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FUNCTION SPACES ON τ-CORSON COMPACTA AND TIGHTNESS OF POLYADIC SPACES

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Abstract. We apply the general theory of τ-Corson Compact spaces to remove an unnecessary hypothesis of zero-dimensionality from a theorem on polyadic spaces of tightness τ. In particular, we prove that polyadic spaces of countable tightness are Uniform Eberlein compact spaces.

Keywords: boolean, polyadic, function space, Corson, compact, $C_p(X)$, Eberlein, tightness

MSC 2000: 54D30, 54C35

0. Introduction

All of our spaces are assumed to be completely regular. For an infinite cardinal κ, let $A_\kappa = \kappa \cup \{\infty\}$ be the one point compactification of the discrete space κ. For a cardinal λ, let A_κ^λ be the product of λ copies of A_κ endowed with the product topology. *Polyadic* spaces, introduced by Mrowka [15], are the continuous images of the spaces A_κ^λ.

The main goal of this paper is to remove the assumption of zero-dimensionality from the hypothesis of the following result by Bell

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M. Bell died on December 9, 2001.

Result 0.1 [4]. *If X is a zero-dimensional polyadic space of tightness τ and cellularity μ, then there exists a closed $F \subset A_\mu^\tau$ such that F continuously maps onto X.*

and so get a useful structure theorem for arbitrary polyadic spaces. Motivation for Result 0.1 came from the problem in Gerlits [8] of whether every polyadic space of tightness τ and cellularity μ is a continuous image of A_μ^τ; we now know (Bell [5]) that this problem has a negative answer—there is a zero-dimensional polyadic space of countable tightness which is not an image of A_μ^ω, for any μ.

Our main problem (of removing zero-dimensionality) belongs to a general class of problems that can be described as follows: For which classes $\mathscr{C}$ of compact spaces is it true that for every $X \in \mathscr{C}$, there exists a zero-dimensional $K \in \mathscr{C}$ such that X is an image of K? That is, when is the family of zero-dimensional members of $\mathscr{C}$ mapping-universal for $\mathscr{C}$? In our case, to solve our problem, we need to develop the general theory of τ-Corson Compact spaces. Since the core of our polyadic result is valid for a larger class of spaces—the continuous images of τ-Valdivia compact spaces (see Section 2 for a definition), we present our main result as a corollary to a Valdivia result.

When the tightness is countable, there is the following important result of Benyamini, Rudin and Wage which we shall relate to.

Result 0.2 [6]. *X is a Uniform Eberlein compact space $\Leftrightarrow$ there exists a cardinal κ and a closed $F \subset A_\kappa^\omega$ such that F continuously maps onto X.*

Let us recall that a space X is a Uniform Eberlein compact space if X is homeomorphic to a weakly compact subset of a Hilbert space.

1. Boolean preliminaries

We denote the algebra of all clopen subsets of a space X by $\mathrm{CO}(X)$. For a collection $\mathscr{C} \subset \mathrm{CO}(X)$, put $\langle\langle \mathscr{C} \rangle\rangle$ equal to the subalgebra of $\mathrm{CO}(X)$ generated by $\mathscr{C}$. A *generating* family for $\mathrm{CO}(X)$ is a $\mathscr{C} \subset \mathrm{CO}(X)$ such that $\langle\langle \mathscr{C} \rangle\rangle = \mathrm{CO}(X)$. A Boolean space is a compact space X which has $\mathrm{CO}(X)$ as a basis. If $\mathscr{B}$ is a boolean algebra, then $\mathrm{st}(\mathscr{B})$ is the Stone space of all ultrafilters of $\mathscr{B}$ which uses $\{B^+ \colon B \in \mathscr{B}\}$ as a basis where for $B \in \mathscr{B}$, $B^+ = \{p \in \mathrm{st}(\mathscr{B}) \colon B \in p\}$. If $\mathscr{C}$ is a subalgebra of $\mathscr{B}$ then the Stone map $\alpha \colon \mathrm{st}(\mathscr{B}) \to \mathrm{st}(\mathscr{C})$ is defined by $\alpha(p) = p \cap \mathscr{C}$ and is the canonical continuous surjection.

Given a set A and a cardinal τ, we denote by $\Sigma_\tau(\mathbb{R}^A)$ $\big(\Sigma_\tau(2^A)\big)$ the subspace of the product $\mathbb{R}^A$ (2^A) consisting of all points $x \in \mathbb{R}^A$ $(x \in 2^A)$ such that $|\{a \in A \colon x(a) \neq 0\}| \leqslant \tau$.

Lemma 1.1. *Let S be a set and τ be an infinite cardinal. Then, for every closed subset A of $\Sigma_\tau(2^S)$ we have*
(i) *A is C-embedded in 2^S,*
(ii) *$\mathrm{CO}(A)$ has a point-τ generating family,*
(iii) *subsets of A of cardinality at most τ have compact closure in A.*

Proof. (i) Corson [7] has proved that $\Sigma_\omega(2^S)$ is C-embedded in 2^S. Since $\Sigma_\omega(2^S)$ is dense in $\Sigma_\tau(2^S)$ it follows that Σ_τ is C-embedded in 2^S. It remains to observe that A is C-embedded in $\Sigma_\tau(2^S)$, because by the result of Kombarov [12] $\Sigma_\tau(2^S)$ is normal.

(ii) Let K be a closure of A in 2^S. For each $s \in S$, let $B_s = \{f \in K\colon\ f(s) = 1\}$. The family $\mathscr{C} = \{B_s \cap A\colon\ s \in S\}$ is a point-τ family of clopen subsets of A. We claim that $\mathrm{CO}(A) = \langle\langle \mathscr{C} \rangle\rangle$. To see this, let $b \in \mathrm{CO}(A)$. Invoking (i), let $f\colon K \to \mathbb{R}$ be a continuous extension of χ_b. Because A is dense in K, $f\colon K \to \{0,1\}$. Hence $f^{-1}(1)$ is a clopen subset of K such that $f^{-1}(1) \cap A = b$. As $f^{-1}(1) \in \langle\langle \{B_s\colon\ s \in S\} \rangle\rangle$, it follows, upon intersection with A, that $b \in \langle\langle \mathscr{C} \rangle\rangle$.

(iii) This follows because for a subset B of A of cardinality at most τ, the closure of B in A equals the closure of B in 2^S. □

2. τ-Corson and τ-Valdivia compact spaces

Let τ be an infinite cardinal number. We say that a compact space K is a *τ-Corson compact* space if K can be embedded in $\Sigma_\tau(\mathbb{R}^\eta)$, for some cardinal η (see also Section 8). The class of ω-Corson compacta is a well-known class of Corson compact spaces.

Proposition 2.1. *Let K be a compact space. K is a τ-Corson compact space $\Leftrightarrow$ there exists a T_0-separating point-τ family of cozero-sets in K.*

Proof. Suppose that K is a compact subset of $\Sigma_\tau(\mathbb{R}^\eta)$. Let $\mathbb{Q}^+$ denote the set of positive rationals. For every $\alpha \in \eta$ and $q \in \mathbb{Q}^+$ we define the following cozero-sets in K: $V^+_{\alpha,q} = \{x \in K\colon\ x(\alpha) > q\}$ and $V^-_{\alpha,q} = \{x \in K\colon\ x(\alpha) < -q\}$. One can easily verify that the family $\{V^+_{\alpha,q}, V^-_{\alpha,q}\colon\ \alpha \in \eta,\ q \in \mathbb{Q}^+\}$ is T_0-separating and point-τ.

Let $\mathscr{V} = \{V_\alpha\colon\ \alpha \in \eta\}$ be a T_0-separating point-τ family of cozero-sets in a compact space K. For every $\alpha \in \eta$, take $f_\alpha\colon K \to \mathbb{R}$ such that $f^{-1}(\{0\}) = K \setminus V_\alpha$. Let F be the diagonal map $\triangle_{\alpha\in\eta} f_\alpha\colon K \to \mathbb{R}^\eta$. The map F is injective since $\mathscr{V}$ is T_0-separating and the point-τ property of $\mathscr{V}$ guarantees that $F(K) \subset \Sigma_\tau(\mathbb{R}^\eta)$. □

Proposition 2.2. *For a compact space K the following are equivalent:*

(i) *K is a zero-dimensional τ-Corson compact space,*

(ii) *K has a T_0-separating point-τ family of clopen subsets,*

(iii) $\mathrm{CO}(K)$ *has a point-τ generating family,*

(iv) *K embeds in $\Sigma_\tau(2^\eta)$, for some cardinal η.*

Proof. (i) ⇒ (ii) Use Proposition 2.1 and the fact that every cozero-set in a compact zero-dimensional space K is a countable union of clopen subsets of K.

(ii) ⇒ (iv) Suppose that $\mathscr{U} = \{U_\alpha\colon \alpha \in \eta\}$ is a T_0-separating point-τ family of clopen subsets of K. Let $g_\alpha = \chi_{U_\alpha}$, for $\alpha \in \eta$. Then the diagonal map $G = \triangle_{\alpha\in\eta} g_\alpha\colon K \to 2^\eta$ is an embedding with $G(K) \subset \Sigma_\tau(2^\eta)$.

(ii) ⇒ (iii) For a family of clopen subsets of a compact space K terms "generating $\mathrm{CO}(K)$" and "T_0-separating" are equivalent.

(iv) ⇒ (i) Trivial. □

All τ-Corson compact spaces have tightness $\leqslant \tau$ by the following result of Kombarov and Malyhin:

Result 2.3 [13]. *For every infinite cardinals τ and η, the space $\Sigma_\tau(\mathbb{R}^\eta)$ has tightness $\leqslant \tau$.*

Let K be a compact space. We call K a τ-*Valdivia compact* space if, for some cardinal η, there exists an embedding $i\colon K \to \mathbb{R}^\eta$ such that the intersection $i(K) \cap \Sigma_\tau(\mathbb{R}^\eta)$ is dense in $i(K)$. For $\tau = \omega$, we obtain the standard class of Valdivia compact spaces.

Proposition 2.4. *Let K be a τ-Valdivia compact space. Then K is a continuous image of a closed subset L of some product 2^S, such that $L \cap \Sigma_\tau(2^S)$ is dense in L.*

Proof. Let i be an embedding of K into $\mathbb{R}^\eta$ such that $i(K)\cap\Sigma_\tau(\mathbb{R}^\eta)$ is dense in $i(K)$. Since K is compact, we may assume that $i(K) \subset I^\eta$ where $I = [0,1]$. Let f be a continuous map of 2^ω onto I such that $f^{-1}(0) = \{\overline{0}\}$ where $\overline{0}$ is the zero function in 2^ω. If F is the product map $\Pi f\colon (2^\omega)^\eta = 2^\eta \twoheadrightarrow I^\eta$, then $F^{-1}\big(\Sigma_\tau(I^\eta)\big) = \Sigma_\tau(2^\eta)$. Thus, if we put $L = \overline{F^{-1}(i(K) \cap \Sigma_\tau(I^\eta))}$, then L has the required properties. □

3. Boolean preimages

If $\varphi\colon K \twoheadrightarrow X$ is a continuous surjection and Y is a space, then we say that φ *factors thru* Y if there exist continuous surjections $\alpha\colon K \twoheadrightarrow Y$ and $\beta\colon Y \twoheadrightarrow X$ such that $\varphi = \beta \circ \alpha$.

Theorem 3.1. *Let K be a Boolean space, $\varphi\colon K \twoheadrightarrow X$ a continuous surjection and $\mathscr{C}$ a T_0-separating family of open subsets of X. Assume that to every $C \in \mathscr{C}$ there is assigned $g(C) \subset \mathrm{CO}(K)$ such that $\varphi^{-1}(C) = \bigcup g(C)$. Let $\mathscr{G} = \bigcup\limits_{C\in\mathscr{C}} g(C)$, $\mathscr{B} = \langle\!\langle \mathscr{G} \rangle\!\rangle$ and $Y = \mathrm{st}(\mathscr{B})$. Then, φ factors thru Y.*

Proof. Let α be the Stone map from K onto Y and define $\beta\colon Y \to X$ by $\beta(p) =$ the unique point in $\bigcap\limits_{B\in p} \varphi(B)$.

We first show that for every $x \in X$ the following holds:

$$\bigcap\{\varphi(B)\colon\ \varphi^{-1}(x) \subset B \in \mathscr{B}\} = \{x\}. \tag{3.1}$$

Let $x \in X$ and $y \in X \setminus \{x\}$. Choose $C \in \mathscr{C}$ such that C contains exactly one of x, y. If $x \in C \not\ni y$, then $\varphi^{-1}(x) \subset \varphi^{-1}(C) = \bigcup g(C)$ and we can get $B \in \mathscr{B}$ such that $\varphi^{-1}(x) \subset B \subset \varphi^{-1}(C)$. Then $\varphi(B) \subset C$ and hence, $y \notin \varphi(B)$. If $y \in C \not\ni x$, then $\varphi^{-1}(y) \subset \varphi^{-1}(C) = \bigcup g(C)$ and we can get $B \in \mathscr{B}$ such that $\varphi^{-1}(y) \subset B \subset \varphi^{-1}(C)$. Hence, $y \notin \varphi(K \setminus B)$ and $\varphi^{-1}(x) \subset K \setminus B \in \mathscr{B}$.

We now show that if $p \in Y$, then $\Big|\bigcap\limits_{B\in p} \varphi(B)\Big| = 1$ and hence β is well-defined. Fix $p \in Y$. Compactness of X implies that we can choose $x \in \bigcap\limits_{B\in p} \varphi(B)$. Let $B \in \mathscr{B}$ satisfy that $\varphi^{-1}(x) \subset B$. If $C \in p$, then $x \in \varphi(C)$, so $\varphi^{-1}(x) \cap C \neq \emptyset$. Hence $B \cap C \neq \emptyset$. As p is an ultrafilter, we get that $B \in p$. We have proven that $\{B\colon\ \varphi^{-1}(x) \subset B \in \mathscr{B}\} \subset p$. Hence $\bigcap\limits_{B\in p} \varphi(B) \subset \bigcap\{\varphi(B)\colon\ \varphi^{-1}(x) \subset B \in \mathscr{B}\}$ which by Equation 3.1 above is $\{x\}$.

We now show that β is continuous. Assume that $\beta(p) = x \in U$, where U is open in X. Since $\bigcup\limits_{B\in p} \varphi(B) \subset U$, we can get a finite $F \subset p$ such that $\bigcap\limits_{B\in F} \varphi(B) \subset U$. Let $C = \bigcap\{B\colon\ B \in F\}$. Then $C \in p$ and $\varphi(C) \subset U$. Thus C^+ is a neighbourhood of p and $\beta(C^+) \subset \varphi(C) \subset U$.

Finally, we show that $\varphi = \beta \circ \alpha$. Let $z \in K$. Then, $\alpha(z) = \{B\colon\ z \in B \in \mathscr{B}\}$ and $\beta(\alpha(z)) =$ the unique point in $\bigcap\limits_{z\in B\in\mathscr{B}} \varphi(B)$. As $\bigcap\limits_{z\in B\in\mathscr{B}} \varphi(B) \subset \bigcap\{\varphi(B)\colon\ \varphi^{-1}(\varphi(z)) \subset B \in \mathscr{B}\}$, Equation 3.1 implies that this unique point is $\varphi(z)$. In particular this shows that β is onto. $\square$

Our first consequence is an alternate proof of the Mardesic Factorization Theorem [14] for the special case when the dimension of K is zero.

Proposition 3.2. *Let φ be a continuous map from a Boolean space K onto a compact space X of weight κ. Then, φ factors thru a zero-dimensional compact space Y of weight κ.*

P r o o f. Let $\mathscr{C}$ be a T_0-separating family of cozero-sets of X of cardinality κ, the weight of X. For every $C \in \mathscr{C}$, choose a countable family $g(C) \subset \mathrm{CO}(K)$ such that $\varphi^{-1}(C) = \bigcup g(C)$. Put $\mathscr{G} = \bigcup\limits_{C \in \mathscr{C}} g(C)$ and $\mathscr{B} = \langle\!\langle \mathscr{G} \rangle\!\rangle$. $Y = \mathrm{st}(\mathscr{B})$ is a zero-dimensional compact space of weight $|\mathscr{B}| = \kappa$ and by Theorem 3.1, φ factors thru Y. $\square$

Proposition 3.3. *Let φ be a continuous map from a Boolean space K onto a τ-Corson compact space X. Then, φ factors thru a zero-dimensional τ-Corson compact space Y.*

P r o o f. We will use the characterization of τ-Corson compact spaces from Proposition 2.1. Referring to the proof of Proposition 3.2 we now further assume that $\mathscr{C}$ is a point-τ family. Then the $\mathscr{G}$ defined there is now a point-τ generating family of $\langle\!\langle \mathscr{G} \rangle\!\rangle$ and hence by Proposition 2.2, $Y = \mathrm{st}(\langle\!\langle \mathscr{G} \rangle\!\rangle)$ is a τ-Corson compact space. $\square$

Proposition 3.4. *A τ-Corson compact space X is a continuous image of a zero-dimensional τ-Corson compact space Y.*

P r o o f. Start with any Boolean space K that admits a continuous map φ onto X and then apply Proposition 3.3 to factor φ thru a zero-dimensional τ-Corson compact space Y. $\square$

4. The class $\mathscr{L}_\tau$

Given infinite cardinal numbers η and τ by $L_\tau(\eta)$ we will denote the set $\eta \cup \{\infty\}$ topologized as follows: all points $\alpha \in \eta$ are isolated in $L_\tau(\eta)$ and neighborhoods of ∞ have the form $\{\infty\} \cup (\eta \setminus A)$, where A is a subset of η of cardinality $\leqslant \tau$. By $\mathscr{L}_\tau$ we denote the class of all spaces which are continuous images of closed subsets of the countable product $L_\tau(\eta)^\omega$, for some cardinal η.

Proposition 4.1. *For every infinite cardinal τ the class $\mathscr{L}_\tau$ is closed under taking continuous images, closed subspaces, countable products and countable unions.*

P r o o f. Only the last property requires some explanation. Let $X = \bigcup\limits_{n \in \omega} X_n$, where each X_n is a continuous image of a closed subset $A_n \subset L_\tau(\eta_n)^\omega$, for some cardinal η_n. Let $\eta = \sup\limits_{n \in \omega} \eta_n$. Observe that ω is a closed subset of $L_\tau(\eta)$, so $\omega \times L_\tau(\eta)^\omega$ can be identified with a closed subset of $L_\tau(\eta)^\omega$. Therefore the discrete union $\bigoplus\limits_{n \in \omega} A_n$

can be embedded as a closed subset of $L_\tau(\eta)^\omega$. Obviously $\bigcup_{n\in\omega} X_n$ is a continuous image of $\bigoplus_{n\in\omega} A_n$. $\square$

For a space X we denote the Lindelöf number of X by $l(X)$. Corollary 4.2 from Noble [16] implies that $l(L_\tau(\eta)^\omega) \leqslant \tau$, for every η. Hence we obtain

Proposition 4.2. *Let τ be an infinite cardinal. For every space $X \in \mathscr{L}_\tau$ we have $l(X) \leqslant \tau$.*

5. Spaces of continuous functions

For spaces X and Y, by $C_p(X,Y)$ we denote the space of all continuous maps from X into Y equipped with the pointwise convergence topology, i.e. we consider $C_p(X,Y)$ as a subspace of the product Y^X. We will deal mainly with the case when $Y=\mathbb{R}$, and we will use standard notation $C_p(X)$ for the function space $C_p(X,\mathbb{R})$. For the sake of completeness we include the proofs of the following two standard facts (see [2]).

Proposition 5.1. *Let $\varphi\colon X \twoheadrightarrow Y$ be a continuous surjection. Then $C_p(Y)$ embeds into $C_p(X)$. If additionally the map φ is quotient then $C_p(Y)$ is homeomorphic to a closed subset of $C_p(X)$.*

Proof. Let $\Phi\colon C_p(Y) \to C_p(X)$ be defined by $\Phi(f) = f\circ\varphi$, for $f \in C_p(Y)$. Standard verification shows that Φ is a topological embedding. If φ is quotient then one can easily check that $\Phi(C_p(Y))$ is a closed subspace of $C_p(X)$ consisting of all functions constant on fibers of φ. $\square$

Proposition 5.2. *Every compact space K embeds into $C_p(C_p(K))$.*

Proof. For every $k\in K$ define $\varphi_k\colon C_p(K)\to\mathbb{R}$ by the formula $\varphi_k(f) = f(k)$, for $f\in C_p(K)$. The map $k\mapsto\varphi_k$ is the required embedding. $\square$

Proposition 5.3. *Let X be a C-embedded subspace of a Boolean space Y. Then $C_p(X)$ is a continuous image of the product $C_p(X,2)^\omega\times\omega^\omega$.*

Proof. Let $C(2^\omega)$ be the Banach space of all continuous real valued functions on the Cantor set 2^ω endowed with the standard supremum norm. Being a separable completely metrizable space, $C(2^\omega)$ is a continuous image of ω^ω. The product $C_p(X,2)^\omega$ can be identified with the space $C_p(X,2^\omega)$. Therefore it is enough to show that $C_p(X)$ is a continuous image of $C_p(X,2^\omega)\times C(2^\omega)$. Let

$\Phi\colon C_p(X,2^\omega)\times C(2^\omega)\to C_p(X)$ be defined by $\Phi(f,g)=g\circ f$, for $f\in C_p(X,2^\omega)$ and $g\in C(2^\omega)$. It is obvious that Φ is well defined and one can verify that Φ is continuous. Given $h\in C_p(X)$ we can extend it to a continuous function $H\colon Y\to\mathbb{R}$. By Proposition 3.2 we can factor H thru 2^ω, i.e. we can find continuous maps $\alpha\colon Y\to 2^\omega$ and $\beta\colon 2^\omega\to\mathbb{R}$ such that $H=\beta\circ\alpha$. Then $\Phi(\alpha|X,\beta)=h$, so the map Φ is a surjection. □

Proposition 5.4. *Let X be a space such that* $\mathrm{CO}(X)$ *has a point-τ generating family $\mathscr{A}$. Then $C_p(X,2)\in\mathscr{L}_\tau$.*

P r o o f. Observe that every function $f\in C_p(X,2)$ has the form χ_A, for some $A\in\mathrm{CO}(X)$. Define maps $\Psi\colon C_p(X,2)\times C_p(X,2)\to C_p(X,2)$ and $\Phi\colon C_p(X,2)\to C_p(X,2)$ as follows: $\Psi(\chi_A,\chi_B)=\chi_{A\cup B}$ and $\Phi(\chi_A)=\chi_{X\setminus A}$, for $A,B\in\mathrm{CO}(X)$. It is clear that these maps are continuous. We consider the following subspace $S=\{\chi_A\colon A\in\mathscr{A}\}\cup\{\mathbf{0}\}\subset C_p(X,2)$, where $\mathbf{0}=\chi_\emptyset$ is the constant zero function on X. Let $F\subset X$ be a finite set and V_F be a basic neighborhood of $\mathbf{0}$ in S determined by F, i.e. $V_F=\{f\in S\colon f(x)=0$, for $x\in F\}=\{\mathbf{0}\}\cup\{\chi_A\colon A\in\mathscr{A}$ and $A\cap F=\emptyset\}$. Since $\mathscr{A}$ is point-τ it follows that $|S\setminus V_F|\leqslant\tau$. Therefore S is a continuous image of the space $L_\tau(|\mathscr{A}|)$, so it belongs to the class $\mathscr{L}_\tau$. Now, define $T_0=S$ and $T_{n+1}=\Psi(T_n\times T_n)\cup\Phi(T_n)$, for $n\in\omega$. By Proposition 4.1 every $T_n\in\mathscr{L}_\tau$, hence also $\bigcup_{n\in\omega}T_n\in\mathscr{L}_\tau$. It remains to observe that $C_p(X,2)=\bigcup_{n\in\omega}T_n$ since $\langle\!\langle\mathscr{A}\rangle\!\rangle=\mathrm{CO}(X)$. □

From Propositions 4.1, 5.3 and 5.4 it immediately follows

Colloraly 5.5. *Let X be a C-embedded subspace of a Boolean space Y such that* $\mathrm{CO}(X)$ *has a point-τ generating family. Then $C_p(X)\in\mathscr{L}_\tau$.*

6. Function spaces on τ-Corson compacta

In this section we prove the following characterization of τ-Corson compact spaces:

Theorem 6.1. *Let K be a compact space. K is a τ-Corson compact space $\Leftrightarrow$ $C_p(K)\in\mathscr{L}_\tau$.*

For $\tau=\omega$, the above characterization of Corson compacta has been proved by R. Pol in [18, §3, Thm. 1.1] (see also [9] and [1]). Our proof follows closely Pol's argument from [18] (cf., [2, Chapter IV. 3]).

For a subset A of $L_\tau(\eta)$ such that $\infty \in A$ we define a retraction $r_A\colon L_\tau(\eta) \twoheadrightarrow A$ as follows:

$$r_A(x) = \begin{cases} x & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

We put $R_A = \prod r_A\colon L_\tau(\eta)^\omega \twoheadrightarrow A^\omega$. Clearly, r_A and R_A are continuous. Given $n \in \omega$ we define the map $P_n\colon L_\tau(\eta)^\omega \to L_\tau(\eta)^\omega$ by $P_n(x_0, x_1, \ldots) = (x_0, x_1 \ldots x_n, \infty, \infty, \ldots)$.

Lemma 6.2. *Let F be a closed subset of $L_\tau(\eta)^\omega$ and A a subset of $L_\tau(\eta)$ such that $\infty \in A$ and $|A| \geqslant \tau$. Then there exists a set $B \subset L_\tau(\eta)$ such that $A \subset B$, $|B| = |A|$ and $R_B(F) \subset F$.*

Proof. For every $z \in L_\tau(\eta)^\omega \setminus F$ we fix a basic neighborhood $V^z = V_1^z \times \ldots \times V_k^z \times L_\tau(\eta) \times L_\tau(\eta) \times \ldots$ of z disjoint from F and such that $k = k(z)$ is minimal for all such neighborhoods. We put $C_z = \bigcup\{L_\tau(\eta) \setminus V_i^z\colon \infty \in V_i^z, i \leqslant k\}$. Observe that $|C_z| \leqslant \tau$. We choose the set B by induction. We start with $B_0 = A$, and put $B_{n+1} = \bigcup\{C_z\colon z \in P_n(B_n^\omega) \setminus F\} \cup B_n$, for $n \in \omega$. Finally, we define $B = \bigcup\limits_{n\in\omega} B_n$. One can easily compute that $|B| = |A|$. We will show that $R_B(F) \subset F$.

Let $x = (x_i) \in F$ and suppose that $y = (y_i) = R_B(x) \notin F$. Take $z = (z_i) = P_k(y)$, where $k = k(y)$. Then $z_i = y_i$, for $i \leqslant k$, hence $z \notin F$ and $k(z) = k$. Choose $n \geqslant k$ such that $z_i \in B_n$, for $i \leqslant k$. Observe that $C_z \subset B$. For each $i \leqslant k$ we have $z_i = y_i = r_B(x_i)$ and we can consider two possibilities: First, $z_i \neq \infty$, then $x_i = r_B(x_i)$ by the definition of r_B, so $x_i = z_i \in V_i^z$. Second, $z_i = \infty$, then $x_i \notin C_z$. Indeed, if $x_i \in C_z \subset B$ then $r_B(x_i) = x_i \neq \infty$. Therefore again $x_i \in V_i^z$. This shows that x belongs to V^z which is disjoint from F—a contradiction. □

Lemma 6.3. *Let F be a closed subset of $L_\tau(\eta)^\omega$, where $\eta > \tau$. Then there exists a family $\{A_\alpha\colon \alpha \in [\tau, \eta]\}$ of subsets of $L_\tau(\eta)$ satisfying:*

(i) $|A_\alpha| = |\alpha|$, *for every* α,

(ii) $A_\alpha \subset A_\beta$, *for* $\alpha < \beta$,

(iii) $A_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$, *for limit* $\alpha > \tau$,

(iv) $A_\eta = L_\tau(\eta)$,

(v) $R_{A_\alpha}(F) \subset F$, *for every* α.

Proof. If $\alpha = \beta + 1$ or $\alpha = \tau$ then we construct the set A_α applying Lemma 6.2 for the set $A_\beta \cup \alpha$ or $\tau \cup \{\infty\}$, respectively. If $\alpha > \tau$ is a limit ordinal, we simply put $A_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$. We only need to verify the condition (v) for limit $\alpha > \tau$. Observe that, for every $y \in L_\tau(\eta)$, there is $\gamma \in [\tau, \alpha)$ such that $r_{A_\alpha}(y) = r_{A_\beta}(y)$, for all $\beta > \gamma$. Hence, for every $x \in L_\tau(\eta)^\omega$, we obtain

$$R_{A_\alpha}(x) = \lim_{\beta<\alpha} R_{A_\beta}(x). \tag{6.1}$$

This together with (v) for nonlimit β implies that for each x in closed set F we have $R_{A_\alpha}(x) \in F$. □

Proposition 6.4. *For every closed $F \subset L_\tau(\eta)^\omega$ there is a continuous linear injection $T\colon C_p(F) \to \Sigma_\tau(\mathbb{R}^\eta)$.*

P r o o f. We will prove this proposition by transfinite induction on η.

First, assume that $\eta \leqslant \tau$. Then the space $L_\tau(\eta)$ is discrete and the subset $F \subset L_\tau(\eta)^\omega$ is a metrizable space of weight $\leqslant \eta$. Take a dense subset $D \subset F$ of cardinality $\leqslant \eta$. Let $T\colon C_p(F) \to \mathbb{R}^D$ be the restriction map, i.e. $T(f) = f\big|D$, for $f \in C_p(F)$. Density of D implies that T is an injection and clearly we can identify $\mathbb{R}^D$ with a subset of $\Sigma_\tau(\mathbb{R}^\eta)$, as $|D| \leqslant \eta$.

Now, assume that the proposition holds for every $\varrho < \eta$. Let F be a closed subset of $L_\tau(\eta)^\omega$ and let A_α be the family of subsets of $L_\tau(\eta)$ given by Lemma 6.3 for this F. Let R_α be the restriction of the retraction R_{A_α} to F and let $F_\alpha = R_\alpha(F)$. The map $f \mapsto f \circ R_\alpha$, where $f \in C_p(F_\alpha)$, is a linear homeomorphism of $C_p(F_\alpha)$ onto the subspace $C_\alpha = \{f \circ R_\alpha\colon\ f \in C_p(F_\alpha)\}$ of $C_p(F)$. Observe that from the condition (ii) of Lemma 6.3 it follows that $R_\alpha = R_\alpha \circ R_\beta$, for $\alpha < \beta$, hence $C_\alpha \subset C_\beta$. By the condition (v) we have $F_\alpha = F \cap A_\alpha^\omega$ and the condition (i) implies that we can use our inductive hypothesis to get a continuous linear injection $T_\alpha\colon\ C_\alpha \to \Sigma_\tau(\mathbb{R}^\alpha)$, for every $\alpha \in [\tau, \eta)$. Let S be the disjoint union of the family of sets $\{\tau\} \cup \{\alpha + 1\colon\ \alpha \in [\tau, \eta)\}$. Obviously we have $|S| = \eta$. We define a map $T\colon\ C_p(F) \to \Sigma_\tau(\mathbb{R}^\tau) \times \prod_{\alpha\in[\tau,\eta)} \Sigma_\tau(\mathbb{R}^{\alpha+1})$ by

$$T(f) = (T_\tau(f \circ R_\tau), (T_{\alpha+1}(f \circ R_{\alpha+1} - f \circ R_\alpha))_{\alpha\in[\tau,\eta)}), \quad \text{for } f \in C_p(F).$$

We can identify $\Sigma_\tau(\mathbb{R}^\tau) \times \prod_{\alpha\in[\tau,\eta)} \Sigma_\tau(\mathbb{R}^\alpha)$ with a subspace of $\mathbb{R}^S$. It is clear that T is continuous and linear; we need to prove that T is injective and $T(C_p(F)) \subset \Sigma_\tau(\mathbb{R}^S)$.

Take distinct $f, g \in C_p(F)$ and let $\alpha = \inf\{\beta \in [\tau, \eta]\colon\ f \circ R_\alpha \neq g \circ R_\alpha\}$ (such β exists since $R_\eta = \mathrm{id}_F$). From the equality (6.1) it follows that $\alpha = \tau$ or $\alpha = \beta + 1$, for some $\beta < \eta$. In the first case we have $T_\tau(f \circ R_\tau) \neq T_\tau(f \circ R_\tau)$ by injectivity of

T_τ, so $T(f) \neq T(g)$. If $\alpha = \beta + 1$ then we have

$$T_{\beta+1}(f \circ R_{\beta+1} - f \circ R_\beta) - T_{\beta+1}(g \circ R_{\beta+1} - g \circ R_\beta) \\ = T_{\beta+1}(f \circ R_{\beta+1}) - T_{\beta+1}(g \circ R_{\beta+1}) \neq 0$$

since $T_{\beta+1}$ is injective and linear. Again, this shows that $T(f) \neq T(g)$.

Now, fix $f \in C_p(F)$. We will show that the set $\{\alpha\colon f \circ R_{\alpha+1} \neq f \circ R_\alpha\}$ has cardinality $\leqslant \tau$, therefore $T(f) \in \Sigma_\tau(\mathbb{R}^S)$.

Suppose to the contrary that we can find a set $T \subset [\tau, \eta)$ of cardinality τ^+ such that, for every $\alpha \in T$, there exists $x^\alpha = (x_i^\alpha) \in F$ with $f \circ R_{\alpha+1}(x^\alpha) \neq f \circ R_\alpha(x^\alpha)$. Since τ^+ is a regular cardinal we can assume, without loss of generality, that $|f \circ R_{\alpha+1}(x^\alpha) - f \circ R_\alpha(x^\alpha)| > \varepsilon$, for some positive ε. Using Proposition 4.2 for F, we can find a point $x = (x_i) \in F$ such that, for every neighborhood U of x we have $|\{\alpha \in T\colon R_\alpha(x^\alpha) \in U\}| = \tau^+$. Let $V = V_1 \times \ldots \times V_k \times L_\tau(\eta) \times L_\tau(\eta) \times \ldots$ be a neighborhood of x in $L_\tau(\eta)^\omega$ such that $\operatorname{diam} f(V \cap F) < \varepsilon$ and $V_i = \{x_i\}$, if $x_i \neq \infty$. Let $C = \bigcup\{L_\tau(\eta) \setminus V_i\colon \infty \in V_i,\ i \leqslant k\}$. Since $|C| \leqslant \tau$ and we have τ^+ many points $R_\alpha(x^\alpha)$ in V with $\alpha \in T$, we can get $\alpha \in T$ such that $R_\alpha(x^\alpha) \in V$ and the set $A_{\alpha+1} \setminus A_\alpha$ is disjoint from C. Then we also have $R_{\alpha+1}(x^\alpha) \in V$. Indeed, we can show that $R_{\alpha+1}(x_\alpha)_i \in V_i$, for all $i \leqslant k$. If $\infty \notin V_i$ then $R_\alpha(x^\alpha)_i = r_{A_\alpha}(x_i^\alpha) = x_i$ hence $x_i^\alpha \in A_\alpha \subset A_{\alpha+1}$ and $r_{A_{\alpha+1}}(x_i^\alpha) = x_i$. If $\infty \in V_i$ then $(A_{\alpha+1} \setminus A_\alpha) \subset V_i$ by the definition of C, so if $x_i^\alpha \in (A_{\alpha+1} \setminus A_\alpha)$ then $R_{\alpha+1}(x^\alpha)_i = x_i^\alpha \in V_i$. For $x_i^\alpha \notin (A_{\alpha+1} \setminus A_\alpha)$ we have $R_{\alpha+1}(x^\alpha)_i = R_\alpha(x^\alpha)_i \in V_i$ (notice that $r_{A_{\alpha+1}}(y) \neq r_{A_\alpha}(y)$ iff $y \in (A_{\alpha+1} \setminus A_\alpha)$). Finally, observe that by the choice of V we have $|f(R_{\alpha+1}(x^\alpha)) - f(R_\alpha(x^\alpha))| < \varepsilon$—a contradiction. □

Proof of Theorem 6.1. Let K be a τ-Corson compact space. By Proposition 3.4, K is a continuous image of a zero-dimensional τ-Corson compact space L. Proposition 5.1 implies that $C_p(K)$ embeds as a closed subset of $C_p(L)$. Therefore it is enough to show that $C_p(L) \in \mathscr{L}_\tau$. By Proposition 2.2, $\mathrm{CO}(L)$ has a point-τ generating family. Corollary 5.5 (with $X = Y = L$) implies that $C_p(L) \in \mathscr{L}_\tau$.

Suppose that $C_p(K) \in \mathscr{L}_\tau$, for a compact space K. Then $C_p(K)$ is a continuous image of a closed subset F of the product $L_\tau(\eta)^\omega$, for some cardinal η. By Proposition 5.1, $C_p(C_p(K))$ embeds into $C_p(F)$ and Proposition 6.4 gives us a continuous injection of $C_p(F)$ into $\Sigma_\tau(\mathbb{R}^\eta)$. By Proposition 5.2 we also have an embedding of K into $C_p(C_p(K))$. Finally, composing these three injections together we can embed the compact space K into $\Sigma_\tau(\mathbb{R}^\eta)$. □

Theorem 6.5. *X is a τ-Corson compact space $\Leftrightarrow$ X is a compact quotient of a C-embedded subspace of $\Sigma_\tau(2^\eta)$, for some cardinal η.*

P r o o f. Let X be a τ-Corson compact space. By Proposition 3.4 get a zero-dimensional τ-Corson compact space Y such that Y continuously maps onto X. So, X is a compact quotient of Y. By Proposition 2.2, Y can be embedded in $\Sigma_\tau(2^\eta)$, for some cardinal η; as Y is compact, Y is C-embedded in $\Sigma_\tau(2^\eta)$.

Let Y be a C-embedded subspace of $\Sigma_\tau(2^\eta)$, for some cardinal η and let φ be a quotient map of Y onto the compact space X. From Proposition 5.1 we get that $C_p(X)$ is homeomorphic to a closed subset of $C_p(Y)$. The restriction map $f \mapsto f\big|Y$ is a continuous map from $C_p\big(\Sigma_\tau(2^\eta)\big)$ onto $C_p(Y)$. Lemma 1.1 implies that $\mathrm{CO}\Big(\Sigma_\tau(2^\eta)\Big)$ has a point-τ generating family and that $\Sigma_\tau(2^\eta)$ is C-embedded in 2^η. Corollary 5.5 now implies that $C_p\big(\Sigma_\tau(2^\eta)\big) \in \mathscr{L}_\tau$ and hence by the preservation properties of $\mathscr{L}_\tau$ in Proposition 4.1 we get that $C_p(X) \in \mathscr{L}_\tau$. Since X is compact, X is a τ-Corson compact space by Theorem 6.1. □

7. Centered and polyadic applications

Theorem 7.1. *A space X which is a continuous image of a τ-Valdivia compact space and has tightness τ is a τ-Corson compact space.*

P r o o f. By Proposition 2.4 we may assume that, for some set S, there exists a closed subset $L \subset 2^S$ with $D = L \cap \Sigma_\tau(2^S)$ dense in L and a continuous surjection $\varphi\colon L \twoheadrightarrow X$. Put $f = \varphi\big|D$. Let us prove that f is a closed map from D onto X. Let F be closed in D and let $x \in \overline{f(F)}$. Since $t(X) = \tau$, choose $A \subset f(F)$ with $|A| \leqslant \tau$ such that $x \in \overline{A}$. Get $A' \subset F$ with $|A'| \leqslant \tau$ such that $f(A') = A$. Then $\overline{A'}$ is a compact subset of D ((iii) of Lemma 1.1) and since F is closed in D we have that $\overline{A'} \subset F$. Since $f(\overline{A'})$ is compact, $x \in \overline{A} = \overline{f(A')} \subset f(\overline{A'}) \subset f(F)$. So $\overline{f(F)} = f(F)$, $f(F)$ is closed and f is a closed map. Since D is dense in L, $f(D)$ is dense in X; as f is closed, we get that f is onto X.

Since f is a quotient map of D onto X and D is C-embedded in $\Sigma_\tau(2^S)$ (because D is C-embedded in 2^S by (i) of Lemma 1.1), Theorem 6.5 implies that X is a τ-Corson compact space. □

Let us point out that the most interesting case of Theorem 7.1, for $\tau = \omega$, follows from the result of Kalenda [10, Thm. 1].

For a collection S of sets put $\mathrm{Cen}(S) = \{\chi_T\colon T \text{ is a centered subcollection of } S\}$ and give $\mathrm{Cen}(S)$ the subspace topology from 2^S. Since $\mathrm{Cen}(S)$ is closed in 2^S (note that the empty set $\emptyset$ is a centered subcollection of S) we have that $\mathrm{Cen}(S)$ is a Boolean space. For the reader's benefit we remark that the spaces $\mathrm{Cen}(S)$'s are just a convenient description of the Adequate Compact spaces of Talagrand [19]. A space is *centered* if it is a continuous image of a $\mathrm{Cen}(S)$, for some collection S. We refer the reader to [3] and [17] for more information about centered spaces. In particular, for the simple fact that polyadic spaces are centered. As $\mathrm{Cen}(S) \cap \Sigma_\omega(2^S)$ is dense in $\mathrm{Cen}(S)$, we see that the spaces $\mathrm{Cen}(S)$ are Valdivia compact spaces.

Corollary 7.2. *A centered space of tightness τ is a τ-Corson compact space. In particular, a centered space of countable tightness is a Corson compact space.*

Corollary 7.3. *A polyadic space X of tightness τ and cellularity μ is a continuous image of a closed $F \subset A_\mu^\tau$.*

P r o o f. Since the cellularity of X is μ, by Gerlits [8, Theorem B of Section 0] we can choose a λ and a continuous $\varphi\colon A_\mu^\lambda \twoheadrightarrow X$. A polyadic space is centered, so Corollary 7.2 implies that X is a τ-Corson compact space. Apply Proposition 3.3 and factor φ to get a zero-dimensional τ-Corson compact space Y, a continuous $\alpha\colon A_\mu^\lambda \twoheadrightarrow Y$ and a continuous $\beta\colon Y \twoheadrightarrow X$ such that $\varphi = \beta \circ \alpha$. Then, Y is a polyadic space of tightness τ and cellularity μ. Invoking Result 0.1, choose a closed $F \subset A_\mu^\tau$ and a continuous $\gamma\colon F \twoheadrightarrow Y$. Then, $\beta \circ \gamma$ maps F continuously onto X. $\square$

Collary 7.4. *A polyadic space of countable tightness is a Uniform Eberlein compact space.*

P r o o f. This follows from Corollary 7.3 with $\tau = \omega$ by using Result 0.2. $\square$

8. Remarks

It should be noted that there is another possible way of generalizing Corson and Valdivia compacta.

Given a set X and a function $f\colon X \to \mathbb{R}$, we denote the set $\{x \in X\colon f(x) \neq 0\}$ by $\mathrm{supp}(f)$. For a set A and a cardinal τ, we denote by $\Sigma_{<\tau}(\mathbb{R}^A)$ the space $\{x \in \mathbb{R}^A\colon |\mathrm{supp}(x)| < \tau\}$. For a generalization of Corson compacta one can consider the classes of compact subsets of spaces $\Sigma_{<\tau}(\mathbb{R}^A)$, rather than $\Sigma_\tau(\mathbb{R}^A)$, see [11]. We decided to work with our (more narrow) notion of τ-Corson compact space because we were unable to generalize Pol's characterization (Theorem 6.1) for compact subsets of $\Sigma_{<\tau}(\mathbb{R}^A)$.

Let η and τ be infinite cardinals. Put $M_\tau(\eta) = \{\infty\} \cup \eta$, where points $\alpha \in \eta$ are isolated and neighborhoods of ∞ have complements of the cardinality $< \tau$. We do not know the answer to the following:

Question 8.1. *Let K be a compact space and τ an infinite cardinal. Is K homeomorphic to a subspace of $\Sigma_{<\tau}(\mathbb{R}^A)$, for some set A, if and only if $C_p(K)$ is a continuous image of a closed subset of the product $M_\tau(\eta)^\omega$ for some cardinal η?*

Let us indicate that Pol's argument from the proof of Theorem 6.1 cannot be extended to answer the above question. The following proposition shows that the counterpart of Proposition 6.4 does not hold true in this case.

Proposition 8.2. *There is no continuous linear injection of $C_p(M_{\omega_\omega}(\omega_\omega))$ into $\Sigma_{<\omega_\omega}(\mathbb{R}^S)$.*

Proof. Assume towards a contradiction that there exists a continuous linear injection $T\colon C_p(M_{\omega_\omega}(\omega_\omega)) \to \Sigma_{<\omega_\omega}(\mathbb{R}^S)$, for some set S.

Take a sequence (A_n) of pairwise disjoint subsets of ω_ω such that $|A_n| = \omega_n$. For every n, let $K_n = \{f\colon M_{\omega_\omega}(\omega_\omega) \to [0,1/n]\colon \operatorname{supp}(f) \subset A_n\}$. K_n is a topological copy of the cube $[0,1]^{\omega_n}$ in $C_p(M_{\omega_\omega}(\omega_\omega))$. The space K_n cannot be embedded in $\Sigma_{<\omega_n}(\mathbb{R}^S)$, since $t(K_n) = \omega_n$. Therefore, we can find, for every n, a function $f_n \in K_n$ such that $|\operatorname{supp}(T(f_n))| \geqslant \omega_n$. By our assumption on T we also have $|\operatorname{supp}(T(f_n))| < \omega_\omega$. Take a subsequence (f_{n_k}) such that the corresponding sequence $(|\operatorname{supp}(T(f_{n_k}))|)$ is increasing. Put $g_k = f_{n_k}$. Observe that for every sequence (t_k) such that $t_k \in [-1,1]$, the sequence $\left(\sum_{k=0}^{m} t_k g_k\right)_m$ converges uniformly (hence also pointwise) to a function $\sum_{k=0}^{\infty} t_k g_k$. Therefore $\sum_{k=0}^{\infty} t_k g_k$ belongs to $C_p(M_{\omega_\omega}(\omega_\omega))$. By continuity and linearity of T we have

$$T\left(\sum_{k=0}^{\infty} t_k g_k\right)(s) = \sum_{k=0}^{\infty} t_k[T(g_k)(s)] \quad \text{for all } s \in S. \tag{8.1}$$

Given $s \in S$, we can take (t_k) such that $t_k[T(g_k)(s)] = |T(g_k)(s)|$. Then the above formula shows that the series $\sum_{k=0}^{\infty} T(g_k)(s)$ is absolutely convergent.

For every k, we can find an $\varepsilon_k > 0$ and a set $S_k \subset S$ such that $|S_k| = |\operatorname{supp}(T(g_k))|$ and

$$|T(g_k)(s)| > \varepsilon_k \quad \text{for all } s \in S_k. \tag{8.2}$$

Refining sets S_k if necessary, we may additionally require that

$$S_k \cap \bigcup_{i=0}^{k-1} \operatorname{supp}(T(g_i)) = \emptyset. \tag{8.3}$$

Next, for every k, we choose a positive number M_k and a subset $T_k \subset S_k$ such that $|T_k| = |S_k|$ and

$$(8.4) \qquad \sum_{i=k+1}^{\infty} |T(g_i)(s)| < M_k \quad \text{for all } s \in T_k.$$

Finally, we choose by induction a decreasing sequence (t_k) of numbers from $(0,1]$ such that $t_{k+1} < t_k\varepsilon_k/(2M_k)$ for all k. Then, for every k and $s \in T_k$, formulas 8.2, 8.3 and 8.4 imply that

$$\left|\sum_{i=0}^{\infty} t_i[T(g_i)(s)]\right| = \left|\sum_{i=k}^{\infty} t_i[T(g_i)(s)]\right| \geqslant |t_k[T(g_k)(s)| - \left|\sum_{i=k+1}^{\infty} t_i[T(g_i)(s)]\right|$$
$$> t_k\varepsilon_k - t_{k+1}\sum_{i=k+1}^{\infty} |T(g_i)(s)| > t_k\varepsilon_k - \frac{t_k\varepsilon_k}{2M_k}M_k = \frac{t_k\varepsilon_k}{2} > 0.$$

Therefore, by the formula 8.1 we obtain the inclusion

$$\bigcup_{k=0}^{\infty} T_k \subset \operatorname{supp}\Big(T\Big(\sum_{k=0}^{\infty} t_k g_k\Big)\Big),$$

which shows that $\left|\operatorname{supp}\Big(T\Big(\sum\limits_{k=0}^{\infty} t_k g_k\Big)\Big)\right| = \omega_\omega$, a contradiction. □

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