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ON THE MINUS DOMINATION NUMBER OF GRAPHS

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Abstract. Let $G=(V,E)$ be a simple graph. A 3-valued function $f\colon V(G)\to\{-1,0,1\}$ is said to be a minus dominating function if for every vertex $v\in V$, $f(N[v])=\sum\limits_{u\in N[v]} f(u)\geqslant 1$, where $N[v]$ is the closed neighborhood of v. The weight of a minus dominating function f on G is $f(V)=\sum\limits_{v\in V} f(v)$. The minus domination number of a graph G, denoted by $\gamma^-(G)$, equals the minimum weight of a minus dominating function on G. In this paper, the following two results are obtained.

(1) If G is a bipartite graph of order n, then

$$\gamma^-(G)\geqslant 4\bigl(\sqrt{n+1}-1\bigr)-n.$$

(2) For any negative integer k and any positive integer $m\geqslant 3$, there exists a graph G with girth m such that $\gamma^-(G)\leqslant k$. Therefore, two open problems about minus domination number are solved.

Keywords: minus dominating function, minus domination number

MSC 2000: 05C69

1. Introduction

Let $G=(V,E)$ be a simple graph. The girth of G is the length of a shortest cycle in G. For a vertex v of G, the closed neighborhood of v is the set $N[v]$ consisting of v together with all vertices of G adjacent to v. Let f be a real valued function on V. For a non-empty subset S of V, we define $f(S)=\sum\limits_{v\in S} f(v)$. The minus dominating function is a function $f\colon V(G)\to\{-1,0,1\}$ such that $f(N[v])\geqslant 1$ for

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all $v \in V(G)$. The minus domination number for a graph G is $\gamma^-(G) = \min\{f(V)\colon f$ is a minus dominating function on $G\}$. The problem of finding $\gamma^-(G)$ seems to be very difficult. Even if we restrict G to be bipartite, the corresponding decision problem is also NP-complete. In [3], the following two open problems about the minus domination number of a graph were posed.

Conjecture 1 ([3]). *If G is a bipartite graph of order n, then*

$$\gamma^-(G) \geqslant 4\bigl(\sqrt{n+1}-1\bigr) - n.$$

Problem 1 ([3]). For every negative integer k and positive integer m, does there exist a graph G with girth m and $\gamma^-(G) \leqslant k$?

In Section 2, we will prove that Conjecture 1 is true. And in Section 3 we will give a positive answer to Problem 1.

2. Minus domination of bipartite graphs

In this section, we will give a proof for Conjecture 1. A bipartite graph $B = (X, Y)$ is an (a, b)-bipartite graph if every vertex in X has degree a and every vertex in Y has degree b. If $B = (X, Y)$ is an (a, b)-bipartite graph, then $a|X| = b|Y|$.

Let $\mathscr{F}_s$ be a family of bipartite graphs of order $n = 4s(s+1)$ in which each bipartite graph $B = (X, Y)$ satisfies the following two properties:

(1) $X = X_1 \cup X_2$ is a partition of X such that $|X_1| = 2s$ and $|X_2| = 2s^2$, and $Y = Y_1 \cup Y_2$ is a partition of Y such that $|Y_1| = 2s$ and $|Y_2| = 2s^2$.

(2) Both $G[X_1 \cup Y_2]$ and $G[Y_1 \cup X_2]$ are $(2s, 2)$-bipartite graphs, $G[X_1 \cup Y_1] = K_{2s,2s}$ is an $(2s, 2s)$-bipartite graph, and $G[X_2 \cup Y_2]$ contains no edges.

Since $K_{2,2s}$ is a $(2s, 2)$-bipartite graph, the family $\mathscr{F}_s$ is not empty for any positive integer s.

It is easy to prove the following lemma.

Lemma 1. *For all positive integers n, the inequality $4\bigl(\sqrt{n+1}-1\bigr) - n \leqslant 1$ holds and it becomes an equality only for $n = 3$.*

Theorem 1. *If G is a bipartite graph of order n, then*

$$\gamma^-(G) \geqslant 4\bigl(\sqrt{n+1}-1\bigr) - n.$$

Further, a bipartite graph G satisfies $\gamma^-(G) = 4\bigl(\sqrt{n+1}-1\bigr) - n$ if and only if G is $K_{1,2}$ or G is a bipartite graph in $\mathscr{F}_s$ where $n = 4s(s+1)$.

P r o o f. Let f be a minimum minus dominating function on G. Let X and Y be the bipartite sets of G. Denote $X^+ = \{v \in X \mid f(v) = 1\}$. $X^- = \{v \in X \mid f(v) = -1\}$ and $X^0 = \{v \in X \mid f(v) = 0\}$. Denote $Y^+ = \{v \in Y \mid f(v) = 1\}$, $Y^- = \{v \in Y \mid f(v) = -1\}$ and $Y^0 = \{v \in Y \mid f(v) = 0\}$. Let $P = X^+ \cup Y^+$, $M = X^- \cup Y^-$ and $W = V(G) - P - M = X^0 \cup Y^0$. Furthermore, let $|X^+| = x_1$, $|X^-| = x_2$, $|Y^+| = y_1$, $|Y^-| = y_2$, $|P| = p$, $|M| = m$ and $|W| = w = n - p - m$. It is obvious that $x_1 + y_1 = p > 0$, and $w \geqslant 0$.

Case 1: $x_1 = 0$ or $y_1 = 0$.

If $x_1 = 0$, then we have that $y_1 > 0$ and $y_2 = 0$. Furthermore, we have $x_2 = 0$. Otherwise, we assume that there exists a vertex $u \in X^- \neq \emptyset$. Since $f(N[u]) \geqslant 1$, we have $N[u] \cap Y^+ \neq \emptyset$. For any $v \in N[u] \cap Y^+$, since $X^+ = \emptyset$, we have $f(N[v]) \leqslant 0$. This contradicts that f is a minus dominating function. Therefore, by Lemma 1, we have $\gamma^-(G) = p - m = x_1 + y_1 - (x_2 + y_2) = y_1 \geqslant 1 \geqslant 4\big(\sqrt{n+1} - 1\big) - n$. For the case $y_1 = 0$, the proof is completely similar. Furthermore, if a bipartite graph G of order n satisfies that $\gamma^-(G) = 1 = 4\big(\sqrt{n+1} - 1\big) - n$, then $n = 3$ and $G = K_{1,2}$.

Case 2: $x_1 > 0$ and $y_1 > 0$.

Since every vertex in X^- must be adjacent to at least two vertices in Y^+, by the pigeon-hole principle, there is a vertex v_0 of Y^+ such that v_0 is adjacent to at least $\lceil 2x_2/y_1 \rceil$ vertices of X^-. Since $1 \leqslant f(N[v_0]) = 1 - |N(v_0) \cap X^-| + |N(v_0) \cap X^+| \leqslant 1 - \lceil 2x_2/y_1 \rceil + |N(v_0) \cap X^+|$, we have that

$$x_1 = |X^+| \geqslant |N(v_0) \cap X^+| \geqslant \lceil 2x_2/y_1 \rceil \geqslant 2x_2/y_1.$$

Thus we obtain that $x_1 y_1 \geqslant 2x_2$. Similarly, we have that $x_1 y_1 \geqslant 2y_2$. Therefore $x_1 y_1 \geqslant x_2 + y_2 = n - p - w$. Since $x_1 y_1 \leqslant \frac{1}{4}(x_1 + y_1)^2 = \frac{1}{4}p^2$, we have that $\frac{1}{4}p^2 \geqslant n - p - w$. Thus we have that $\frac{1}{4}p^2 + p \geqslant n - w$. Since $p = x_1 + y_1 \geqslant 2$ and $w \geqslant 0$, we have that $w(w + 4p - 8) \geqslant 0$. Thus we can obtain that

$$\frac{p^2}{4} + p \geqslant n - w \geqslant n - \frac{(p+2)w}{4} - \frac{w^2}{16}.$$

This follows that

$$\Big(\frac{2p+w}{4} + 1\Big)^2 \geqslant n + 1.$$

Thus we have that

$$2p + w \geqslant 4\big(\sqrt{n+1} - 1\big).$$

Therefore,

$$\gamma^-(G) = p - m = 2p - (n - w) = (2p + w) - n \geqslant 4\big(\sqrt{n+1} - 1\big) - n.$$

Now we assume that G is a bipartite graph of order n such that $\gamma^-(G) = 4(\sqrt{n+1}-1)-n$. Then $2p+w = 4(\sqrt{n+1}-1)$ and $w(w+4p-8) = 0$. Since $p \geqslant 2$, we have that $w = 0$ and $\frac{1}{4}p^2 + p = n$. Thus $x_1y_1 = \frac{1}{4}(x_1+y_1)^2$ and $x_1y_1 = x_2 + y_2$. Therefore, the following properties of G can be obtained:

(1) $x_1 = y_1 = \frac{1}{2}p = \sqrt{n+1} - 1$,

(2) $x_2 = y_2 = \frac{1}{2}x_1y_1 = \frac{1}{8}p^2 = \frac{1}{2}(n - 2\sqrt{n+1} + 2)$,

(3) every vertex in $X_2 \cup Y_2$ has degree 2,

(4) every vertex in X_1 (Y_1) is adjacent to $\sqrt{n+1} - 1$ vertices in Y_2 (X_2), and

(5) $G[X_1 \cup Y_1]$ is a $(\sqrt{n+1}-1, \sqrt{n+1}-1)$ bipartite graph and $G[X_2 \cup Y_2]$ contains no edges.

Since $\sqrt{n+1}$ is an integer and n is even, there exists an s such that $n = 4s(s+1)$. Thus G is a bipartite graph in $\mathscr{F}_s$. Now for any graph G in $\mathscr{F}_s$, we let $f(v) = -1$ if $v \in X_2 \cup Y_2$ and $f(v) = 1$ if $v \in X_1 \cup Y_1$. Then f is a minus dominating function on G. Thus $\gamma^-(G) \leqslant f(V(G)) = |X_1| + |Y_1| - |X_2| - |Y_2| = 4(\sqrt{n+1} - 1) - n$. Therefore any graph G in $\mathscr{F}_s$ satisfies that $\gamma^-(G) = 4(\sqrt{n+1} - 1) - n$. This completes the proof. □

3. Graphs with negative minus domination number and large girth

In this section, we are going to give a positive answer to Problem 1. An s-regular graph with girth m is called an (s, m)-graph.

Lemma 2 ([8, p. 81]). *For any positive integers $s \geqslant 2$, $m \geqslant 3$ and $n \geqslant 3$, there exists a connected (s, m)-graph G such that the order of G is at least n.*

An s-factor of G is an s-regular spanning subgraph of G, and G is s-factorable if there are edge-disjoint s-factors $H_1, H_2, \ldots, H_r$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_r$.

Lemma 3. *For any positive integer r, if G is a $4r$-regular graph, then G is 4-factorable.*

Proof. By a famous theorem of Petersen [7], we have that any regular graph with even degree is 2-factorable. Thus G can be factored into $2r$ 2-factors $F_1, \ldots, F_{2r}$. Let $H_j = F_{2j-1} \cup F_{2j}$, $j = 1, \ldots, r$. Then $H_1, \ldots, H_r$ are r pair-wise edge disjoint 4-factors of G. □

Theorem 2. *For any negative integer k and positive integer $m \geqslant 3$, there exists a graph G with girth m and $\gamma^-(G) \leqslant k$.*

P r o o f. Assume that k is a negative integer and $m \geqslant 3$ is a positive integer. Let n be a positive integer such that $m - n \leqslant k$. By Lemma 2, there exists a connected $(8, m)$-graph H with order at least n. By Lemma 3, H can be factored into two edge disjoint 4-factors H_1 and H_2. Let C be an m-cycle in H. By subdividing all edges in $E(H_1) - E(C)$ we obtain a new graph G from H. Then G is a connected graph with girth m. We denote by T the set of all vertices with degree 2 in G. Then $t = |T| \geqslant 2n - m$, and the order of G is $n + t$. We define a mapping $f\colon V(G) \to \{-1, 0, 1\}$ such that $f(v) = 1$ if $v \in V(G) - T$ and $f(v) = -1$ if $v \in T$. Then it is easy to verify that f is a minus dominating function on G. Thus $\gamma^-(G) \leqslant f(V(G)) = n - t \leqslant n - (2n - m) = m - n \leqslant k$. Therefore, G is a graph satisfying all the conditions of the theorem. This completes the proof. □

References

[1] *J. A. Bondy and U. S. R. Murty*: Graph Theory with Applications. Macmillan, New York, 1976.

[2] *J. E. Dunbar, W. Goddard, S. T. Hedetniemi, M. A. Henning and A. McRae*: The algorithmic complexity of minus domination in graphs. Discrete Appl. Math. *68* (1996), 73–84.

[3] *J. E. Dunbar, S. T. Hedetniemi, M. A. Henning and A. A. McRae*: Minus domination in graphs. Discrete Math. *199* (1999), 35–47.

[4] *J. E. Dunbar, S. T. Hedetniemi, M. A. Henning and A. A. McRae*: Minus domination in regular graphs. Discrete Math. *49* (1996), 311–312.

[5] *J. H. Hattingh, M. A. Henning and P. J. Slater*: Three-valued neighborhood domination in graphs. Australas. J. Combin. *9* (1994), 233–242.

[6] *T. W. Haynes, S. T. Hedetniemi and P. J. Slater*: Fundamentals of Domination in Graphs. Marcel Dekker, New York, 1998.

[7] *J. Petersen*: Die Theorie der regulären Graphs. Acta Math. *15* (1891), 193–220.

[8] *W. T. Tutte*: Connectivity in Graphs. University Press, Toronto, 1966.

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