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A GENERALIZATION OF LERCH’S FORMULA

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Abstract. We give higher-power generalizations of the classical Lerch formula for the gamma function.

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1. Results

About a century ago, Matyas Lerch [6, p. 13] proved a famous formula saying

$$\prod_{n=0}^{\infty} (n + x) = \frac{\sqrt{2\pi}}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function. Here, we use the notation due to Deninger [2] of the regularized product

$$\prod_{n=0}^{\infty} a_n = \exp \left( -\frac{d}{ds} \sum_{n=0}^{\infty} a_n^{-s} \bigg|_{s=0} \right),$$

where $\sum_{n=0}^{\infty} a_n^{-s}$ is analytically continued to a holomorphic function at $s = 0$ (see Manin [7] for a survey). In the case above,

$$\sum_{n=0}^{\infty} (n + x)^{-s}$$

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is nothing but the Hurwitz zeta function $\zeta(s, x)$, and the Lerch formula was originally written as

$$\zeta'(s, x)|_{s=0} = \log \frac{\Gamma(x)}{\sqrt{2\pi}},$$

where the differentiation is with respect to the first variable $s$.

On the other hand, it is not so well-known that Lerch [7, p. 60] also proved that

(2) $$\prod_{n=0}^{\infty} \left((n + x)^2 + y^2\right) = \frac{2\pi}{\Gamma(x + iy)\Gamma(x - iy)}.$$  

This is remarkable from the following two points of view. Firstly, let $x > 0$ and $y = 0$ in (2), then we get (1) again, since

$$\prod_{n=0}^{\infty} (n + x)^2 = \left(\prod_{n=0}^{\infty} (n + x)\right)^2.$$  

Secondly, it gives an example of the non-trivial equality

(3) $$\prod_{n=0}^{\infty} (a_n b_n) = \left(\prod_{n=0}^{\infty} a_n\right) \left(\prod_{n=0}^{\infty} b_n\right)$$

by setting $a_n = n + x + iy$ and $b_n = n + x - iy$. It should be noted that there exist also counter examples to (3). Moreover, it is worth noting that in the study of the higher Selberg zeta function $z_{\Gamma}(s)$ which are studied in [4], the formula (2) becomes essential to obtain the analytic properties, especially the determinant expression of $z_{\Gamma}(s)$.

In this paper we show that the formulas (1) and (2) can be generalized to the higher power case in the following form:

**Theorem.** The following formulas hold:

(i) $$\prod_{n=0}^{\infty} \left((n + x)^m - y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1}^{\infty} \Gamma(x - \zeta y)} \quad (m = 1, 2, \ldots).$$

(ii) $$\prod_{n=0}^{\infty} \left((n + x)^m + y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1}^{\infty} \Gamma(x + \zeta y)} \quad \text{(for odd } m > 0).$$

(iii) $$\prod_{n=0}^{\infty} \left((n + x)^m + y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=-1}^{\infty} \Gamma(x + \zeta y)} \quad \text{(for even } m > 0).$$
Corollary. We have

(i) \[ \prod_{n=0}^{\infty} (n^2 + a^2) = a(e^{\pi a} - e^{-\pi a}) = 2a \sinh(\pi a). \]

(ii) \[ \prod_{n=0}^{\infty} (n^4 + a^4) = a^2 \left( e^{\sqrt{2} \pi a} + e^{-\sqrt{2} \pi a} - 2 \cos(\sqrt{2} \pi a) \right) = 2a^2 \left( \cosh(\sqrt{2} \pi a) - \cos(\sqrt{2} \pi a) \right). \]

Using the theorem and the corollary above one can get a number of interesting explicit identities as follows.

Examples.

(1) \[ \prod_{n=0}^{\infty} (n^2 + 1) = e^{\pi} - e^{-\pi}. \]

(2) \[ \prod_{n=0}^{\infty} (n^2 - n + 1) = e^{\sqrt{3} \pi} + e^{-\sqrt{3} \pi}. \]

(3) \[ \prod_{n=0}^{\infty} (n^3 + 1) = \sqrt{2\pi} \left( e^{\sqrt{3} \pi} + e^{-\sqrt{3} \pi} \right). \]

(4) \[ \prod_{n=0}^{\infty} (n^4 + 1) = 2 \left( \cosh(\sqrt{2} \pi) - \cos(\sqrt{2} \pi) \right). \]

2. Proof of the theorem

In this section, we give a proof of the theorem.

Proof of Theorem. We prove (i). The formulas (ii) and (iii) follow from (i) easily, so we omit them. We assume \( m \geq 2 \) from now on, since the case \( m = 1 \) is nothing but the original result of Lerch. Let

\[ \varphi_m(s, x, y) = \sum_{n=0}^{\infty} ((n + x)^m - y^m)^{-s}, \]

then by Bochner [1], \( \varphi_m(s, x, y) \) is analytically continued to all \( s \in \mathbb{C} \) as a meromorphic function in \( s \), and it is holomorphic around \( s = 0 \) and \( s = 1 \). Hence we have expressions

\[ \varphi_m(s, x, y) = a_0(x, y) + a_1(x, y)s + \ldots \quad (\text{around } s = 0) \]
\[ \varphi_m(s, x, y) = b_0(x, y) + b_1(x, y)(s - 1) + \ldots \quad \text{(around } s = 1) \]

Looking now at the relation
\[ \frac{\partial}{\partial y} \varphi_m(s, x, y) = msy^{m-1} \varphi_m(s + 1, x, y), \]
we see that
\[ \frac{\partial}{\partial y} a_1(x, y) = my^{m-1}b_0(x, y) = \sum_{n=0}^{\infty} \frac{my^{m-1}}{(n + x)^m - y^m}. \]

It is easy to verify the following decomposition:
\[ \frac{my^{m-1}}{(n + x)^m - y^m} = \sum_{\zeta^m=1} \frac{\zeta}{(n + x) - \zeta y}. \]

Hence we have
\[ \frac{\partial}{\partial y} a_1(x, y) = \sum_{\zeta^m=1} \zeta \left[ \frac{1}{x - \zeta y} + \sum_{n=1}^{\infty} \left( \frac{1}{n + (x - \zeta y)} - \frac{1}{n} \right) \right], \]

since
\[ \sum_{\zeta^m=1} \zeta = 0. \]

Moreover, let
\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right) = 0.5772156649\ldots \]
be the Euler constant, then we of course have
\[ \frac{\partial}{\partial y} a_1(x, y) = \sum_{\zeta^m=1} \zeta \left[ \frac{1}{x - \zeta y} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n + (x - \zeta y)} - \frac{1}{n} \right) \right], \]
again by \( \sum_{\zeta^m=1} \zeta = 0. \)

On the other hand, by the infinite product expression for the gamma function
\[ \frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left( 1 + \frac{x}{n} \right) e^{-\frac{x}{n}}, \]
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we see that
\[ -\frac{\Gamma'(x)}{\Gamma(x)} = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n+x} - \frac{1}{n} \right). \]

Thus we obtain
\[
\frac{\partial}{\partial y} a_1(x, y) = \sum_{\zeta^m=1} (-\zeta) \cdot \frac{\Gamma'(x - \zeta y)}{\Gamma(x - \zeta y)} \cdot \frac{\partial}{\partial y} \log \left( \prod_{\zeta^m=1} \Gamma(x - \zeta y) \right) .
\]

Hence
\[
a_1(x, y) = \log \left( C \prod_{\zeta^m=1} \Gamma(x - \zeta y) \right)
\]
for a constant $C$. Since
\[
\prod_{n=0}^{\infty} ((n + x)^m - y^m) = \exp \left( -\frac{\partial}{\partial s} \varphi_m(s, x, y) \bigg|_{s=0} \right)
\]
\[
= \exp (-a_1(x, y)) ,
\]
we get
\[
\prod_{n=0}^{\infty} ((n + x)^m - y^m) = \frac{1}{C \prod_{\zeta^m=1} \Gamma(x - \zeta y)} .
\]
Letting $y \to 0$, we have
\[
C^{-1} = \left( \prod_{n=0}^{\infty} (n + x)^m \right) \prod_{\zeta^m=1} \Gamma(x) \prod_{\zeta^m=1} \Gamma(x)
\]
\[
= \left( \prod_{n=0}^{\infty} (n + x)^m \right) \prod_{\zeta^m=1} \Gamma(x)^m
\]
\[
= (2\pi)^{m/2} .
\]

Here in the last equality we used the Lerch formula (1) in the Introduction. This completes the proof of the statement (i) of the theorem. \qed
3. Calculation for the Corollary and Examples

Proof. First we prove the corollary:

(i) Put $m = 2, x = 0$ and $y = a$ in (iii) of the theorem. Then we have immediately

$$\prod_{n=0}^{\infty} (n^2 + a^2) = \frac{2\pi}{\Gamma(ia)\Gamma(-ia)} = a(e^{\pi a} - e^{-\pi a}),$$

where we used the relation

$$\frac{1}{\Gamma(x)\Gamma(-x)} = -\frac{x \sin(\pi x)}{\pi} = -\frac{x(e^{\pi ix} - e^{-\pi ix})}{2\pi i}.$$

(ii) Put $m = 4, x = 0$ and $y = \frac{1+i}{\sqrt{2}}a$ in (iii) of the theorem. Then we obtain

$$\prod_{n=0}^{\infty} (n^4 + a^4) = \frac{(2\pi)^2}{\prod_{\zeta=1}^{4} \Gamma(-\zeta\frac{1+i}{\sqrt{2}} a)}.$$

The remaining calculation is quite similar to (i) above. This completes the proof of the corollary.

We now make calculation for examples: It would be sufficient to remark on (3). Put $m = 3, x = 0$ and $y = -1$ in (i) of the theorem. Then

$$\prod_{n=0}^{\infty} (n^3 + 1) = \frac{(2\pi)^{3/2}}{\Gamma(1)\Gamma(\omega)\Gamma(\overline{\omega})}$$

where $\omega = \frac{-1+\sqrt{3}i}{2}$. We use

$$\frac{1}{\Gamma(\omega)\Gamma(\overline{\omega})} = \frac{\omega \cdot \overline{\omega}}{\Gamma(\omega + 1)\Gamma(\overline{\omega} + 1)} = \frac{1}{\Gamma(\omega + 1)\Gamma(-\omega)} = \frac{\sin(-\omega\pi)}{\pi}$$

to get the example (3).

We now make small remarks for further study.

Remarks.

(a) From examples (2), (3) and Lerch’s result

$$\prod_{n=0}^{\infty} (n + 1) = \sqrt{2\pi},$$
we see that when \( a_n, b_n \) are given by \( a_n = n + 1, b_n = n^2 - n + 1 \), respectively, the following equality is true.

\[
\prod_{n=0}^{\infty} (a_n b_n) = \left( \prod_{n=0}^{\infty} a_n \right) \cdot \left( \prod_{n=0}^{\infty} b_n \right).
\]

(b) We can obtain a “multiple version” of our theorem using multiple gamma functions (see [3]) instead of the usual gamma function.

(c) To obtain a \( q \)-analogue of the Lerch formula appropriately we need to extend the notion of the (zeta) regularized products. See [5].

References


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