mathrm{germ}_0(\partial/\partial x^1)$ and $K^A\varphi\circ\pi(j^A(\tilde\sigma)) = \pi(j^A(\varphi\circ\tilde\sigma)) = \pi(j^A(\gamma_1(t),t^1,\dots,t^k,0,\dots,0))$. On the other hand, since $p\circ\gamma$ is of rank k near $0 \in \mathbb{R}^k$, there is a diffeomorphism $\psi\colon \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$ such that $p\circ\gamma = \psi\circ(t^1,\dots,t^k,0,\dots,0)$ near $0 \in \mathbb{R}^k$. Then $id_{\mathbb{R}}\times\psi$ preserves $\mathrm{germ}_0(\partial/\partial x^1)$ and sends $\pi(j^A(\gamma_1(t),t^1,\dots,t^k,0,\dots,0))$ into $\pi(j^A(\gamma))$. Hence $\pi(j^A\gamma)$ is in the orbit.

Step 3. $\mathscr{A}$ is sum of a vertical operator and a multiple of the flow operator.

We prove that $\mathscr{A} = \alpha\mathscr{A}^{\langle 1\rangle} + \mathscr{V}$ for some $\alpha \in \mathbb{R}$ and some Π-vertical operator $\mathscr{V}\colon T_{|\mathscr{M}fm} \rightsquigarrow TK^A$, where Π is the bundle functor projection of K^A. Let $\alpha^i \in \mathbb{R}$, $i=1,\dots,m$ be the coordinates of the vector $Z = T\Pi(\mathscr{A}(\partial/\partial x^1)_{|\sigma})$. For $\tau \neq 0$, we take the $\mathcal{M}f_m$-maps $c_\tau\colon \mathbb{R}^m \to \mathbb{R}^m$, $c_\tau(x^1,\dots,x^m) = (\tau x^1,\dots,x^m)$. The maps c_τ preserve σ, send $\partial/\partial x^1$ into $\tau\partial/\partial x^1$ and send Z into $\overline{Z}$, the coordinates of which are $\tau\alpha^1,\alpha^2,\dots,\alpha^m$. Hence $\overline{Z} = T\Pi(\mathscr{A}(\tau\partial/\partial x^1)_{|\sigma})$. For $\tau\to 0$, we obtain $T\Pi(\mathscr{A}(0)_{|\sigma})$, but $\mathscr{A}(0)$ is an absolute operator and, consequently, a Π-vertical operator. Thus, $\alpha^2 = \ldots = \alpha^m = 0$. As the (first) coordinate of the vector $T\Pi(\mathscr{A}^{\langle 1\rangle}(\partial/\partial x^1)_{|\sigma})$ equals 1, $V := \mathscr{A} - \alpha^1\mathscr{A}^{\langle 1\rangle}$ is Π-vertical.

Step 4. The expression of the flow of $\mathscr{V}(\partial/\partial x^1)$.

In view of the previous step of the proof, we shall investigate only the Π-vertical operator $\mathscr{V}$ from now on. We study the flow $F_s = F1_s^{\mathscr{V}(\partial/\partial x^1)}$ of $\mathscr{V}(\partial/\partial x^1)$, and it suffices to study $F_s(\sigma)$ for small s thanks to the step 2. We can write $F_s(\sigma) = \pi(j^A(\tilde{\sigma}+\tilde{\sigma}_s))$, where $\tilde{\sigma}_s\colon \mathbb{R}^k \to \mathbb{R}^m$ is some family of maps smoothly parametrized by s, with $\tilde{\sigma}_s(0)=0$ and $\tilde{\sigma}_0(t)=0$. For $\tau \neq 0$, we take the $\mathcal{M}f_m$-maps $b_\tau\colon \mathbb{R}^m \to \mathbb{R}^m$, $b_\tau(x^1,\dots,x^m) = (x^1,\dots,x^{k+1},\tau x^{k+2},\dots,\tau x^m)$. The maps b_τ preserve σ and $\partial/\partial x^1$. Hence b_τ preserve also $F_s(\sigma)$, which means that $F_s(\sigma) = \pi(j^A(b_\tau\circ(\tilde{\sigma}+\tilde{\sigma}_s)))$. For $\tau \to 0$ we get $F_s(\sigma) = \pi(j^A(\tilde{\sigma}_s^1, t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0))$, where s is so small that $j^A(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0) \in \operatorname{reg} T^A\mathbb{R}^m$.

Step 5. The invariance of $\underline{\mathfrak{i}}$ with respect to $(\varrho_s)^*$.

Let $\varrho_s\colon \mathbb{R}^k \to \mathbb{R}^k$, $\varrho_s(t^1,\dots,t^k) = (t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1})$, $(\varrho_s)^*\colon \mathscr{E}_k \to \mathscr{E}_k$ be the pullback of ϱ_s and $A = \mathscr{E}_k/\underline{\mathfrak{i}}$ the Weil algebra in question. We prove that $(\varrho_s)^*(\underline{\mathfrak{i}}) \subset \underline{\mathfrak{i}}$. We consider a map $\eta\colon \mathbb{R}^k \to \mathbb{R}$ with $\operatorname{germ}_0(\eta) \in \underline{\mathfrak{i}}$. Since $m \geqslant k+2$, we have a diffeomorphism $\chi\colon \mathbb{R}^m \to \mathbb{R}^m$, $\chi(x^1,\dots,x^m) = (x^1,\dots,x^{k+1},x^{k+2}+\eta(x^2,\dots,x^{k+1}),x^{k+3},\dots,x^m)$. Clearly, χ preserves $\partial/\partial x^1$ and χ preserves σ as $\operatorname{germ}_0(\eta) \in \underline{\mathfrak{i}}$. Hence χ preserve also $F_s(\sigma)$. Furthermore, $\chi(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0) = (\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},\eta\circ\varrho_s,0\dots,0)$. Then we have $F_s(\sigma) = \pi(j^A(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0)) = \pi(j^A(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},\eta\circ\varrho_s,0\dots,0))$. Then there is some $\varphi \in \operatorname{Aut} A$ such that $\varphi(j^A(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0)) = j^A((\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},\eta\circ\varrho_s,0\dots,0))$. This means that $j^A(0) = j^A(\eta\circ\varrho_s)$ and that is why $\operatorname{germ}_0(\eta\circ\varrho_s) \in \underline{\mathfrak{i}}$, in other words $(\varrho_s)^*(\underline{\mathfrak{i}}) \subset \underline{\mathfrak{i}}$.

Step 6. The expression of the flow of $\mathscr{V}(\partial/\partial x^1)$ anew.

Let $[(\varrho_s)^*]\colon A \to A$ be the quotient homomorphism. A is finite dimensional and $(\varrho_s)^{-1}$ exists near $0 \in \mathbb{R}^k$ if s is small. Thus $[(\varrho_s)^*] \in \operatorname{Aut}(A)$ and $[(\varrho_s)^*]^{-1} = [((\varrho_s)^{-1})^*]$. Hence $F_s(\sigma) = \pi([(\varrho_s)^*]^{-1}(j^A(\tilde{\sigma}_s^1,t^1+\tilde{\sigma}_s^2,\dots,t^k+\tilde{\sigma}_s^{k+1},0,\dots,0))) = \pi(j^A(\tilde{\sigma}_s^1\circ(\varrho_s)^{-1},t^1,\dots,t^k,0,\dots,0)) = \pi(j^A(\eta_s,t^1,\dots,t^k,0,\dots,0))$, where $\eta_s\colon \mathbb{R}^k \to \mathbb{R}$ is some family, smoothly parametrized by s, with $\eta_s(0)=0$ and $\eta_0(t)=0$.

Step 7. $[\operatorname{germ}_0(\eta_s)]_{|\underline{\mathfrak{i}}}$ belongs to SA.

Let us denote $a_s = [\operatorname{germ}_0(\eta_s)]_{|\underline{\mathfrak{i}}}$. We take a diffeomorphism $\tilde{\varphi}\colon \mathbb{R}^k \to \mathbb{R}^k$ preserving 0 such that $\underline{\mathfrak{i}}$ is invariant with respect to the pullback $\tilde{\varphi}^*\colon \mathscr{E}^k \to \mathscr{E}^k$. Let $\varphi = [\tilde{\varphi}^*]\colon A \to A$ be its quotient map. Then $\varphi^{-1} = [(\tilde{\varphi}^{-1})^*]$ and $\varphi \in \operatorname{Aut} A$. Let $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$, $\Phi(x^1,\dots,x^m) = (x^1,\tilde{\varphi}^1(x^2,\dots,x^{k+1}),\dots,\tilde{\varphi}^k(x^2,\dots,x^{k+1}),x^{k+2},x^m)$. Evidently, $\Phi(0)=0$ and Φ preserves $\partial/\partial x^1$. Φ preserves also σ as $K^A\Phi(\sigma) = \pi(j^A(\Phi\circ\tilde{\sigma})) = \pi(\varphi^{-1}(j^A(\Phi\circ\tilde{\sigma}))) = \pi(j^A(\Phi\circ\tilde{\sigma}\circ\tilde{\varphi}^{-1})) = \pi(j^A(0,t^1,\dots,t^k,0,\dots,0))$. Hence Φ preserves $F_s(\sigma)$. Now $F_s(\sigma) = \pi(j^A(\Phi\circ(\eta_s,t^1,\dots,t^k,0,\dots,0))) = \pi(j^A(\eta_s,\tilde{\varphi}^1,\dots,\tilde{\varphi}^k,0,\dots,0)) = \pi(\varphi^{-1}(j^A(\eta_s,\tilde{\varphi}^1,\dots,\tilde{\varphi}^k,0,\dots,0))) = \pi(j^A(\eta_s\circ\tilde{\varphi}^{-1},t^1,\dots,t^k,0,\dots,0))$. Hence there is some $\psi \in \operatorname{Aut} A$ such that $\psi(j^A(\eta_s,t^1,\dots,t^k,0,\dots,0)) = j^A(\eta_s\circ\tilde{\varphi}^{-1},t^1,\dots,t^k,0,\dots,0)$. It follows that $\psi(j^A\eta_s) = j^A(\eta_s\circ$

$\tilde\varphi^{-1}$). In addition, we obtain $\psi(j^At^1) = j^At^1$, ..., $\psi(j^At^k) = j^At^k$, i.e. ψ is nothing but the identity. Thus, $j^A\eta_s = j^A(\eta_s \circ \tilde\varphi^{-1})$, which means that $\varphi(a_s) = a_s$ for any $\varphi \in \operatorname{Aut} A$. Thus $a_s \in SA$.

Step 8. $\mathscr{A}$ *equals* $\mathscr{A}^{\langle a\rangle}$.

Let $\tilde\eta\colon \mathbb{R}^k \to \mathbb{R}$, $\tilde\eta := \mathrm{d}/\mathrm{d}s\big|_0 \eta_s$, $a := \mathrm{d}/\mathrm{d}s\big|_0 a_s$. Then $a = [\mathrm{germ}_0(\tilde\eta)]_{|\underline{\mathrm{i}}} \in SA$. We have $\mathscr{V}(\partial/\partial x^1)_{|\sigma} = \mathrm{d}/\mathrm{d}s\big|_0 F_s(\sigma) = \mathrm{d}/\mathrm{d}s\big|_0(\pi(j^A(\eta_s, t^1, \dots, t^k, 0, \dots 0))) = \mathrm{d}/\mathrm{d}s\big|_0(\pi(j^A(s\tilde\eta, t^1, \dots, t^k, 0, \dots, 0))) = A^{\langle a\rangle}(\partial/\partial x^1)_{|\sigma}$. Hence $\mathscr{A} = \mathscr{A}^{\langle a\rangle}$ as in the steps 2 and 3.

Step 9. a is uniquely determined.

To prove that a is uniquely determined it suffices to show that $\mathscr{A}^{\langle a\rangle} = 0$ implies $a = 0$. Let $A^{\langle a\rangle} = 0$, $a \in SA$. There exists $\eta\colon \mathbb{R}^k \to \mathbb{R}$ such that $a = [\mathrm{germ}_0(\eta)]_{|\underline{\mathrm{i}}}$. Let φ_s be the flow of $(\partial/\partial x^1)^{\langle a\rangle}$. Then $\varphi_s(\sigma) = \pi(j^A(s\eta, t^1, \dots, t^k, 0, \dots, 0))$. For sufficiently small $s_0 \neq 0$, we have $\varphi_{s_0}(\sigma) = \sigma$ as $\mathscr{A}^{\langle a\rangle} = 0$. We obtain $\varphi(j^A(0, t^1, \dots, t^k, 0, \dots, 0)) = j^A(s_0\eta, t^1, \dots, t^k, 0, \dots, 0)$ for some $\varphi \in \operatorname{Aut} A$. Thus, $j^A\eta = j^A 0$. Hence $a = 0$. □

2.3. Natural affinors on K^A.

The second main result of this paper is the following classification theorem.

Theorem 2. *Let A be a Weil algebra, $m \geqslant w(A) + 2$. Then for every natural affinor $\mathscr{Q}$ on K^A there exists uniquely determined $a \in SA$ such that $\mathscr{Q} = \mathrm{Af}(a)$.*

P r o o f. Using $\mathscr{Q}$ we define natural operator $\mathscr{Q} \circ A^{\langle 1\rangle}\colon T \rightsquigarrow TK^A$. Then there exists a uniquely determined $a \in SA$ such that $\mathscr{Q} \circ \mathscr{A}^{\langle 1\rangle} = \mathscr{A}^{\langle 1\rangle} = \mathrm{Af}(a) \circ \mathscr{A}^{\langle 1\rangle}$. Let $\tilde\sigma$ and σ be as in the proof of Theorem 1. Clearly, $(\partial/\partial x^1)^{(1)}_{|j^A(\tilde\sigma)} \in TT^A\mathbb{R}^m$ has dense orbit. Then $\varrho := (\partial/\partial x^1)^{\langle 1\rangle}_{|\sigma} \in TK^A\mathbb{R}^m$ has dense orbit, too. But $\mathscr{Q}(\varrho) = \mathrm{Af}(a)(\varrho)$. Consequently, $\mathscr{Q} = \mathrm{Af}(a)$. □

2.4. Corollaries.

Corollary 1. *Let A be a Weil algebra, $m \geqslant w(A) + 2$ and $SA = \mathbb{R} \cdot 1$. Then every natural operator $T_{|\mathscr{M}f_m} \rightsquigarrow TK^A$ is a constant multiple of the flow operator.*

Corollary 2. *Let A be a homogeneous Weil algebra and $m \geqslant w(A) + 2$. Then every natural operator $T_{|\mathscr{M}f_m} \rightsquigarrow TK^A$ is a constant multiple of the flow operator.*

Corollary 3. *Let $m \geqslant k_1 + \ldots + k_l + 2$. Then every natural operator $T_{|\mathscr{M}f_m} \to TK^A$, where A is the Weil algebra of the functor $T^{r_1}_{k_1} \circ \ldots \circ T^{r_l}_{k_l}$, is a constant multiple of the flow operator.*

Corollary 4. *Let A be a Weil algebra, $m \geqslant w(A) + 2$. Then every canonical vector field on K^A is the zero vector field.*

Corollary 5. *Let A be a Weil algebra, $m \geqslant w(A) + 2$ and $SA = \mathbb{R} \cdot 1$. Then every natural affinor on K^A is a constant multiple of the identity affinor.*

Corollary 6. *Let A be a homogeneous Weil algebra and $m \geqslant w(A) + 2$. Then every natural affinor on K^A is a constant multiple of the identity affinor.*

Corollary 7. *Let $m \geqslant k_1 + \ldots + k_l + 2$. Then every natural affinor on K^A, where A is the Weil algebra of the functor $T^{r_1}_{k_1} \circ \ldots \circ T^{r_l}_{k_l}$, is a constant multiple of the identity affinor.*

Proofs. Corollary 1 follows from Theorem 1 immediately. Corollary 2 follows from Corollary 1 and Proposition 1. Corollary 3 follows from Corollary 2, because the Weil algebra of the functor $T^{r_1}_{k_1} \circ \ldots \circ T^{r_l}_{k_l}$ is homogeneous as in examples (ii) and (vii). Corollary 4 follows from Theorem 1 immediately. Corollary 5 follows from Theorem 2 immediately. Corollary 6 follows from Corollary 5 and Proposition 1. Corollary 7 follows from Corollary 6. $\square$

Remark 1. Up to now, only the special case $l = 1$ of Corollary 3 has been known, see [7, Proposition 44.4]. As well, up to now, only the special case $l = 1$, of Corollary 7 has been known, see [8].

2.5. The rigidity of K^A.

Corollary 4 shows that the group of all automorphisms $K^A \to K^A$ is discrete. Modifying the steps 4, 5 and 6 of the proof of Theorem 1 we can obtain the following strict result.

Theorem 3 (Rigidity Theorem). *Let A be a Weil algebra, $m \geqslant w(A) + 1$. Then every natural transformation $\mathscr{C}\colon K^A \to K^A$ is the identity one.*

Proof. Define $\sigma = \pi(j^A(\tilde{\sigma})) \in K^A_0\mathbb{R}^m$, $\tilde{\sigma}\colon \mathbb{R}^k \to \mathbb{R}^m$, $\tilde{\sigma}(t^1, \ldots, t^k) = (t^1, \ldots, t^k, 0, \ldots, 0)$. Since $K^A_0\mathbb{R}^m$ is the orbit of σ, it suffices to show that $\mathscr{C}(\sigma) = \sigma$.

We can write $\mathscr{C}(\sigma) = \pi(j^A(\xi))$, where $\xi\colon \mathbb{R}^k \to \mathbb{R}^m$ is of rank k at 0 and $\xi(0) = 0$. Applying (if needed) a linear isomorphism $\mathbb{R}^m \to \mathbb{R}^m$ preserving σ to both sides of the equality $\mathscr{C}(\sigma) = \pi(j^A(\xi))$, we can assume that $\varrho\colon \mathbb{R}^k \to \mathbb{R}^k$, $\varrho = (\xi^1, \ldots, \xi^k)$ is an embedding. Then similarly as in Step 4 of the proof of Theorem 1 we can write $\mathscr{C}(\sigma) = \pi(j^A(\xi^1, \ldots, \xi^k, 0, \ldots, 0))$. Similarly as in Step 5, $(\varrho)^*(\underline{\mathfrak{i}}) \subset \underline{\mathfrak{i}}$. Then similarly as in Step 6 we have $\mathscr{C}(\sigma) = \pi([\varrho^*]^{-1}(j^A(\xi^1, \ldots, \xi^k, 0, \ldots, 0))) = \sigma$. $\square$

Remark 2. In Corollary 4 we can assume that $m \geqslant w(A)+1$. Indeed, if $m \geqslant w(A)+1$, then every one-parameter group of natural automorphisms $K^A \to K^A$ is trivial thanks to the Rigidity Theorem.

Remark 3. Up to now, only the special case of Rigidity Theorem for $A = \mathbb{D}^r_k$ has been known, see [8] (see also [10] for $A = \mathbb{D}^1_k$).

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