Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 54 (2004), No. 4, 941–947

Persistent URL: `http://dml.cz/dmlcz/127942`

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A GENERALIZATION OF LERCH'S FORMULA

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(Received January 4, 2002)

Abstract. We give higher-power generalizations of the classical Lerch formula for the gamma function.

Keywords: Lerch's formula, Hurwitz zeta function, zeta regularized product

MSC 2000: 11M36

1. RESULTS

About a century ago, Matyas Lerch [6, p. 13] proved a famous formula saying

$$\prod_{n=0}^{\infty}(n+x)=\frac{\sqrt{2\pi}}{\Gamma(x)}, \tag{1}$$

where $\Gamma(x)$ is the gamma function. Here, we use the notation due to Deninger [2] of the regularized product

$$\prod_{n=0}^{\infty} a_n=\exp\Bigl(-\frac{\mathrm{d}}{\mathrm{d}s}\sum_{n=0}^{\infty}a_n^{-s}\Big|_{s=0}\Bigr)$$

where $\sum\limits_{n=0}^{\infty} a_n^{-s}$ is analytically continued to a holomorphic function at $s=0$ (see Manin [7] for a survey). In the case above,

$$\sum_{n=0}^{\infty}(n+x)^{-s}$$

Work in part supported by Grant-in Aid for Scientific Research (B) No. 11440010, and by Grant-in Aid for Exploratory Research No. 13874004, Japan Society for the Promotion of Science.

is nothing but the Hurwitz zeta function $\zeta(s,x)$, and the Lerch formula was originally written as

$$\zeta'(s,x)|_{s=0} = \log\frac{\Gamma(x)}{\sqrt{2\pi}},$$

where the differentiation is with respect to the first variable s.

On the other hand, it is not so well-known that Lerch [7, p. 60] also proved that

$$\prod_{n=0}^{\infty}\left((n+x)^2+y^2\right) = \frac{2\pi}{\Gamma(x+\mathrm{i}y)\Gamma(x-\mathrm{i}y)}. \tag{2}$$

This is remarkable from the following two points of view. Firstly, let $x>0$ and $y=0$ in (2), then we get (1) again, since

$$\prod_{n=0}^{\infty}(n+x)^2 = \left(\prod_{n=0}^{\infty}(n+x)\right)^2.$$

Secondly, it gives an example of the non-trivial equality

$$\prod_{n=0}^{\infty}(a_n b_n) = \left(\prod_{n=0}^{\infty}a_n\right)\left(\prod_{n=0}^{\infty}b_n\right) \tag{3}$$

by setting $a_n = n+x+\mathrm{i}y$ and $b_n = n+x-\mathrm{i}y$. It should be noted that there exist also counter examples to (3). Moreover, it is worth noting that in the study of the higher Selberg zeta function $z_\Gamma(s)$ which are studied in [4], the formula (2) becomes essential to obtain the analytic properties, especially the determinant expression of $z_\Gamma(s)$.

In this paper we show that the formulas (1) and (2) can be generalized to the higher power case in the following form:

Theorem. *The following formulas hold:*

(i) $$\prod_{n=0}^{\infty}\left((n+x)^m-y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1}\Gamma(x-\zeta y)} \quad (m=1,2,\ldots).$$

(ii) $$\prod_{n=0}^{\infty}\left((n+x)^m+y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1}\Gamma(x+\zeta y)} \quad (\textit{for odd } m>0).$$

(iii) $$\prod_{n=0}^{\infty}\left((n+x)^m+y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=-1}\Gamma(x+\zeta y)} \quad (\textit{for even } m>0).$$

Corollary. *We have*

$$\text{(i)} \qquad \prod_{n=0}^{\infty} (n^2 + a^2) = a(\mathrm{e}^{\pi a} - \mathrm{e}^{-\pi a}) = 2a \sinh(\pi a).$$

$$\text{(ii)} \qquad \begin{aligned} \prod_{n=0}^{\infty} (n^4 + a^4) &= a^2 \left(\mathrm{e}^{\sqrt{2}\pi a} + \mathrm{e}^{-\sqrt{2}\pi a} - 2\cos\left(\sqrt{2}\pi a\right)\right) \\ &= 2a^2 \left(\cosh\left(\sqrt{2}\pi a\right) - \cos\left(\sqrt{2}\pi a\right)\right). \end{aligned}$$

Using the theorem and the corollary above one can get a number of interesting explicit identities as follows.

Examples.

$$\text{(1)} \qquad \prod_{n=0}^{\infty} (n^2 + 1) = \mathrm{e}^{\pi} - \mathrm{e}^{-\pi}.$$

$$\text{(2)} \qquad \prod_{n=0}^{\infty} (n^2 - n + 1) = \mathrm{e}^{\frac{\sqrt{3}}{2}\pi} + \mathrm{e}^{-\frac{\sqrt{3}}{2}\pi}.$$

$$\text{(3)} \qquad \prod_{n=0}^{\infty} (n^3 + 1) = \sqrt{2\pi} \left(\mathrm{e}^{\frac{\sqrt{3}}{2}\pi} + \mathrm{e}^{-\frac{\sqrt{3}}{2}\pi}\right).$$

$$\text{(4)} \qquad \prod_{n=0}^{\infty} (n^4 + 1) = 2 \left(\cosh\left(\sqrt{2}\pi\right) - \cos\left(\sqrt{2}\pi\right)\right).$$

2. Proof of the theorem

In this section, we give a proof of the theorem.

Proof of Theorem. We prove (i). The formulas (ii) and (iii) follow from (i) easily, so we omit them. We assume $m \geqslant 2$ from now on, since the case $m = 1$ is nothing but the original result of Lerch. Let

$$\varphi_m(s, x, y) = \sum_{n=0}^{\infty} \left((n + x)^m - y^m\right)^{-s},$$

then by Bochner [1], $\varphi_m(s, x, y)$ is analytically continued to all $s \in \mathbb{C}$ as a meromorphic function in s, and it is holomorphic around $s = 0$ and $s = 1$. Hence we have expressions

$$\varphi_m(s, x, y) = a_0(x, y) + a_1(x, y)s + \ldots \quad \text{(around } s = 0\text{)}$$

and

$$\varphi_m(s,x,y) = b_0(x,y) + b_1(x,y)(s-1) + \dots \quad \text{(around } s=1\text{)}.$$

Looking now at the relation

$$\frac{\partial}{\partial y}\varphi_m(s,x,y) = msy^{m-1}\varphi_m(s+1,x,y),$$

we see that

$$\begin{aligned}\frac{\partial}{\partial y}a_1(x,y) &= my^{m-1}b_0(x,y)\\ &= my^{m-1}\varphi_m(1,x,y)\\ &= \sum_{n=0}^{\infty}\frac{my^{m-1}}{(n+x)^m - y^m}.\end{aligned}$$

It is easy to verify the following decomposition:

$$\frac{my^{m-1}}{(n+x)^m - y^m} = \sum_{\zeta^m=1}\frac{\zeta}{(n+x)-\zeta y}.$$

Hence we have

$$\frac{\partial}{\partial y}a_1(x,y) = \sum_{\zeta^m=1}\zeta\left[\frac{1}{x-\zeta y} + \sum_{n=1}^{\infty}\left(\frac{1}{n+(x-\zeta y)} - \frac{1}{n}\right)\right],$$

since

$$\sum_{\zeta^m=1}\zeta = 0.$$

Moreover, let

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n} - \log n\right) = 0.5772156649\dots$$

be the Euler constant, then we of course have

$$\frac{\partial}{\partial y}a_1(x,y) = \sum_{\zeta^m=1}\zeta\left[\frac{1}{x-\zeta y} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+(x-\zeta y)} - \frac{1}{n}\right)\right],$$

again by $\sum_{\zeta^m=1}\zeta = 0$.

On the other hand, by the infinite product expression for the gamma function

$$\frac{1}{\Gamma(x)} = x\mathrm{e}^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)\mathrm{e}^{-\frac{x}{n}},$$

we see that

$$-\frac{\Gamma'}{\Gamma}(x) = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n}\right).$$

Thus we obtain

$$\begin{aligned}\frac{\partial}{\partial y}a_1(x,y) &= \sum_{\zeta^m=1}(-\zeta)\cdot\frac{\Gamma'}{\Gamma}(x-\zeta y)\\ &= \frac{\partial}{\partial y}\log\Bigg(\prod_{\zeta^m=1}\Gamma(x-\zeta y)\Bigg).\end{aligned}$$

Hence

$$a_1(x,y) = \log\Bigg(C\prod_{\zeta^m=1}\Gamma(x-\zeta y)\Bigg)$$

for a constant C. Since

$$\begin{aligned}\prod_{n=0}^{\infty}((n+x)^m - y^m) &= \exp\left(-\frac{\partial}{\partial s}\varphi_m(s,x,y)\Big|_{s=0}\right)\\ &= \exp\left(-a_1(x,y)\right),\end{aligned}$$

we get

$$\prod_{n=0}^{\infty}((n+x)^m - y^m) = \frac{1}{C\prod_{\zeta^m=1}\Gamma(x-\zeta y)}.$$

Letting $y \to 0$, we have

$$\begin{aligned}C^{-1} &= \left(\prod_{n=0}^{\infty}(n+x)^m\right)\prod_{\zeta^m=1}\Gamma(x)\\ &= \left(\prod_{n=0}^{\infty}(n+x)\right)^m\Gamma(x)^m\\ &= (2\pi)^{m/2}.\end{aligned}$$

Here in the last equality we used the Lerch formula (1) in the Introduction. This completes the proof of the statement (i) of the theorem. □

3. Calculation for the corollary and examples

Proof. First we prove the corollary:
(i) Put $m=2, x=0$ and $y=a$ in (iii) of the theorem. Then we have immediately

$$\prod_{n=0}^{\infty}(n^2+a^2)=\frac{2\pi}{\Gamma(\mathrm{i}a)\Gamma(-\mathrm{i}a)}=a(\mathrm{e}^{\pi a}-\mathrm{e}^{-\pi a}),$$

where we used the relation

$$\frac{1}{\Gamma(x)\Gamma(-x)}=-\frac{x\sin(\pi x)}{\pi}=-\frac{x(\mathrm{e}^{\pi\mathrm{i}x}-\mathrm{e}^{-\pi\mathrm{i}x})}{2\pi\mathrm{i}}.$$

(ii) Put $m=4$, $x=0$ and $y=\frac{1+\mathrm{i}}{\sqrt{2}}a$ in (iii) of the theorem. Then we obtain

$$\prod_{n=0}^{\infty}(n^4+a^4)=\frac{(2\pi)^2}{\prod\limits_{\zeta^4=1}\Gamma(-\zeta\frac{1+\mathrm{i}}{\sqrt{2}}a)}.$$

The remaining calculation is quite similar to (i) above. This completes the proof of the corollary. □

We now make calculation for examples: It would be sufficient to remark on (3). Put $m=3$, $x=0$ and $y=-1$ in (i) of the theorem. Then

$$\prod_{n=0}^{\infty}(n^3+1)=\frac{(2\pi)^{3/2}}{\Gamma(1)\Gamma(\omega)\Gamma(\overline{\omega})}$$

where $\omega=\frac{-1+\sqrt{3}\mathrm{i}}{2}$. We use

$$\begin{aligned}\frac{1}{\Gamma(\omega)\Gamma(\overline{\omega})}&=\frac{\omega\cdot\overline{\omega}}{\Gamma(\omega+1)\Gamma(\overline{\omega}+1)}=\frac{1}{\Gamma(\omega+1)\Gamma(\overline{\omega}+1)}\\&=\frac{1}{\Gamma(\omega+1)\Gamma(-\omega)}=\frac{\sin(-\omega\pi)}{\pi}\end{aligned}$$

to get the example (3).

We now make small remarks for further study.

Remarks.
(a) From examples (2), (3) and Lerch's result

$$\prod_{n=0}^{\infty}(n+1)=\sqrt{2\pi},$$

we see that when a_n, b_n are given by $a_n = n+1$, $b_n = n^2 - n + 1$, respectively, the following equality is true.

$$\prod_{n=0}^{\infty}(a_n b_n) = \left(\prod_{n=0}^{\infty} a_n\right) \cdot \left(\prod_{n=0}^{\infty} b_n\right).$$

(b) We can obtain a “multiple version” of our theorem using multiple gamma functions (see [3]) instead of the usual gamma function.

(c) To obtain a q-analogue of the Lerch formula appropriately we need to extend the notion of the (zeta) regularized products. See [5].

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