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NATURAL T-FUNCTIONS ON THE COTANGENT BUNDLE OF A WEIL BUNDLE

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Abstract. A natural T-function on a natural bundle F is a natural operator transforming vector fields on a manifold M into functions on FM. For any Weil algebra A satisfying $\dim M \geqslant \operatorname{width}(A)+1$ we determine all natural T-functions on T^*T^AM, the cotangent bundle to a Weil bundle T^AM.

Keywords: natural bundle, natural operator, Weil bundle

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1.

The aim of this paper is the classification of all natural T-functions defined on the cotangent bundle to a Weil bundle T^*T^A for any Weil algebra A. The starting point is a general result by Kolář, [4], [5], determining all natural operators $T \to TT^A$ transforming vector fields on manifolds to vector fields on a Weil bundle T^A. We also follow the similar classification results of Mikulski, [7] and [8]. Natural operators lifting vector fields to cotangent bundle structures were studied in [9] and also in [3] and [12], where some partial results of our general problem are solved. We follow the basic terminology from [5].

We start from the concept of a natural T-function. For a natural bundle F, a natural T-function f is a natural operator f_M transforming vector fields on a manifold M to functions on FM. The naturality condition reads as follows. For a local diffeomorphism $\varphi\colon M \to N$ between manifolds M, N and for vector fields X

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on M and Y on N satisfying $T\varphi \circ X = Y \circ \varphi$, the equality $f_N Y \circ F\varphi = f_M X$ holds. An absolute natural operator of this kind, i.e. independent of the vector field, is called a natural function on F.

There is a related problem of the classification of all natural operators lifting vector fields on m-dimensional manifolds to T^*T^A. The solution of the second problem is given by the solution of the first one as follows [13]. Natural operators $A_M\colon TM \to TT^*T^AM$ are in the canonical bijection with natural T-functions $g_M\colon T^*T^*T^AM \to \mathbb{R}$ linear on fibers of $T^*(T^*T^AM) \to T^*T^AM$. Using natural equivalences $s\colon TT^* \to T^*T$ by Modugno-Stefani, [10] and $t\colon TT^* \to T^*T^*$ by Kolář-Radziszewski, [6], we obtain the identification of g_M with natural T-functions $f_M\colon T^*TT^AM \to \mathbb{R}$ given by $f_M = g_M \circ t_{T^AM} \circ s^{-1}_{T^AM}$. Thus we investigate natural T-functions defined on $T^*T^{\mathbb{D}\otimes A}M$ to determine all natural operators $T \to TT^*T^A$, where $\mathbb{D}$ denotes the algebra of dual numbers.

We recall the general result of Kolář, [4], [5]. For a Weil algebra A, the Lie group $\mathcal{A}ut A$ of all algebra automorphisms of A has a Lie algebra $\mathcal{A}ut A$ identified with $\operatorname{Der} A$, the algebra of derivations of A. Thus every $D \in \operatorname{Der} A$ determines a one parameter subgroup $d(t)$ and a vector field D_M on T^AM tangent to $(d(t))_M$. Hence we have an absolute natural operator $\lambda_D\colon TM \to TT^AM$ defined by $\lambda_D X = D_M$ for any vector field X on M. For a natural bundle F, let $\mathcal{F}$ denote the corresponding flow operator, [5]. Further, let $L_M\colon A \times TT^AM \to TT^AM$ denote the natural affinor of Koszul, [4], [5]. Then the result of Kolář reads

$$\text{All natural operators } T \to TT^A \text{ are of the form } L(c)\mathcal{T}^A + \lambda_D$$
$$\text{for some } c \in A \text{ and } D \in \operatorname{Der} A.$$

Let $\xi\colon M \to TM$ be a vector field. Kolář in [3] defined an operation $\tilde{}$ transforming a vector field on a manifold M into a function on T^*M by $\tilde{\xi}(\omega) = \langle \xi(p(\omega)), \omega \rangle$, where p is the cotangent bundle projection and $\omega \in T^*M$. One can immediately verify that for a natural bundle F and a natural operator $A_M\colon TM \to TFM$ we have a natural T-function $\tilde{A}_M\colon T^*FM \to \mathbb{R}$ defined by $\tilde{A}_M(X) = \widetilde{A_M X}$ for any vector field $X\colon M \to TM$.

2.

In this section, we find all natural T-functions $f_M\colon T^*T^AM \to \mathbb{R}$ for any manifold M for $m = \dim M \geqslant \operatorname{width}(A) + 1$. For some Weil algebras A, [13], all natural T-functions in question are of the form

$$h(\widetilde{L(c)\mathcal{T}^A}, \widetilde{\lambda_D})) \quad c \in C, \;\; D \in \mathcal{D}$$

where C is a basis of A, $\mathcal{D}$ is a basis of $\operatorname{Der} A$ and h is any smooth function $\mathbb{R}^{\dim A+\dim \operatorname{Der} A} \to \mathbb{R}$. Let $\mathbb{D}_k^r$ denote the algebra of jets $J_0^r(\mathbb{R}^k, \mathbb{R})$. It can be also considered as the algebra of polynomials of variables $\tau_1, \dots, \tau_k$. By [5], any Weil algebra A is obtained as the factor of $\mathbb{D}_k^r$ by an ideal I, i.e. $A = \mathbb{D}_k^r / I$.

The contravariant approach to the definition of a Weil bundle by Morimoto sets $M_A = \operatorname{Hom}(C^\infty(M, \mathbb{R}), A)$ and was studied by many authors, e.g. Muriel, Munoz, Rodriguez, Alonso [1], [11]. The covariant approach (Kolář, [3], [5]) defines $T^A M$ as the space of A-velocities. Let $\varphi, \psi\colon \mathbb{R}^k \to M$, $\varphi(0) = \psi(0)$. Then φ and ψ are said to be I-equivalent iff for any $\operatorname{germ}_x f$, $f\colon M \to \mathbb{R}$ the inclusion $\operatorname{germ}(f \circ \varphi - f \circ \psi) \in I$ holds. Classes of such an equivalence $j^A \varphi$ are said to be A-velocities. For a smooth map $g\colon M \to N$ define $T^A g(j^A \varphi) = j^A(g \circ \varphi)$. Since T^A preserves products, we have $T^A \mathbb{R} = A$, $T^A \mathbb{R}^m = A^m$. The identification $F\colon M_A \to T^A M$ between those two approaches to the definition of a Weil bundle is given by

$$(1) \qquad F(j^A\varphi)(f) = j^A(f \circ \varphi) \qquad \text{for any } f \in C^\infty(M, \mathbb{R}).$$

We are going to construct natural T-functions defined on T^*T^A from natural operators $T \to TT_k^r$, since there are some additional ones on T^*T^A, which cannot be constructed from natural operators $T \to TT^A$.

Let $p\colon \mathbb{D}_k^r \to A$ be the projection Weil algebra homomorphism inducing the natural transformation $\tilde{p}_M\colon T_k^r M \to T^A M$. There is a linear map $\iota\colon A \to \mathbb{D}_k^r$ such that $p \circ \iota = \mathrm{id}_A$. By means of ι we construct an embedding $T^A M \to T_k^r M$. Consider any $j^A \varphi \in T^A M$ as an element of $\operatorname{Hom}(C^\infty(M, \mathbb{R}), A)$. Then domains of $j^A\varphi \in T_{x_0}^A M$ can be replaced by $J_{x_0}^r(M, \mathbb{R})$. Indeed, for any $f \in C^\infty(M, \mathbb{R})$, $j^A\varphi(f) = j^A(f \circ \varphi) = [\operatorname{germ}_{x_0} f \circ \operatorname{germ}_0 \varphi]_I$, where $x_0 = \varphi(0)$, $0 \in \mathbb{R}^k$. Since any ideal I in the algebra $E(k)$ of finite codimension contains the rth power of the maximal ideal of $E(k)$, the last expression can be replaced by $[j_0^r(f \circ \varphi)]_J = j^A\varphi(j_{x_0}^r f)$, where J is an ideal of $\mathbb{D}_k^r$ corresponding to I.

Further, any element $j_{x_0}^r f \in J_{x_0}^r(M, \mathbb{R})$ can be decomposed into $f(x_0) + j_{x_0}^r(t_{f(x_0)}^{-1} \circ f) = f(x_0) + j_{x_0}^r \tilde{f}$, where $t_y\colon \mathbb{R} \to \mathbb{R}$ denotes in general a translation mapping 0 into y. The second expression is an element of the bundle of covelocities of type $(1, r)$, namely an element of $(T^{r*})_{x_0} M = (T_1^{r*})_{x_0} M$, the bundle of covelocities of type (k, r) being defined as $T_k^{r*} M = J^r(M, \mathbb{R}^k)_0$, [5].

Select any minimal set of generators $\mathcal{B}_{x_0}$ of the algebra $T_{x_0}^{r*} M$. For any $j_{x_0}^r \tilde{f} \in \mathcal{B}_{x_0}$ define $\tilde{\iota}_{x_0}\colon T_{x_0}^A M \to (T_k^r)_{x_0} M$ by $(\tilde{\iota}_{x_0}(j^A\varphi))(j_{x_0}^r \tilde{f}) = \tilde{\iota}((j^A\varphi)(j_{x_0}^r \tilde{f}))$. In the second step, $\tilde{\iota}$ can be extended to a homomorphism $J_{x_0}^r(M, \mathbb{R}) \to \mathbb{D}_k^r$.

We extend the map $\tilde{\iota}_{x_0}$ to $\tilde{\iota}\colon T^A M \to T_k^r M$. For a general Weil algebra B we show that any element $j^B\varphi \in T_{\overline{x}}^B M$ corresponds bijectively to some element $j^B\varphi_0 \in T_{x_0}^B M$. Indeed, $j^B\varphi(j_{\overline{x}}^r f) = j^B(f \circ \varphi) = j^B(f \circ t_{\overline{x}}^{-1} \circ t_{\overline{x}} \circ \varphi_0) = j^B\varphi_0(j_{x_0}^r f_0)$.

This general property extends $\tilde{\iota}_{x_0}$ to $\tilde{\iota}\colon T^AM \to T^r_kM$. The map $\tilde{\iota}$ is not a natural transformation and for a manifold M, it depends on the selection of the algebra basis $\mathcal{B}_{x_0}$ at $x_0 \in M$. To stress this we shall use sometimes the notation $\tilde{\iota}_{\mathcal{B}_{x_0}}$ for $\tilde{\iota}$. We have proved the following assertion.

Proposition 1. *Let $A = \mathbb{D}^r_k/I$ be a Weil algebra, $p\colon \mathbb{D}^r_k \to A$ the projection homomorphism with its associated natural transformation $\tilde{p}\colon T^r_k \to T^A$ and $\iota\colon A \to \mathbb{D}^r_k$ a linear map satisfying $p \circ \iota = \mathrm{id}_A$. For a manifold M and $x_0 \in M$ let $\mathcal{B}_{x_0}$ be a minimal set of generators of the algebra $J^r_{x_0}(M,\mathbb{R})_0 = T^{r*}_{x_0}M$. Then there is an embedding $\tilde{\iota}_{\mathcal{B}_{x_0}}\colon T^AM \to T^r_kM$ satisfying $\tilde{p}_M \circ \tilde{\iota}_{\mathcal{B}_{x_0}} = \mathrm{id}_{T^AM}$ such that $(\tilde{\iota}_{\mathcal{B}_{x_0}}(j^A\varphi))(j^r_{x_0}\tilde{f}) = \iota((j^A\varphi)(j^r_{x_0}\tilde{f}))$ for any $j^A\varphi \in T^A_{x_0}M$ and $j^r_{x_0}\tilde{f} \in \mathcal{B}_{x_0}$.*

In the following investigations, we shall need coordinates on T^AM and T^*T^AM. We introduce them and using Proposition 1, we give a relation between them and those on T^r_kM to be right now recalled. Consider a polynomial form of elements from $\mathbb{D}^r_k$, namely $\frac{1}{\alpha!}x_\alpha\tau^\alpha$ for $0 \leqslant |\alpha| \leqslant r$. Since Weil bundles preserve products, we have canonical coordinates x^i_α on $T^r_k\mathbb{R}^m = (\mathbb{D}^r_k)^m$ for $1 \leqslant i \leqslant m$ and $0 \leqslant |\alpha| \leqslant r$. Consider the system $\mathcal{S}$ formed by non-zero images $p(\tau^\alpha)$ of all $\tau^\alpha \in \mathbb{D}^r_k$ forming its monomial linear basis. Take a maximal linearly independent subset $\mathcal{S}_0$ of $\mathcal{S}$ (a linear basis of A). Then any element $d \in \mathcal{S} - \mathcal{S}_0$ is uniquely expressed as $c^d_a a$ for $a \in x_0$. For any element $b \in \mathcal{S}$, select a monomial representative τ^β having a minimal multiindex among all of them. Then there is such a basis $\mathcal{S}_0 \subseteq \mathcal{S}$ that any $c^d_a = c^\delta_\alpha$ satisfy $|\delta| \geqslant |\alpha|$ for the minimal representatives τ^α of $p^{-1}(a)$ and τ^δ of $p^{-1}(d)$. Define the map $\iota\colon A \to \mathbb{D}^r_k$ by $\iota(a) = \tau^\alpha$ for a minimal representative τ^α of $a \in \mathcal{S}_0$ and $\iota(d) = c^\delta_\alpha\tau^\alpha$ for other elements $d \in \mathcal{S}$ and their minimal representatives τ^δ. Hence ι is a linear map satisfying $p\circ\iota = \mathrm{id}_A$ from Proposition 1. It introduces the coordinates y^i_α on T^AM by

$$\tilde{\iota}\Big(\tilde{p}\Big(\frac{1}{\gamma!}x^i_\gamma\tau^\gamma\Big)\Big) = \frac{1}{\alpha!}y^i_\alpha\tau^\alpha. \tag{2}$$

The following formula gives the relation between the coordinates y^i_α of $\tilde{p}(\frac{1}{\gamma!}x^i_\gamma\tau^\gamma)$ and x^i_α of the projected element of T^r_kM. It is of the form

$$y^i_\alpha = x^i_\alpha + \frac{\alpha!}{\delta!}x^i_\delta c^\delta_\alpha. \tag{3}$$

The transformation laws for the action of the jet group G^r_k on the standard fiber $(T^*T^A)_0\mathbb{R}^m$ are of the form

$$\bar{y}^i_\alpha = a^i_{l_1\dots l_s}y^{l_1}_{\alpha_1}\dots y^{l_s}_{\alpha_s} + \frac{\alpha!}{\delta!}a^i_{h_1\dots h_t}y^{h_1}_{\delta_1}\dots y^{h_t}_{\delta_t}c^\delta_\alpha. \tag{4}$$

Further, we define the additional coordinates p_i^α on T^*T^AM by $p_i^\alpha dy_\alpha^i$. The transformation laws for the action of G_m^{r+1} on the additional coordinates satisfies

$$(5)\qquad \bar{p}_j^\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\tilde{a}^l_{jl_1\dots l_s}\bar{y}^{l_1}_{\alpha_1}\dots\bar{y}^{l_s}_{\alpha_s}p_l^{\alpha\beta} + \frac{\gamma!}{\delta!\beta!}\tilde{a}^l_{jh_1\dots h_t}\bar{y}^{h_1}_{\delta_1}\dots\bar{y}^{h_t}_{\delta_t}c_\gamma^{\delta\beta}p_l^\gamma.$$

The relation between p_i^α and the additional coordinates q_i^γ on $T^*T_k^rM$ defined by $q_i^\gamma dx_\gamma^i$ is given by

$$(6)\qquad q_i^\gamma = p_i^\gamma \quad \text{for } \tau^\gamma \in \mathcal{S}_0 \qquad \text{and} \qquad q_i^\gamma = \frac{\alpha!}{\gamma!}p_i^\alpha c_\alpha^\gamma \quad \text{otherwise.}$$

Without loss of generality, we can suppose the following form of generators of the ideal I. Let $\pi_s^r\colon \mathbb{D}_k^r \to \mathbb{D}_k^s$ be the canonical projection of Weil algebras. Then there is such a set of generators of I that each of them either gets mapped to zero by π_1^r or is a linear monomial. In the following investigations, such an ideal will be called a normal ideal. It is easy to see that for any Weil algebra A there is a Weil algebra A_0 with this property and an algebra isomorphism $\varphi\colon A \to A_0$. Then every natural operator $D_M^A\colon TM \to TT^AM$ is bijectively assigned a natural operator $D_M^{A_0}\colon TM \to TT^{A_0}M$ by

$$D_M^AX(y) := T\tilde{\varphi}_0^{-1} \circ D_M^{A_0}X \circ \tilde{\varphi}_0(y)$$

for a vector field X on M, $y \in T^AM$. The notation $\tilde{\varphi}_0$ indicates the natural equivalence $T^A \to T^{A_0}$ induced by the isomorphism $\varphi_0\colon A \to A_0$.

For a manifold M and an algebra basis $\mathcal{B}_{x_0}$ of the algebra of covelocities $T_{x_0}^{r*}M$ with the source at $x_0 \in M$, let us define operators $TM \to TT^AM$ by means of $\tilde{\iota}_{\mathcal{B}_{x_0}}$ and natural operators $T \to TT_k^r$ as follows. Every natural operator $l\colon T \to TT_k^r$ defines an operator

$$(7)\qquad \Lambda = \Lambda_{M,\mathcal{B}_{x_0}}\colon TM \to TT^AM \qquad \text{by} \quad \Lambda_{M;\mathcal{B}_{x_0}} = T\tilde{p} \circ \lambda \circ \tilde{\iota}_{\mathcal{B}_{x_0}}$$

which does not have to be natural and neither do the functions $\tilde{\Lambda} = \tilde{\Lambda}_{M;\mathcal{B}_{x_0}}\colon T^*T^AM \to \mathbb{R}$. Consider a basis of natural operators $T \to TT_k^r$.

The non-absolute natural operators λ together with some of the absolute ones in this basis induce natural operators $\Lambda\colon T \to TT^A$, while the others will be used for the construction of natural functions $T^*T^AM \to R$, i.e. those functions $T^*T^AM \to \mathbb{R}$ which become free of the selection of $x_0 \in M$ and $\mathcal{B}_{x_0} \in T_{x_0}^{*r}M$.

By general theory, [5], searching for natural T-functions defined on T^*T^A, we are going to investigate G_m^{r+2}-invariant functions defined on $(J^{r+1}T)_0\mathbb{R}^m \times (T^*T^A)_0\mathbb{R}^m$. Therefore we state some assertions, concerning the action of G_m^{r+2} and some of its subgroups on this space. It will be necessary to consider the coordinate expression of this action as well as that of base operators $\Lambda\colon TM \to TT^AM$ and their associated functions $\tilde{\Lambda}\colon T^*T^AM \to \mathbb{R}$.

Denote by λ_j^β a natural operator $\lambda_{D_j^\beta}$ associated to a derivation of $\mathbb{D}_k^r$ defined by $\tau_i \to \delta_i^j \tau^\beta$ for $j \in \{1, \dots, k\}$ and $1 \leqslant |\beta| \leqslant r$. Then we have coordinate forms of λ_j^β, Λ_j^β and $\tilde{\Lambda}_j^\beta$. We have

$$\text{(8)}\quad \lambda_j^\beta = \frac{\gamma!}{(\gamma-\beta)!} x^i_{j\gamma-\beta} \frac{\partial}{\partial x^i_\gamma}, \quad \Lambda_j^\beta = \Big(\frac{\alpha!}{(\alpha-\beta)!} y^i_{j\alpha-\beta} + \frac{\alpha!}{(\delta-\beta)!} y^i_{j\delta-\beta} c^\delta_\alpha\Big) \frac{\partial}{\partial y^i_\alpha},$$

$$\tilde{\Lambda}_j^\beta = \Big(\frac{\alpha!}{(\alpha-\beta)!} y^i_{j\alpha-\beta} + \frac{\alpha!}{(\delta-\beta)!} y^i_{j\delta-\beta} c^\delta_\alpha\Big) p_i^\alpha. \tag{9}$$

Let k be the width of the Weil algebra A. For $m \geqslant k$, define an immersion element $i \in T_0^A\mathbb{R}^m$ as follows. For $m \geqslant k$, let $i_k^m\colon \mathbb{R}^k \to \mathbb{R}^m$ defined by $i_k^m = \mathrm{id}_{\mathbb{R}^k} \times (0)^{m-k}$ be the canonical inclusion of $\mathbb{R}^k$ into $\mathbb{R}^m$. Then define $i \in T_0^A\mathbb{R}^m$ by

$$i = j^A i_k^m. \tag{10}$$

In coordinates, it satisfies $y^i_\alpha = 0$ whenever $|\alpha| \geqslant 2$ and $y^i_j = \delta^i_j$.

Consider the jet group G_k^r, [5]. It can be identified with $\mathrm{Aut}\,\mathbb{D}_k^r$, the group of automorphisms of the algebra $\mathbb{D}_k^r$, as follows. For $j_0^r g \in G_k^r$ and $j_0^r\varphi \in \mathbb{D}_k^r$ define

$$j_0^r g(j_0^r\varphi) = j_0^r\varphi \circ (j_0^r g)^{-1}. \tag{11}$$

For a Weil algebra $p\colon \mathbb{D}_k^r \to A = \mathbb{D}_k^r/I$ Alonso in [1] defined subgroups G_A and G^A of G_k^r as follows. $G_A = \{j_0^r g \in G_k^r;\ p \circ j_0^r g = p\}$ and $G^A = \{j_0^r g \in G_k^r;\ \mathrm{Ker}(p \circ j_0^r g) = \mathrm{Ker}(p)\}$. He also proved that G_A is a normal subgroup of G^A and the property $G^A/G_A \simeq \mathrm{Aut}\,A$.

In the following investigations, we shall need the concept of a regular A-point and thus we recall it. An element $\varphi \in M_A$ is said to be regular (a regular A-point) if and only if its image coincides with A, [1]. Taking into account the identificatin (1), such a concept can be extended to an A-velocity $j^A\varphi \in T^AM$. Clearly, it is regular if and only if φ is an immersion in $0 \in \mathbb{R}^k$, where k is the width of A. Further, it must hold that $\dim M \geqslant k$. In the case $m = k$ the concept of regularity coincides with that of invertibility. The map $\tilde{\imath}$ from Proposition 1 preserves regularity and thus $\tilde{\imath}\colon A^k \to \mathbb{R}^k$ can be restricted to $\mathrm{reg}(N^k) \to G_k^r$, where N denotes the nilpotent ideal of A.

The following lemma characterizes G_A as the stability subgroup of the immersion element i.

Lemma 2. *Let $A = \mathbb{D}^r_m/I$ be a Weil algebra of width k with the projection homomorphism p and a normal ideal I of $\mathbb{D}^r_m$. Let $\mathrm{St}(i) \subseteq G^r_m$ be the stability subgroup of the immersion element $i \in T^A_0\mathbb{R}^m$ under the canonical left action of G^r_m. Then $G_A = \mathrm{St}(i) = \mathrm{Ker}\,\tilde{p} \cap G^r_m$, if we consider the restriction of $\tilde{p}_{\mathbb{R}^m}$ to G^r_m.*

Proof. The formula (11) implies that every element of G^r_m stabilizes i if and only if $a^i_j = \delta^i_j$ for $j \in \{1, \dots, k\}$ and $a^i_\alpha + \frac{\alpha!}{\delta!} a^i_\delta c^\delta_\alpha = 0$ whenever $|\alpha| \geqslant 2$ and $\tau^\alpha \in \langle \tau_1, \dots, \tau_k \rangle$.

On the other hand, $G_A = \{j^r_0 g \in G^r_m;\ p \circ j^r_0\varphi \circ (j^r_0 g)^{-1} = p \circ j^r_0\varphi\ \forall j^r_0\varphi \in \mathbb{D}^r_m\}$. The transformation law for the action of $j^r_0 g \in \mathrm{Aut}\,\mathbb{D}^r_m$ on $j^r_0\varphi \in \mathbb{D}^r_m$ (in the coordinates x_α) is given by

$$\bar{x}_\alpha = x_{l_1 \dots l_q} \tilde{a}^{l_1}_{\alpha_1} \dots \tilde{a}^{l_q}_{\alpha_q} \tag{12}$$

for all decompositions $\alpha_1 \dots \alpha_q$ of α. Further, the application of (3) on (12) yields the identity

$$\bar{y}_\alpha = x_{l_1 \dots l_q} \tilde{a}^{l_1}_{\alpha_1} \dots \tilde{a}^{l_q}_{\alpha_q} + \frac{\alpha!}{\delta!} x_{h_1 \dots h_t} \tilde{a}^{h_1}_{\delta_1} \dots \tilde{a}^{h_t}_{\delta_t} c^\delta_\alpha, \tag{13}$$

satisfied for any admissible $\bar{y}_\alpha$, x_γ.

Substituting the ith projection pr_i for φ in (13), we obtain $0 = \bar{y}_\alpha = \tilde{a}^i_\alpha + \frac{\alpha!}{\delta!} \tilde{a}^i_\delta c^\delta_\alpha$ for $|\alpha| \geqslant 2$, $\tau^\alpha \notin I$ and $\tau^\alpha \in \langle \tau_1 \dots, \tau_k \rangle$. Moreover we obtain $\tilde{a}^i_j = a^i_j = \delta^i_j$ for $j \in \{1, \dots, k\}$. This proves that $G_A \subseteq \mathrm{St}(i)$. The converse inclusion follows from the coordinate characterization of $\mathrm{St}(i)$ in the very beginning of the proof, the fact that the functions pr_i fulfill the condition from the definition of G_A and from an application of the automorphisms from the definition of G_A. This proves our claim.

The second assertion follows from the formulas (3), (4) and the definition of the coordinates y^i_α, which completes the proof. □

Let $A = \mathbb{D}^r_m/I$ be a Weil algebra, $\dim M \geqslant m+1$. In the proof of the main result, we need to describe the stability group of $j^{r+1}_0(\partial/\partial y^{m+1})$. The transformation laws for the action of G^{r+2}_{m+1} on $(J^{r+1}T)_0\mathbb{R}^m$ have the coordinate expression

$$\overline{X}^i_\alpha = a^i_{l\gamma_1} X^l_{\gamma_2} \tilde{a}^\gamma_\alpha, \tag{14}$$

where X^i_α, $|\alpha| \leqslant r+1$ denote the canonical coordinates of $j^{r+1}_0(\partial/\partial y^{m+1})$. Further, any multiindex γ including the empty one is decomposed into γ_1, γ_2 and the notation $\tilde{a}^\gamma_\alpha$ denotes the system of all $\tilde{a}^{l_1}_{\alpha_1} \dots \tilde{a}^{l_s}_{\alpha_s}$ for $l_1, \dots, l_s$ forming the multiindex γ and decompositions $\alpha_1, \dots, \alpha_s$ forming α. It follows that in coordinates any element of G^{r+2}_{m+1} must satisfy $a^i_j = \delta^i_{m+1}$ and $a^i_\alpha = 0$ whenever the multiindex α formed by all of $1, \dots, m+1$ contains at least one $m+1$ for $|\alpha| \geqslant 2$. To describe the stability group of $j_0^{r+1}(\partial/\partial y^{m+1})$ in terms of Lemma 2, denote by A^s_{m+1} the Weil algebra of $\mathbb{D}^s_{m+1}/J$ for $J = \langle \tau_{m+1}\tau^\alpha \rangle$, $|\alpha| \geqslant 1$. Thus we have proved the following lemma.

Lemma 3. *The stability group of $j_0^{r+1}(\partial/\partial y^{m+1})$ in G^{r+2}_{m+1} is of the form*

$$\tilde{\iota}((A^{r+2}_{m+1})^{m+1}) \cap G^{r+2}_{m+1}.$$

Moreover, the stability group of $j_0^{r+1}(\partial/\partial y^{m+1})$ and the immersion element $i \in T_0^A\mathbb{R}^{m+1}$ is of the form

$$G_{A;m+1} = G_A \cap \tilde{\iota}((A^{r+2}_{m+1})^{m+1}).$$

Let us consider the basis $\tilde{\mathcal{B}}$ of all T-functions $\tilde{\Lambda}$ defined on T^*T^AM (not natural in general), constructed from the non-absolute natural operators $L(\tau^\alpha)\mathcal{T}^A$ and from the absolute operators Λ^β_j with the coordinate expression given by (8). Let $\tilde{\mathcal{B}}_1$ denote the subbasis of $\tilde{\mathcal{B}}$ formed by natural T-functions $T^*T^A \to \mathbb{R}$.

Alonso in [1] proved that there is a structure of a fiber bundle on $\operatorname{reg} T^AM$ with the standard fiber G^r_m/G_A over an m-dimensional manifold M and therefore $\operatorname{reg} T^A_0\mathbb{R}^m$ is identified with G^r_m/G_A. The elements of $\operatorname{reg}(T^A)_0\mathbb{R}^m$ are the left classes $j^r_0 g G_A$.

Let $A = \mathbb{D}^s_m/I$ be a Weil algebra of width $k \leqslant m$, where I is a normal ideal. Define a map $\tilde{\iota}^*\colon A^m \to G^s_m$ by

$$\tilde{\iota}^* := (\tilde{\iota} \circ \tilde{p}^k) \times \mathrm{id}_{\mathbb{R}^{m-k}}\,. \tag{15}$$

Then we have a map $\mathrm{Imm}\colon T^*(\operatorname{reg} T^A)_0\mathbb{R}^m \to (T^*_i T^A)_0\mathbb{R}^m$ defined by

$$\mathrm{Imm}(w) = l((\tilde{\iota}^*(q(w)))^{-1}, w), \tag{16}$$

for $w \in T^* \operatorname{reg} T^A_0\mathbb{R}^m$ and the cotangent bundle projection q.

In the following assertion we prove that the map Imm preserves the value of any $\tilde{\Lambda}\colon T^*T^A\mathbb{R}^m \to \mathbb{R}$ induced by a natural function $\tilde{\lambda}\colon T^*A \to \mathbb{R}$.

Proposition 4. *Let $A = \mathbb{D}^r_m/I$ be Weil algebra of width k with the normal ideal I and $(T^*(\operatorname{reg} T^A))_0\mathbb{R}^m \to (\operatorname{reg} T^A)_0\mathbb{R}^m$ be the restriction of the natural bundle $T^*T^A\mathbb{R}^m \to T^A\mathbb{R}^m$ to the open submanifold $(\operatorname{reg} T^A)_0\mathbb{R}^m$. Then all operators from $\tilde{\mathcal{B}} - \tilde{\mathcal{B}}_0$ are G^{r+2}_m-invariant with respect to the map* Imm.

P r o o f. We prove that for any $\tilde{\Lambda}^\beta_j\colon (T^*T^A)_0\mathbb{R}^m$ and for any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^m$ the values of $\tilde{\Lambda}^\beta_j(w)$ and $\tilde{\Lambda}^\beta_j(\mathrm{Imm}(w))$ coincide. We use the coordinates from (2) and (5) and the transformation laws from (4) and (5) for the action of G^{r+2}_m on

$(T^*T^A)_0\mathbb{R}^m$. To emphasize $\operatorname{Imm}(w)$ as a transformed value under this action use $\bar{p}_i^\alpha$ for the additional cordinates of $\operatorname{Imm}(w)$ (obviously, the cordinates $\bar{y}_\alpha^i$ indicate those of the immersion element i). Then the formula (5) reduces to

$$\bar{p}_j^\beta = \frac{(\alpha+\beta)!}{\alpha!\beta!}\,\tilde{a}_{j\alpha}^l p_l^{\alpha\beta} + \frac{\gamma!}{\delta!\beta!}\,\tilde{a}_{j\delta}^l c_\gamma^{\delta\beta} p_l^\gamma. \tag{17}$$

We have $\beta!\bar{p}_j^\beta = \tilde{\Lambda}_j^\beta(\operatorname{Imm}(w)) = \tilde{\Lambda}_j^\beta(\bar{y}_\alpha^i, \bar{p}_i^\gamma)$, which follows from the formula (9). The coincidence of $\tilde{\Lambda}_j^\beta(w)$ with $\tilde{\Lambda}_j^\beta(\operatorname{Imm}(w))$ will be proved if there is an element $j_0^{r+2}g \in \tilde{\imath}^*(A^m)$ the coordinates of which satisfy the equation determined by the formulas (17) and by the second formula from (9) multiplied by $\beta!$. Clearly, it suffices to put $\tilde{a}_\gamma^i = y_\gamma^i$ and complete the other coordinates of $j_0^{r+2}g$ so that it belongs to $\tilde{\imath}^*(A^m)$. This proves our claim. □

The following lemma specifies a certain class of functions, among which all investigated ones must be contained.

Lemma 5. *Let A be a normal Weil algebra of width k and height r considered as $\mathbb{D}_{m+1}^{r+2}/I$ for $m \geqslant k$. Then every $\widetilde{G_{m+1}^{r+2}}$-invariant function $f\colon (J^{r+1}T)_0\mathbb{R}^{m+1} \times T^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$ is of the form $h(L\widetilde{(\tau^\alpha)}\mathcal{T}^A, \tilde{\Lambda}_j^\beta)$ for some smooth function h of a suitable type.*

P r o o f. By the general lemma from [5, Chapter VI], every G_{m+1}^1-invariant function defined on $(J^{r+1}T)_0\mathbb{R}^{m+1} \times T^*T^A\mathbb{R}^{m+1}$ must satisfy $f(j_0^{r+1}X, w) = h(X_\gamma^i p_i^\beta, y_\alpha^i p_i^\beta)$ for any non-zero $j_0^{r+1}X$ of a vector field X on $\mathbb{R}^{m+1}$, if we use again the coordinates y_α^i and p_i^α. The last expression can be considered in the form $h(L\widetilde{(\tau^\alpha)}\mathcal{T}^A, X_\gamma^i p_i^\beta, \tilde{\Lambda}_j^\beta, y_\delta^i p_i^\beta)$ for $|\beta| \geqslant 0$, $|\gamma| \geqslant 1$ and $|\delta| \geqslant 2$. Identify $q(w)$ with $j^A g$ for any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, i.e. $q(w) = l(\tilde{\imath}^*(j^A g), i)$ and put $j_0^{r+1}Y = l((\tilde{\imath}^*(j^A g))^{-1}, j_0^{r+1}X)$. Then $f(j_0^{r+1}X, w) = h(L\widetilde{(\tau^\alpha)}\mathcal{T}^A, Y_\gamma^i \bar{p}_i^\beta, \tilde{\Lambda}_j^\beta, 0, \bar{p}_i^0)$ for $|\gamma| \geqslant 1$ and $i \in \{1, \dots, k\}$. Here $\bar{p}_i^\beta$ indicate the transformed values of p_i^β under the map Imm. The last identity follows from Proposition 5. Further, there is $j_0^{r+2}g \in G_A \cap G_{A_{m+1}^{r+2}}$ such that $l(j_0^{r+1}g, j_0^{r+1}(\partial/\partial y^{m+1})) = j_0^{r+1}Y$. Then we have $f(j_0^{r+1}X, w) = h(L\widetilde{(\tau^\alpha)}\mathcal{T}^A, 0, \tilde{\Lambda}_j^\beta, p_i^0)$ for $i \in \{1, \dots, k\}$. The excessive coordinates p_i^0 are annihilated by an element of $\operatorname{Ker}\pi_r^{r+1} \cap \tilde{\imath}((A_{m+1}^{r+2})^{m+1})$, namely by an element satisfying in coordinates $a_\alpha^i = 0$ except of $\alpha = \underbrace{(i, \dots, i)}_{(r+1)\text{-times}}$. Such an element stabilizes $j_0^{r+1}(\partial/\partial y^{m+1})$ as well as i, which completes the proof. □

Searching for all natural T-functions $T^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$ among those from Lemma 5, we introduce a basis $\mathcal{B}$ of functions, defined on $T_i^*T^A\mathbb{R}^{m+1}$ which shall be identified with $\tilde{\mathcal{B}}$ as follows. By general theory, [5], every natural T-function defined on $T^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$ is determined by its values over $j_0^{r+1}(\partial/\partial y^{m+1})$ and $(T^*T^A)_0\mathbb{R}^{m+1}$. Further, Lemma 3 and the formula (16) imply that the map Imm stabilizes $j_0^{r+1}(\partial/\partial y^{m+1})$ in the following sense. For any $w \in T^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, the action of $\tilde{\imath}^*(q(w))$ on $(J^{r+1}T)_0\mathbb{R}^{m+1}$ stabilizes $j_0^{r+1}(\partial/\partial y^{m+1})$.

Thus we have the basis $\mathcal{B}$ of functions defined on $T_i^*T^A\mathbb{R}^{m+1}$ obtained by the restriction of $\tilde{\mathcal{B}}$ to $j_0^{r+1}(\partial/\partial y^{m+1})$ and $T_i^*T^A\mathbb{R}^{m+1}$. Conversely, $\mathcal{B}$ determines $\tilde{\mathcal{B}}$ by

$$\tilde{\mathcal{B}}\left(j_0^{r+1}\left(\frac{\partial}{\partial y^{m+1}}\right), w\right) = \mathcal{B} \circ \operatorname{Imm}(w). \tag{18}$$

Analogously, we construct $\mathcal{B}_1$ from $\tilde{\mathcal{B}}_1$. Moreover, for any $w \in T_i^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$, the values formed by $\mathcal{B}(w)$ coincide with the cordinates p_j^β of w for $j = 1, \dots, k$ and $|\beta| \geqslant 1$ in case of the absolute operators and p_{m+1}^β in case of the non-absolute ones. Thus any base T-function of $\mathcal{B}$ defined on $T_i^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1}$ corresponds to some projection $\operatorname{pr}_j^\beta \colon T_i^*(\operatorname{reg} T^A)_0\mathbb{R}^{m+1} \to \mathbb{R}$.

It follows from Lemma 3 and the naturality of $\widetilde{L(\tau^\alpha)\mathcal{T}^A}$ that all natural T-functions $(T^*T^A)\mathbb{R}^{m+1} \to \mathbb{R}$ from Lemma 5 are in a canonical bijection with G_A-invariant functions defined on $T_i^*T^A\mathbb{R}^{m+1}$ which are of the form $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(\tilde{\Lambda}_j^\beta)$ for $\tilde{\Lambda}_j^\beta \colon T_i^*T^A\mathbb{R}^{m+1} \to \mathbb{R}$. Using coordinates, we find all G_A-invariants of p_j^β, $j \in \{1, \dots, k\}$, $|\beta| \geqslant 1$. Then we identify the functions $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(p_j^\beta)$ with $h(\widetilde{L(\tau^\alpha)\mathcal{T}^A})(\tilde{\Lambda}_j^\beta)$ and by (17), we obtain all natural T-functions on $T^*T^A\mathbb{R}^{m+1}$.

This way we have deduced that our problem can be reduced to the problem of finding all G_A-invariant functions defined on $T_i^*T^A\mathbb{R}^{m+1}$. The coordinate expression for the action of G_A on $T_i^*T^A\mathbb{R}^{m+1}$ is given by (17). It follows that $T_i^*T^A\mathbb{R}^{m+1}$ is identified with the space R^N endowed with such an action. Thus we are searching for G_A-invariant functions defined on $\mathbb{R}^N$.

We are going to investigate $G_A \cap G_{m+1}^r$-orbits on R^N, since only p_j^0 depend on B_{m+1}^{r+1} and they can be annihilated by this subgroup. For those orbits, we construct all functions distinguishing them and then we express the corresponding invariants in terms of elements from $\tilde{\mathcal{B}}$.

The following assertion describes an important property of $(G_A \cap \operatorname{Ker} \pi_s^r)$-orbits which is needed in the proof of the main result. Denote by $\mathcal{B}_s \subseteq \mathcal{B}$ the set of all $(G_A \cap \operatorname{Ker} \pi_s^r)$-invariants selected from $\mathcal{B}$ and denote by N_s the number of elements in $\mathcal{B}_s$. Clearly, $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots \subseteq \mathcal{B}_{r-1} \subseteq \mathcal{B}_r$. Further, denote $\mathcal{B}_t^s = \mathcal{B}_s - \mathcal{B}_t$ and $N_t^s = N_s - N_t$. Then we have

Proposition 7. *Let $w \in \mathbb{R}^N$ and let $\operatorname{Orb}_s(w)$ be its $(G_A \cap \operatorname{Ker}\pi^r_s)$-orbit. Then $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$ has the structure of an affine subspace of $R^{N^{s+1}_s}$, the modelling vector space of which is $(B^{s+1}_{m+1} \cap G_A)/H$ for a normal Lie subgroup $H \subseteq B^{s+1}_{m+1} \cap G_A$. The canonical injection i_0 of such a vector space into the vector space $R^{N^{s+1}_s}$ and the sum of a point with a vector are given by*

$$(19) \qquad i_0([j^{s+1}_0\varphi]_H) = \ell(j^{s+1}_0\varphi, w) - w \quad \text{and} \quad w + [j^{s+1}_0\varphi]_H = \ell(j^{s+1}_0\varphi, w),$$

respectively for $[j^{s+1}_0\varphi]_H \in (B^{s+1}_{m+1} \cap G_A)/H$ and any element w of $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$, where ℓ denotes the canonical left action of a jet group on the standard fiber.

Proof. The proof is done directly applying the formula (17) restricted to $B^{s+1}_{m+1} \cap G_A$. Let w_1 and w_2 be elements of $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$. Then w_1 can be obtained from w by the action of an element of $B^{s+1}_{m+1} \cap G_A$. The cordinate expression for such a transformation is given by $\bar p^\beta_j = p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}\tilde a^l_{j\alpha}p^{\alpha\beta}_l + \frac{\gamma!}{\delta!\beta!}\tilde a^l_{j\delta}c^{\delta\beta}_\gamma p^\gamma_l = p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}\tilde a^l_{j\alpha}p^{\alpha\beta}_l + \frac{(\delta+\beta)!}{\delta!\beta!}\tilde a^l_{j\delta}q^{\delta\beta}_l$ using the formula (6). Analogously for w_1 and w_2, we have $\bar{\bar p}^\beta_j = \bar p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}\tilde b^l_{j\alpha}\bar p^{\alpha\beta}_l + \frac{\gamma!}{\delta!\beta!}\tilde b^l_{j\delta}c^{\delta\beta}_\gamma \bar p^\gamma_l = \bar p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}\tilde b^l_{j\delta}\bar p^{\alpha\beta}_l + \frac{(\delta+\beta)!}{\delta!\beta!}\tilde b^l_{j\delta}\bar q^{\delta\beta}_l$. It follows from the definition of $\operatorname{Orb}_s(w)$, $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$, the formula (6) and the transformation laws for the action of G^{r+1}_{m+1} on $T^*T^r_kM$ that $q^{\delta\beta}_l$ are $\operatorname{Ker}\pi^{r+1}_1$-invariants. Then we have $\bar{\bar p}^\beta_j = p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}(\tilde a^l_{j\alpha} + \tilde b^l_{j\alpha})p^{\alpha\beta}_l + \frac{(\delta+\beta)!}{\delta!\beta!}(\tilde a^l_{j\delta} + \tilde b^l_{j\delta})q^{\delta\beta}_l + \frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!}\tilde a^h_{l\gamma}\tilde b^l_{j\alpha}p^{\alpha\beta\gamma}_h + \frac{(\alpha+\beta+\varepsilon)!}{\alpha!\beta!\varepsilon!}\tilde a^h_{l\varepsilon}\tilde b^l_{j\alpha}q^{\alpha\beta\varepsilon}_h$. Consider $j^{s+1}_0\psi \in B^{s+1}_{m+1}$ and $j^{s+1}_0\varphi \in B^{s+1}_{m+1}$. Let $\tilde a^i_\gamma$, $\tilde b^i_\gamma$ and $\tilde c^i_\omega$ denote the coordinates of $j^{s+1}_0\varphi^{-1}$, $j^{s+1}_0\psi^{-1}$ and $j^{2s+1}_0(i^{2s+1}_s(j^{s+1}_0\varphi^{-1}) \circ i^{2s+1}_s(j^{s+1}_0\psi^{-1}))$ for $|\gamma| = s+1$ and $|\omega| = 2s+1$, where $i^r_s\colon J^s \to J^r$ denotes in general the canonical inclusion of jet functors of order s, r, for $s \leqslant r$. Then any c^h_ω is in fact a sum of some $\tilde a^h_{l\eta}\tilde b^l_\gamma$ for all admissible decompositions of ω containing γ and η. Then the transformation laws for p^β_j depend on B^{2s+1}, which is a contradiction with $p^\beta_j \in \mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$. Finally, we have $\bar{\bar p}^\beta_j = p^\beta_j + \frac{(\alpha+\beta)!}{\alpha!\beta!}(\tilde a^l_{j\alpha} + \tilde b^l_{j\alpha})p^{\alpha\beta}_l + \frac{(\delta+\beta)!}{\delta!\beta!}(\tilde a^l_{j\delta} + \tilde b^l_{j\delta})q^{\delta\beta}_l$, which implies $\overrightarrow{ww_2} = \overrightarrow{ww_1} + \overrightarrow{w_1w_2}$.

In the second step, we are going to prove the uniqueness of an element of $B^{s+1}_{m+1} \cap G_A$ determined by the couple of elements of $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$. This follows from the fact that if an element of $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$ is stabilized by $j^{s+1}_0 g \in B^{s+1}_{m+1}$ under the canonical left action then the whole $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$ is stabilized. Denote $H = \operatorname{St}^{s+1}_{s;m+1} \subseteq G_A \cap B^{s+1}_{m+1}$ the stability group of $\mathcal{B}^{s+1}_s(\operatorname{Orb}_s(w))$. Clearly, H is a closed and normal subgroup of $G_A \cap B^{s+1}_{m+1}$, which completes the proof. □

The first formula from (19), giving the definition of the vector space structure on $(B^{s+1}_{m+1} \cap G_A)/H$ also allows us to introduce the scalar product on it, induced by the scalar product on $R^{N^{s+1}_s}$. It will be used in the construction of a basis $\tilde{\mathcal{D}}$ of additional natural functions. The construction is given by a procedure, generating step by step a basis of G_A-invariants. As a matter of fact, they are functions defined on $T_i^* T^A \mathbb{R}^{m+1}$ corresponding in the sense of (18) to base natural T^*T^A-functions, which are in fact functions of the elements of $\tilde{\mathcal{B}}$.

We start the procedure by selecting the elements of $\mathcal{B}_1$ and puting $\tilde{D}_1 = \tilde{\mathcal{B}}_1$. For any $w \in T_i^* T^A \mathbb{R}^{m+1}$, consider its orbit $\mathrm{Orb}(w) = \mathrm{Orb}_1(w)$.

In the second step, consider $\mathcal{B}_1^2(\mathrm{Orb}_1(w))$, which is by Proposition 7 a k_2-dimensional affine subspace of the affine space $\mathbb{R}^{N_1^2}$ for some $k_2 \leqslant N_1^2$. Consider the orthogonal complement $\mathbb{V}_2^C$ in the vector space $\mathbb{R}^{N_1^2}$ to $\mathbb{V}_2 = (B_{m+1}^2 \cap G_A)/H_1^2$, where H_1^2 corresponds to the normal subgroup H of $B_{m+1}^{s+1} \cap G_A$ from Proposition 7. The new G_A-invariants are obtained as the components of the unique point P_2 given by the intersection of $\mathcal{B}_1^2(\mathrm{Orb}_1(w))$ with the affine subspace of $\mathbb{R}^{N_1^2}$ containing the origin and the modelling vector space of which being $\mathbb{V}_2^C$. For almost every G_A-orbit in the sense of density, the maximal dimension K_2 is attained and so it suffices to select only $N_1^2 - K_2$ components forming the basis of the additional G_A-invariants from the second step.

We are going to give their expressions in formulas. Select a linear basis of $\mathbb{V}_2$ formed by the elements $[j_0^2\varphi_1^1]_{H_1^2}, \dots [j_0^2\varphi_1^{K_2}]_{H_1^2}$. Denote by $\mathrm{Ort}_i^2([j_0^2\varphi_2]_{H_1^2})$ the orthogonal complement to the sequence obtained from this basis by omitting the ith element. Then for any $w \in T_i^* T^A \mathbb{R}^{m+1}$ we have

$$(20) \qquad P_2(w) = \mathcal{B}_1^2(w) + \frac{((\mathcal{B}_1^2(w), [j_0^2\varphi_2]_{H_1^2}), \mathrm{Ort}_i^2([j_0^2\varphi_2]_{H_1^2})}{(([j_0^2\varphi_2]_{H_1^2}, [j_0^2\varphi_2^i]_{H_1^2}), \mathrm{Ort}_i^2([j_0^2\varphi_2]_{H_1^2})}[j_0^2\varphi_2^i]_{H_1^2}$$

using the vector form of the notation and the symbol $(\ ,\ )$ for the scalar product. Taking into account the identification (18) and selecting $N_1^2 - K_2$ components of P_2, we obtain the base natural functions $\tilde{I}_2^1, \dots, \tilde{I}_2^{N_1^2 - K_2}$ and the basis $\tilde{\mathcal{D}}_2 = \tilde{\mathcal{D}}_1 \cup \tilde{I}_2^1, \dots, \tilde{I}_2^{N_1^2 - K_2}$ of natural T^*T^A-functions after the second step of the procedure.

Further, we used the uniquely determined element $\alpha_2(w)$ of $\mathbb{V}_2 = (B_{m+1}^2 \cap G_A)/H_1^2$ to obtain P_2 and so the element $w \in T_i^* T^A \mathbb{R}^{m+1}$ is after the second step transformed into $w_2 = \ell(\alpha_2(w), w)$.

In the $(s+1)$th step of the procedure we start from the basis $\tilde{\mathcal{D}}_s$ of natural functions and an element $w_s = \ell(\alpha_s) \circ \dots \circ \ell(\alpha_2)(w)$ instead of the w from the second step.

Consider $\mathcal{B}_s^{s+1}(\mathrm{Orb}_s(w))$, which is by Proposition 7 a k_s-dimensional affine subspace of the affine space $\mathbb{R}^{N_s^{s+1}}$ for some $k_{s+1} \leqslant N_s^{s+1}$. Consider the orthogonal complement $\mathbb{V}_{s+1}^C$ in the vector space $\mathbb{R}^{N_s^{s+1}}$ to $\mathbb{V}_{s+1} = (B_{m+1}^{s+1} \cap G_A)/H_s^{s+1}$, where H_s^{s+1} corresponds to the normal subgroup H of $B_{m+1}^{s+1} \cap G_A$ from Proposition 7. The new G_A-invariants are obtained as the components of the unique point P_{s+1} given by the intersection of $\mathcal{B}_s^{s+1}(\mathrm{Orb}_s(w_s))$ with the affine subspace of $\mathbb{R}^{N_s^{s+1}}$ containing the origin and the modelling vector space of which being $\mathbb{V}_s^C$. For almost every G_A-orbit in the sense of density, the maximal dimension K_{s+1} is attained and so it suffices to select only $N_1^2 - K_2$ components forming the basis of the additional G_A-invariants from the $(s+1)$th step.

Let us express them in formulas. Select a linear basis of $\mathbb{V}_{s+1}$ formed by the elements $[j_0^{s+1}\varphi_{s+1}^1]_{H_s^{s+1}}, \ldots, [j_0^{s+1}\varphi_{s+1}^{K_{s+1}}]_{H_s^{s+1}}$. Denote by $\mathrm{Ort}_i^{s+1}([j_0^{s+1}\varphi_{s+1}]_{H_s^{s+1}})$ the orthogonal complement to the sequence obtained from this basis by omitting the ith element. Then for any $w \in T_i^*T^A\mathbb{R}^{m+1}$ we have

$$P_{s+1}(w_s) = \mathcal{B}_s^{s+1}(w_s) + C_i^{s+1}[j_0^{s+1}\varphi_{s+1}^i]_{H_s^{s+1}} \tag{21}$$

if we put

$$C_i^{s+1} = \frac{((\mathcal{B}_s^{s+1}(w_s), [j_0^{s+1}\varphi_{s+1}]_{H_s^{s+1}}), \mathrm{Ort}_i^{s+1}([j_0^{s+1}\varphi_{s+1}]_{H_s^{s+1}})}{(([j_0^{s+1}\varphi_{s+1}]_{H_s^{s+1}}, [j_0^{s+1}\varphi_{s+1}^i]_{H_s^{s+1}}), \mathrm{Ort}_i^{s+1}([j_0^{s+1}\varphi_{s+1}]_{H_s^{s+1}})} \tag{22}$$

where $(\ ,\ )$ denotes the scalar product and we use the vector form of the notation. Taking into account the identification (18), we obtain the N_s^{s+1}-tuple of natural T^*T^A-functions given by

$$\tilde{I}_{s+1}(w) \simeq P_{s+1}(\ell(\alpha_{s+1}) \circ \ldots \circ \ell(\alpha_2)(w)). \tag{23}$$

Selecting $N_s^{s+1} - K_{s+1}$ components of P_{s+1}, we obtain the base natural functions $\tilde{I}_{s+1}^1, \ldots, \tilde{I}_{s+1}^{N_s^{s+1}-K_{s+1}}$ and the basis $\tilde{\mathcal{D}}_{s+1} = \tilde{\mathcal{D}}_s \cup \tilde{I}_{s+1}^1, \ldots, \tilde{I}_s^{N_s^{s+1}-K_{s+1}}$ of natural T^*T^A-functions after the $(s+1)$th step of the procedure.

This generating alghoritm is finished if in the $(s+2)$th step the inequality $k_{s+2} \geqslant N_{s+1}^{s+2}$. This means that the excessive coordinates can be annihilated by the action of $B_{m+1}^{s+1} \cap G_A$. Clearly, $s \leqslant r-1$.

In the case of the $(s+2)$th step, we start from w_{s+1} obtained as follows. We used the uniquely determined element $\alpha_{s+1}(w_s)$ of $\mathbb{V}_{s+1} = (B_{m+1}^{s+1} \cap G_A)/H_s^{s+1}$ to obtain P_{s+1} and so the element $w_s \in T_i^*T^A\mathbb{R}^{m+1}$ is after the $(s+1)$th step transformed into $w_{s+1} = \ell(\alpha_{s+1}(w_s), w_s)$.

We have proved the main result given in the following classification theorem

Theorem 8. *Let $A = \mathbb{D}_k^r/I$ be a Weil algebra of width k, $\dim M = m \geqslant k+1$. Let $\tilde{\iota}_{\mathcal{B}_{x_0}}\colon T^AM \to T_k^rM$ be the embedding presented in Proposition* 1. *Consider a basis C of A and a basis $\mathcal{B}_0$ of $\mathrm{Der}(\mathbb{D}_k^r)$. Further, let $\tilde{\mathcal{B}}$ be a basis of functions defined on T^*T^AM constructed from operators $T\tilde{p} \circ \lambda_D \circ \tilde{\iota}_{\mathcal{B}_{x_0}}$ by the operation $\tilde{}$ defined at the very end of Section* 1, *$D \in \mathcal{B}_0$. Then all natural T-functions $f_M\colon T^*T^AM \to \mathbb{R}$ are of the form*

$$h(L_M\tilde{(c)}\mathcal{T}_M^A, \tilde{I}_{M;h}^1, \tilde{I}_{M;2}^1, \ldots, \tilde{I}_{M;2}^{N_1^2-K_2}, \tilde{I}_{M;r}^1, \ldots \tilde{I}_{M;r}^{N_{r-1}^r-K_r}) \tag{24}$$

where h is any smooth function of a suitable type, $\tilde{I}_{h_1}$ are natural functions selected directly from $\tilde{\mathcal{B}}$ and $\tilde{I}^{l_s}_{M;s}$ are obtained in the sth step of the recurrent procedure.

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