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ON DOMINATION NUMBER OF 4-REGULAR GRAPHS

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Abstract. Let G be a simple graph. A subset $S \subseteq V$ is a dominating set of G, if for any vertex $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. In this paper we prove that if G is a 4-regular graph with order n, then $\gamma(G) \leq \frac{4}{11}n$.

Keywords: regular graph, dominating set, domination number

MSC 2000: 05C69

1. INTRODUCTION

Let G = (V(G), E(G)) be a simple graph. For a vertex $v \in V(G)$, denote by N(v)the open neighborhood of v. Let $N[v] = N(v) \cup \{v\}$. Denote by $\delta(G)$ the minimum degree of G. For a subset S of V(G), denote by G[S] the subgraph induced by S. A subset $S \subseteq V$ is a dominating set of G, if for any vertex $u \in V - S$ there exists a vertex $v \in S$ such that $uv \in E(G)$. The domination number, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set S of G is a γ -set if $|S| = \gamma(G)$. Some bounds on $\gamma(G)$ with minimum degree conditions have been obtained as follows.

Theorem 1 [3]. If a graph G has no isolated vertices, then $\gamma(G) \leq n/2$.

McGuaig and Shepherd made another improvement on the upper bound. Let \mathscr{A} be the collection of graphs in Figure 1.

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Theorem 2 [2]. If G is a connected graph with $\delta(G) \ge 2$ and $G \notin \mathscr{A}$, then $\gamma(G) \le 2n/5$.

Reed again improved the bound by increasing the minimum degree requirement.

Theorem 3 [4]. If G is a connected graph with $\delta(G) \ge 3$, then $\gamma(G) \le 3n/8$.

Motivated by the above conclusions, Haynes et al. [1] conjectured that

Conjecture 1 [1]. For any graph G with $\delta(G) \ge k$, $\gamma(G) \le k(3k-1)^{-1}n$.

The question still remains open for graphs G having $4 \leq \delta(G) \leq 6$. In the next section, we will prove that $\gamma(G) \leq 4n/11$ for any 4-regular graph G with order n.

2. Main results

First, we give some definitions and symbols needed for the proof of Theorem 4.

Let S be a γ -set of G, let $N_i(S) = \{u \in V - S : |N(u) \cap S| = i\}$ where $1 \leq i \leq 4$. For any vertex $v \in S$, let $N_i(v, S) = N(v) \cap N_i(S)$. Denote by $\lambda(S)$ the number of isolates in G[S]. Let $\mu(S) = |N_1(S)|$ and $\eta(S) = |N_2(S)|$. Let $J_0 = \{v \in S : |N_1(v, S)| = 0\}, J_1 = \{v \in S : |N_1(v, S)| = 1\}$ and $J_2 = \{v \in S : |N_1(v, S)| \geq 2\}$. Let $B = \{v \in J_0 : N(v) \cap N_3(S) \neq \emptyset\}$ and $R = \{u \in N_3(S) : N(u) \cap B \neq \emptyset\}$. For any vertex $v \in J_1$ there exists only one vertex $u \in V - S$ such that $u \in N_1(v, S)$; we write P(v) for u.

For any two vertex subsets $C, D \subseteq V$, we denote the set of edges between C and D by E[C, D].

Theorem 4. If G is a 4-regular graph with order n, then $\gamma(G) \leq \frac{4}{11}n$.

Proof. Among all γ -sets of G, let S be chosen so that

- (1) $\lambda(S)$ is maximized;
- (2) subject to $(1), \mu(S)$ is minimized;
- (3) subject to $(2), \eta(S)$ is minimized.

Before proceeding further, we prove the following claims.

Claim 1. Each vertex $v \in J_0 \cup J_1$ is an isolate in G[S].

Proof. Suppose to the contrary that v is not isolated in G[S]. If $v \in J_0$, then $S' = S - \{v\}$ is a domination set of G. This contradicts the fact that S is a γ -set of G. If $v \in J_1$, then $S' = (S - \{v\}) \cup \{P(v)\}$ is a γ -set of G with $\lambda(S') > \lambda(S)$. This contradicts our choice of S.

Claim 2. For any vertex $v \in J_1$, if $|N_2(v, S)| = 0$ then $|N(P(v)) \cap N_1(S)| = 0$.

Proof. Suppose to the contrary that $|N(P(v)) \cap N_1(S)| > 0$, then $S' = (S - \{v\}) \cup \{P(v)\}$ is also a γ -set of G with $\lambda(S') = \lambda(S)$, $\mu(S') < \mu(S)$, a contradiction.

Claim 3. For any $u \in V - S$, if $v_1, v_2 \in N(u) \cap J_0$ then $|N_2(v_1, S) \cap N_2(v_2, S)| \ge 2$.

Proof. Suppose to the contrary that $|N_2(v_1, S) \cap N_2(v_2, S)| < 2$. Then if $|N_2(v_1, S) \cap N_2(v_2, S)| = 0$, then $S' = (S - \{v_1, v_2\}) \cup \{u\}$ is a dominating set of G with |S'| < |S|, a contradiction. If $|N_2(v_1, S) \cap N_2(v_2, S)| = 1$ then $S' = (S - \{v_1, v_2\}) \cup (N_2(v_1, S) \cap N_2(v_2, S))$ is a dominating set of G with |S'| < |S|, a contradiction. \Box

Claim 4. Assume that $v \in J_1$ and $|N_2(v, S)| = 0$. For $1 \le t \le 3$, if $|N_4(v, S)| = t$, then $|N(P(v)) \cap N_3(S)| \ge t$.

Proof. Suppose to the contrary that $|N(P(v)) \cap N_3(S)| < t$. Then we have $|N(P(v)) \cap N_2(S)| = 3 - |N(P(v)) \cap N_3(S)| > 3 - t$. Thus $S' = (S - \{v\}) \cup \{P(v)\}$ is a γ -set of G with $\lambda(S') = \lambda(S), \mu(S') = \mu(S)$ and $\eta(S') < \eta(S)$. This contradicts the choice of S.

Now we define a function $f: E[V - S, V] \rightarrow \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ as follows. For any $v \in S$, define

$$f(uv) = \begin{cases} 1, & u \in N_1(v, S), \\ \frac{1}{2}, & u \in N_2(v, S), \\ \frac{1}{4}, & u \in N_3(v, S) \cup N_4(v, S), \\ 0, & \text{otherwise.} \end{cases}$$

For any $u \in N_3(S)$, define

$$f(uw) = \begin{cases} \frac{1}{2}, & w \in N_3(S), \\ \frac{1}{4}, & w \in V - N_3(S). \end{cases}$$

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For any $u \in V - S - N_3(S)$, define

$$f(uw) = \begin{cases} \frac{1}{4}, & w \in N_3(S), \\ 0, & w \in V - S - N_3(S). \end{cases}$$

In order to prove the theorem, note that

$$n - |S| = |V - S| = \sum_{uv \in E[V - S, V]} f(uv),$$

so we need only to prove that

$$\sum_{uv \in E[V-S,V]} f(uv) \ge \frac{7}{4}|S|.$$

If we can find a function g: E[V - S, S] satisfying the conditions

(1)
$$\sum_{uv \in E[V-S,V]} f(uv) \ge \sum_{uv \in E[V-S,S]} g(uv)$$

(2)
$$\sum_{uv \in E[V-S,S]} g(uv) \ge \frac{7}{4}|S|,$$

the conclusion will follow immediately.

For convenience, for any $v \in S$ we define $h(v) = \sum_{u \in N(v) \cap (V-S)} g(uv)$.

Note that

$$\sum_{uv \in E[V-S,S]} g(uv) = \sum_{v \in S} \left(\sum_{u \in N(v) \cap (V-S)} g(uv) \right) = \sum_{v \in S} h(v).$$

If the following condition holds, then condition (2) holds as well:

For any vertex
$$v \in S$$
, $h(v) \ge \frac{7}{4}$.

In the following, we will define a function $g: E[V-S,S] \to \{0,\frac{1}{4},\frac{3}{8},\frac{1}{2}\}$ satisfying conditions (1) and (3).

For any vertex $v \in J_2$ and $uv \in E[V - S, S]$, define

$$g(uv) = \begin{cases} 1, & u \in N_1(v, S), \\ 0, & \text{otherwise.} \end{cases}$$

Assuming that $w_1, w_2 \in N_1(v, S)$, we have $h(v) \ge g(w_1v) + g(w_2v) = 2$.

For any vertex $v \in J_1$ and $uv \in E[V - S, S]$, define

$$g(uv) = \begin{cases} 1, & u \in N_1(v, S), \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Thus we have

$$h(v) = g(P(v)v) + \sum_{u \in N(v) - \{P(v)\}} g(uv) = 1 + \frac{3}{4} = \frac{7}{4}.$$

For any $v \in J_0$ and $uv \in E[V - S, S]$, if $u \in N_2(v, S)$, define

$$g(uv) = f(uv) = \frac{1}{2}.$$

Before proceeding further, we introduce the following notation: Let $K = \{v \in J_0: N(v) \cap N(J_0 - \{v\}) = \emptyset\}$ and $L = J_0 - K$. Denote

$$M_1 = \{y \in N(K) \cap N_4(S) | N(y) - K \subseteq J_1$$

and for any vertex $x \in N(y) - K$, $N_2(x, S) = \emptyset\},$
$$Q_1 = \{y \in N(K) \cap N_4(S) | N(y) - K \subseteq J_1$$

and there exist two vertices $x_1, x_2 \in N(y) - K$

such that $N_2(x_1, S) = \emptyset$ and $N_2(x_2, S) = \emptyset$ and an other vertex $x_3 \in N(y) - K$ such that $N_2(x_3, S) \neq \emptyset$.

Now, we consider the following two cases.

- Case 1. $v \in K$.
- **Case 1.1.** $|N(v) \cap Q_1| \leq 1$.
- Case 1.1.1. $N(v) \cap M_1 = \emptyset$.

For any $u \in N(v) - Q_1 - N_2(S)$, if $u \in N_3(S)$ then there exists a vertex $x \in V - S$ such that $ux \in E(G)$. Define $g(uv) = f(uv) + f(ux) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. If $u \in N_4(S)$, then $(N(u) - \{v\}) \subseteq J_1 \cup J_2$. If $(N(u) - \{v\}) \cap J_2 \neq \emptyset$, then there exists a vertex $x \in N(u) \cap J_2$. Define $g(uv) = f(uv) + f(ux) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Otherwise, there exist two vertices $x_1, x_2 \in N(u) - \{v\}$ such that $N_2(x_1, S) \neq \emptyset$ and $N_2(x_2, S) \neq \emptyset$. Assume that $w_1 \in N_2(x_1, S)$ and $w_2 \in N_2(x_2, S)$ and define

$$g(uv) = f(uv) + \frac{1}{2}\left(f(w_1x_1) - \frac{1}{4}\right) + \frac{1}{2}\left(f(w_2x_2) - \frac{1}{4}\right) = \frac{1}{2}.$$

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If $N(v) \cap Q_1 \neq \emptyset$, for any vertex $u \in N(v) \cap Q_1$ define $g(uv) = f(uv) = \frac{1}{4}$. Thus

$$h(v) = \sum_{u \in N(v) \cap (V-S)} g(uv) \ge \sum_{u \in N(v) \cap (V-S-Q_1)} g(uv) + \sum_{u \in N(v) \cap Q_1} g(uv) \ge \frac{7}{4}$$

Case 1.1.2. $N(v) \cap M_1 \neq \emptyset$.

There exists a vertex $u \in N(v) \cap M_1$. For any vertex $x \in N(u) - \{v\}$, by Claim 4, we can select a vertex $z \in N(P(x)) \cap N_3(S)$. We claim that $z \in N_3(S) - (R - N(v))$. Suppose to the contrary that $z \in R - N(v)$, then there exists a vertex $b \in B$ such that $bz \in E(G)$ and $b \neq v$. Since $N(v) \cap N(b) = \emptyset$, we let $S' = (S - \{b, v, x\}) \cup \{z, u\}$. Then S' is a dominating set of G with cardinality less than S, a contradiction. Assume that $N(u) - \{v\} = \{x_1, x_2, x_3\}$. Then for $1 \leq i \leq 3$ there exist $z_i \in N(P(x_i)) \cap (N_3(S) - (R - N(v)))$. Assume $N(v) = \{u_1, u_2, u_3, u_4\}$. For i = 1, 2, 3, define

$$g(u_i v) = f(u_i v) + f(z_i P(x_i)) \ge \frac{1}{2}.$$

Moreover, define $g(u_4v) = f(u_4v) \ge \frac{1}{4}$. Thus we have

$$h(v) = \sum_{i=1}^{4} g(u_i v) \ge \frac{7}{4}$$

Case 1.2. $|N(v) \cap Q_1| \ge 2$.

Then there exist two vertices $u, u' \in N(v) \cap Q_1$. Further, there exist $x_1, x_2 \in N(u) \cap J_1$ and $x'_1, x'_2 \in N(u') \cap J_1$ such that $N_2(x_1, S) = \emptyset$, $N_2(x_2, S) = \emptyset$, $N_2(x'_1, S) = \emptyset$ and $N_2(x'_2, S) = \emptyset$. By Claim 4 there exist $z_i \in N(P(x_i)) \cap (N_3(S) - (R - N(v)))$ and $z'_i \in N(P(x'_i)) \cap (N_3(S) - (R - N(v)))$ where i = 1, 2. Assume $N(v) = \{u_1, u_2, u_3, u_4\}$. For i = 1, 2, define

$$g(u_i v) = f(u_i v) + f(z_i P(x_i)) \ge \frac{1}{2}.$$

Moreover, define $g(u_4v) = f(u_4v) + f(z'_1P(x'_1)) \ge \frac{1}{2}$ and $g(u_4v) = f(u_4v)$. Thus we have

$$h(v) = \sum_{i=1}^{4} g(u_i v) \ge \frac{7}{4}$$

Case 2. $v \in L$.

By the definition of L and Claim 3 there exists $v' \in L$ such that $|N_2(v, S) \cap N_2(v', S)| \ge 2$.

If $|N_2(v, S) \cap N_2(v', S)| = 4$, then

$$h(v) = \sum_{u \in N(v)} g(uv) = 4 \times \frac{1}{2} = 2$$
 and $h(v') = \sum_{u' \in N(v')} g(u'v') = 4 \times \frac{1}{2} = 2.$

If $|N_2(v,S) \cap N_2(v',S)| = 3$, then for $u \in N(v) - N_2(v,S)$ and $u' \in N(v') - N_2(v',S)$, define $g(uv) = f(uv) = \frac{1}{4}$ and $g(u'v') = f(u'v') = \frac{1}{4}$, thus

$$h(v) = \sum_{y \in N(v)} g(yv) = 3 \times \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \quad \text{and} \quad h(v') = \sum_{u' \in N(v')} g(u'v') = 3 \times \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

If $|N_2(v, S) \cap N_2(v', S)| = 2$, then we assume that $u_1, u_2 \in N_2(v, S) \cap N_2(v', S)$ and distinguish the following cases.

Case 2.1. $|(N(v) \cup N(v')) \cap (N_3(S) \cup N(J_2))| \ge 2.$

Case 2.1.1. $|(N(v) \cup N(v')) \cap N(J_2)| \ge 2.$

Without loss of generality, there exist $y \in N(v) \cap N(J_2)$ and $y' \in N(v') \cap N(J_2)$. Then there exist vertices $x, x' \in J_2$ such that $xy \in E(G)$ and $x'y' \in E(G)$. Define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $g(y'v') = f(y'v') + f(y'x') = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(yv') + g(zv') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

Case 2.1.2. $|(N(v) \cup N(v')) \cap N_3(S)| \ge 2.$

Without loss of generality, there exist vertices $y \in N(v) \cap N_3(S)$ and $y' \in N(v') \cap N_3(S)$. Then there exist vertices $x, x' \in V - S$ such that $xy \in E(G)$ and $x'y' \in E(G)$. If $x \in N_3(S)$, define $g(yv) = f(yv) + \frac{1}{2}f(yx) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Otherwise, define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Similarly, we can define $g(y'v') = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(yv') + g(zv') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.$$

Case 2.3. $|(N(v) \cup N(v')) \cap N(J_2)| = 1$ and $|(N(v) \cup N(v')) \cap N_3(S)| = 1$.

Without loss of generality, there exist vertices $y \in N(v) \cap N_3(S)$ and $y' \in N(v') \cap N(J_2)$. Then there exist a vertex $x \in V - S$ such that $xy \in E(G)$ and a vertex $x' \in J_2$ such that $x'y' \in E(G)$. If $x \in N_3(S)$, define $g(yv) = f(yv) + \frac{1}{2}f(yx) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Otherwise, define $g(yv) = f(yv) + f(yx) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Define $g(y'v') = f(y'v') + f(y'x') = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. For $z \in N(v) - \{u_1, u_2, y\}$ and $z' \in N(v') - \{u_1, u_2, y'\}$, define $g(zv) = f(zv) = \frac{1}{4}$ and $g(z'v') = f(z'v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(yv) + g(zv) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(y'v') + g(z'v') = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

Case 2.2. $|(N(v) \cup N(v')) \cap (N_3(S) \cup N(J_2))| \leq 1.$

Denote

$$\begin{split} M_2 &= \{ y \in N(L) \cap N_4(S) | N(y) - L \subseteq J_1 \text{ and there exist two vertices} \\ & x_1, x_2 \in N(y) - L \text{ such that } N_2(x_1, S) = \emptyset, \ N_2(x_2, S) = \emptyset \}, \\ Q_2 &= \{ y \in N(L) \cap N_4(S) | N(y) - L \subseteq J_1 \text{ and } | N(y) - L | = 2 \\ & \text{ and for } x_1, x_2 \in N(y) - L, \ N_2(x_1, S) = \emptyset, \ N_2(x_2, S) \neq \emptyset \}. \end{split}$$

Case 2.2.1. $|N(v) \cup N(v') \cap Q_2| \leq 1$.

Case 2.2.1.1. $(N(v) \cup N(v')) \cap M_2 = \emptyset$.

If $N(v) \cap Q_2 \neq \emptyset$, assume $u_3 \in N(v) \cap Q_2$ and $u_4 \in N(v) - \{u_1, u_2, u_3\}$ and $u_5 \in N(v') - \{u_1, u_2, u_3\}$. Then, without loss of generality, there exist vertices x_1, x_2, x_3 and x_4 such that $x_1, x_2 \in N(u_3) - L$, $x_3, x_4 \in N(u_4) - L$ and $N_2(x_1, S) = \emptyset$, $N_2(x_2, S) \neq \emptyset$, $N_2(x_3, S) \neq \emptyset$ and $N_2(x_4, S) \neq \emptyset$. Assume $w_i \in N_2(x_i, S)$ for i = 2, 3, 4. By Claim 4 we can select $z_1 \in (N(P(x_1)) \cap (N_3(S) - (R - (N(v) \cap N(v')))))$. Define $g(u_3v') = f(u_3v) + f(z_1x_1) = \frac{1}{2}, g(u_4v) = f(u_4v) = \frac{1}{4}$ and $g(u_3v) = f(u_3v) = \frac{1}{4}, g(u_5v') = f(u_5v') + \frac{1}{2} \times (f(w_2x_2) - \frac{1}{4}) + \frac{1}{2} \times (f(w_3x_3) - \frac{1}{4}) + \frac{1}{2} \times (f(w_4x_4) - \frac{1}{4}) \ge \frac{1}{2}$.

Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(u_3v) + g(u_4v) \ge \frac{7}{4},$$

$$h(v') = g(u_1v') + g(u_2v') + g(u_3v') + g(u_5v') \ge \frac{7}{4}.$$

If $N(v) \cap Q_2 = \emptyset$, assume $u_3, u_4 \in N(v) - \{u_1, u_2\}$ and $u_5, u_6 \in N(v') - \{u_1, u_2\}$. Then, without loss of generality, there exist vertices x_1, x_2, x_3 and x_4 such that $x_1, x_2 \in N(u_3) - L, x_3, x_4 \in N(u_4) - L$ and $N_2(x_1, S) \neq \emptyset, N_2(x_2, S) \neq \emptyset, N_2(x_3, S) \neq \emptyset$ and $N_2(x_4, S) \neq \emptyset$. Assume $w_i \in N_2(x_i, S)$ for i = 1, 2, 3, 4.

Define $g(u_3v) = f(u_3v) + \frac{1}{2} \times (f(w_1x_1) - \frac{1}{4}) + \frac{1}{2} \times (f(w_2x_2) - \frac{1}{4}) = \frac{1}{2}, g(u_4v) = f(u_4v) = \frac{1}{4}$ and $g(u_5v') = f(u_5v') + \frac{1}{2} \times (f(w_3x_3) - \frac{1}{4}) + \frac{1}{2} \times (f(w_4x_4) - \frac{1}{4}) = \frac{1}{2}, g(u_6v') = f(u_6v') = \frac{1}{4}$. Therefore, we have

$$h(v) = g(u_1v) + g(u_2v) + g(u_3v) + g(u_4v) \ge \frac{7}{4},$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(u_5v') + g(u_6v') \ge \frac{7}{4}.$$

Case 2.2.1.2. $(N(v) \cup N(v')) \cap M_2 \neq \emptyset$.

Assume $u \in (N(v) \cup N(v')) \cap M_2$, then there exist two vertices $x_1, x_2 \in N(u) - \{v, v'\} \subseteq J_1$ such that $N_2(x_1, S) = \emptyset$, $N_2(x_2, S) = \emptyset$. By the same argument as in case 1, we can select two vertices $z_1 \in N(P(x_1)) \cap N_3(S), z_2 \in N(P(x_2)) \cap N_3(S)$; we claim that $z_1, z_2 \in N_3(S) - (R - (N(v) \cup N(v')))$. Without loss of generality, suppose to the contrary that there exists a vertex $b \in B$ such that $bz_1 \in E(G)$ and $b \notin \{v, v'\}$. Since $N(v) \cap N(b) = \emptyset$, $S' = S - \{b, v, x_1\} \cup \{z_1, u\}$ is a dominating set of G with cardinality less than S, a contradiction. Assume that $y_1, y_2 \in N(v) - \{u_1, u_2\}$, $y'_1, y'_2 \in N(v') - \{u_1, u_2\}$, define $g(y_1v) = f(y_1v) + f(z_1P(x_1)) \ge \frac{1}{2}, g(y_2v) = f(y_2v) \ge \frac{1}{4}$ and $g(y'_1v') = f(y'_1v') + f(z_2P(x_2)) \ge \frac{1}{2}, g(y'_2v') = f(y'_2v') \ge \frac{1}{4}$. Thus we have

$$h(v) = g(u_1v) + g(u_2v) + g(y_1v) + g(y_2v) \ge \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

and

$$h(v') = g(u_1v') + g(u_2v') + g(y'_1v') + g(y'_2v') \ge \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

Case 2.2.2. $|N(v) \cup N(v') \cap Q_2| \ge 2$.

Then there exist two distinct vertices $y, y' \in (N(v) \cup N(v')) \cap Q_2$, so there exist $x_1, x_2 \in (N(y) \cup N(y')) \cap J_1$ such that $N_2(x_1, S) = \emptyset$ and $N_2(x_2, S) = \emptyset$. (Note that this is possible for $x_1 = x_2$). By Claim 4, we can select $z_1 \in (N(P(x_1)) \cap (N_3(S) - (R - (N(v) \cap N(v')))), z_2 \in N(P(x_2)) \cap (N_3(S) - (R - (N(v) \cap N(v'))))$. We can argue in the same way as before, and conclude that $h(v) \ge \frac{7}{4}$ and $h(v') \ge \frac{7}{4}$.

Thus we complete the definition of the function g. It is easy to find that g satisfies conditions (1) and (3). This completes the proof of the theorem.

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