Francis Jordan; Frédéric Mynard

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# PRODUCTIVELY FRÉCHET SPACES

Francis Jordan and Frédéric Mynard, Statesboro

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*Abstract.* We solve the long standing problem of characterizing the class of strongly Fréchet spaces whose product with every strongly Fréchet space is also Fréchet.

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A topological space $X$ is *Fréchet* if for every subset $A$ of $X$, $x \in \operatorname{cl} A$ implies the existence of a sequence on $A$ that converges to $x$. A topological space $X$ is *strongly Fréchet*[1] if whenever $x \in \bigcap_{n\in\omega} \operatorname{cl} A_n$ for a decreasing sequence of subsets $A_n$ of $X$, there exists a sequence $(x_n)_{n\in\omega}$ such that $x_n \in A_n$ and $x \in \lim(x_n)_{n\in\omega}$. Because of the instability of these properties under product, the quest for conditions on Fréchet spaces $X$ and $Y$ to ensure the Fréchetness of the product $X \times Y$ has attracted lots of attention for years [13], [1], [2], [18], [19], [20], [23], [16], [8], [17], [22], [9], [10], [7], [6]. Even the square of a compact Fréchet space need not be Fréchet [22]. It has been noticed [13] that if $X$ and $Y$ are two Fréchet spaces both containing a convergent free sequence, and if $X \times Y$ is Fréchet, then both $X$ and $Y$ are strongly Fréchet. The countable fan $S_\omega$ is an example of a Fréchet space which is not strongly Fréchet.[2] Therefore, a space whose product with every Fréchet space is Fréchet cannot contain a free convergent sequence. Actually, such spaces are exactly the *finitely generated* ones, that is, the topological spaces in which each point admits

Discussions with Iwo Labuda (University of Mississippi) about [11] are at the origin of the idea of productively Fréchet spaces. We are grateful to him for valuable suggestions.

[1] Also called countably bisequential and characterized as Fréchet $\alpha_4$ spaces.

[2] $S_\omega$ is obtained from the disjoint sum of countably many convergent sequences by identifying the limit points. The resulting set is equipped with the corresponding quotient topology.

a smallest neighborhood [14]. This class is trivial in the sense that Hausdorff finitely generated spaces are discrete.

The natural question is consequently to characterize spaces whose product with every *strongly* Fréchet space is Fréchet. It is known that a product of two strongly Fréchet spaces need not be Fréchet (e.g., [22], [2]), even not sequential[3] [24], so we have to find a convenient proper subclass of strongly Fréchet spaces. The first result in this direction is due to E. Michael [13, Proposition 4.D.4] and states that the product of a bisequential space with a strongly Fréchet space is strongly Fréchet. Two families of subsets $\mathscr{A}$ and $\mathscr{B}$ *mesh*, in symbol $\mathscr{A} \# \mathscr{B}$, if $A \cap B \neq \emptyset$ for every $A \in \mathscr{A}$ and every $B \in \mathscr{B}$. A topological space is *bisequential* [13] if whenever $x$ is a cluster point for a filter $\mathscr{F}$, there exists a countably based filter $\mathscr{H}$ that meshes with $\mathscr{F}$ and converges to $x$. It is well known that a countable product of bisequential spaces is bisequential [13].

Let $\varphi_\omega$ denote the class of countably based filters. We introduce two types of adherence (or set of cluster points) of a filter:

$$\operatorname{adh} \mathscr{F} = \bigcup_{\mathscr{G} \# \mathscr{F}} \lim \mathscr{G}, \quad \operatorname{adh}_{\mathrm{First}} \mathscr{F} = \bigcup_{\varphi_\omega \ni \mathscr{G} \# \mathscr{F}} \lim \mathscr{G}.$$

The definitions of strongly Fréchet and bisequential spaces can be nicely rephrased in terms of these two types of adherence. Indeed, $X$ is strongly Fréchet if and only if

$$\operatorname{adh} \mathscr{F} \subset \operatorname{adh}_{\mathrm{First}} \mathscr{F}, \tag{1}$$

for every countably based filter $\mathscr{F}$. Analogously, $X$ is bisequential if and only if (1) holds for every filter $\mathscr{F}$. Notice that $X$ is Fréchet if and only if (1) holds for every principal filter.

A. V. Arhangel'skii extended Michael's result from bisequential to $\aleph_0$-*bisequential* spaces, i.e., spaces for which each countable subset is bisequential and for every $x \in X$ and $C \subset \bigcup_{B \subset X, |B| \leqslant \aleph_0} \operatorname{cl}_{bX} B$ (where $bX$ denotes a compactification of $X$) such that $x \in \operatorname{cl}_{bX} C$, there exists a countable subset $D$ of $C$ such that $x \in \operatorname{cl}_{bX} D$ [1].[4]

**Theorem 1** [1, Proposition 6.27]**.** *The product of an $\aleph_0$-bisequential space and of a strongly Fréchet space is strongly Fréchet.*

Also, S. Dolecki and T. Nogura proved

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[3] A topology is *sequential* if every sequentially closed set is closed.

[4] In particular, each space of countable tightness in which the closure of each countable subset is compact and bisequential is $\aleph_0$-bisequential [1, Proposition 6.20].

**Theorem 2** [6, part of Thorem 4]**.** *The product of a regular $\beta_3$ Fréchet q-space with a strongly Fréchet space is Fréchet.*

Recall that $X$ is called a *q-space* if for every $x \in X$, there exists a sequence $(Q_n)_{n\in\omega}$ of neighborhoods of $x$ such that every sequence $x_n \in Q_n$ has non empty adherence. On the other hand, a topology is $\beta_3$ [6] if for every convergent free (i.e., $x_n \neq x$) bisequence

$$x_{n,k} \to_k x_n \to_n x,$$

there exists a compact metrizable subset $C$ of $\{x\} \cup \{x_n\colon\ n \in \omega\} \cup \{x_{n,k}\colon\ n, k \in \omega\}$ such that

$$|\{n\colon\ |C \cap \{x_{n,k}\colon\ k \in \omega\}| = \omega\}| = \omega.$$

A filter $\mathscr{F}$ on a set $X$ is called *strongly Fréchet* [12], [4] if for every countably based filter $\mathscr{H}$ such that $\mathscr{H} \# \mathscr{F}$, there exists a countably based (equivalently a sequence) filter $\mathscr{L}$ such that $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$. Notice that a space is strongly Fréchet if and only if all its neighborhood filters are strongly Fréchet. Analogously, a filter $\mathscr{F}$ on a set $X$ is *productively Fréchet* if there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$ whenever $\mathscr{H}$ is a strongly Fréchet filter meshing with $\mathscr{F}$. A topological space is called *productively Fréchet* if all its neighborhood filters are productively Fréchet, or equivalently if (1) holds for every strongly Fréchet filter. Bisequential spaces can also be characterized by a similar property of their neighborhood filters: a filter $\mathscr{F}$ is *bisequential* if there exists a countably based filter $\mathscr{L}$ such that $\mathscr{L} \geqslant \mathscr{F}$ and $\mathscr{L} \# \mathscr{H}$ whenever $\mathscr{H}$ is a filter meshing with $\mathscr{F}$. As every countably based filter is bisequential, hence strongly Fréchet, every bisequential space is productively Fréchet and every productively Fréchet space is strongly Fréchet.

**Remark 3.** In view of the definitions of strongly Fréchet and productively Fréchet filters, one may be tempted to iterate the process to generate new classes of filters in the following way: $\varphi_\omega^0 = \varphi_\omega$ is the class of countably based filters, and $\mathscr{F} \in \varphi_\omega^{n+1}$ if for every $\mathscr{H} \in \varphi_\omega^n$ such that $\mathscr{H} \# \mathscr{F}$, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}$. Then the class $\varphi_\omega^1 = \varphi_\omega^3$ of strongly Fréchet filters and the class $\varphi_\omega^2$ of productively Fréchet filters are the only two new classes generated.

If $\mathscr{F}$ is a filter on a set $X$, $\mathscr{G}$ is a filter on a set $Y$ and $\mathscr{H}$ is a filter on $X \times Y$, we denote by $\mathscr{H}\mathscr{F}$ the (possibly degenerate) filter on $Y$ generated by the sets

$$HF = \{y\colon\ \exists x \in F, (x, y) \in H\},$$

for $H \in \mathscr{H}$ and $F \in \mathscr{F}$ and by $\mathscr{H}^-\mathscr{G}$ the filter on $X$ generated by the sets

$$H^-G = \{x\colon\ \exists y \in G, (x, y) \in H\},$$

for $H \in \mathscr{H}$ and $G \in \mathscr{G}$. Notice that

$$\mathscr{H} \# (\mathscr{F} \times \mathscr{G}) \Longleftrightarrow \mathscr{H}\mathscr{F} \# \mathscr{G} \Longleftrightarrow \mathscr{F} \# \mathscr{H}^{-}\mathscr{G}.$$

As a convention, we consider that degenerate filters are in any class of filters we may consider.

**Lemma 4.** *If $\mathscr{F}$ is a strongly Fréchet filter on $X$ and $\mathscr{H}$ is a countably based filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is strongly Fréchet.*

Proof. Let $\mathscr{G}$ be a countably based filter such that $\mathscr{G} \# \mathscr{H}\mathscr{F}$. Then $\mathscr{H}^{-}\mathscr{G}$ is a countably based filter that meshes with $\mathscr{F}$. As $\mathscr{F}$ is strongly Fréchet, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{F} \vee \mathscr{H}^{-}\mathscr{G}$. Then, $\mathscr{H}\mathscr{L} \# \mathscr{G}$ and $\mathscr{H}\mathscr{L} \geqslant \mathscr{H}\mathscr{F}$, so that $\mathscr{H}\mathscr{L} \vee \mathscr{G}$ is a countably based filter finer than $\mathscr{H}\mathscr{F} \vee \mathscr{G}$. □

Using the same argument and Lemma 4, we obtain also:

**Lemma 5.** *If $\mathscr{F}$ is a productively Fréchet filter on $X$ and $\mathscr{H}$ is a countably based filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is productively Fréchet.*

**Lemma 6.** *If $\mathscr{F}$ is a productively Fréchet filter on $X$ and $\mathscr{H}$ is a strongly Fréchet filter on $X \times Y$, then the filter $\mathscr{H}\mathscr{F}$ is strongly Fréchet.*

Proof. First assume that $\mathscr{F}$ is countably based. Let $\mathscr{G}$ be a countably based filter such that $\mathscr{G} \# \mathscr{H}\mathscr{F}$. Then $\mathscr{F} \times \mathscr{G}$ is a countably based filter meshing with $\mathscr{H}$. As $\mathscr{H}$ is strongly Fréchet, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{H} \vee (\mathscr{F} \times \mathscr{G})$. The filter $\mathscr{L}\mathscr{F}$ is countably based and $\mathscr{L}\mathscr{F} \geqslant \mathscr{H}\mathscr{F}$ and $\mathscr{L}\mathscr{F} \# \mathscr{G}$, so there exists a countably based filter $\mathscr{M} \geqslant \mathscr{H}\mathscr{F} \vee \mathscr{G}$. Hence, if $\mathscr{F}$ is countably based and $\mathscr{H}$ is strongly Fréchet, then $\mathscr{H}\mathscr{F}$ is strongly Fréchet.

More generally, assume that $\mathscr{F}$ is productively Fréchet, that $\mathscr{H}$ is strongly Fréchet and that $\mathscr{G}$ is a countably based filter that meshes with $\mathscr{H}\mathscr{F}$. By the previous case, $\mathscr{H}^{-}\mathscr{G}$ is a strongly Fréchet filter that meshes with $\mathscr{F}$. As $\mathscr{F}$ is productively Fréchet, there exists a countably based filter $\mathscr{M} \geqslant \mathscr{F} \vee \mathscr{H}^{-}\mathscr{G}$. Again by the previous case, $\mathscr{H}\mathscr{M}$ is a strongly Fréchet filter finer than $\mathscr{H}\mathscr{F}$ and that meshes with $\mathscr{G}$. Thus, there exists a countably based filter $\mathscr{L} \geqslant \mathscr{H}\mathscr{M} \vee \mathscr{G}$ so that $\mathscr{L} \geqslant \mathscr{H}\mathscr{F} \vee \mathscr{G}$. □

In the next lemma $\mathbb{D}$ and $\mathbb{J}$ denote generic classes of filters (like that of countably based, strongly Fréchet or productively Fréchet filters). $\mathbb{D}(X)$ denotes the corresponding family of filters on the set $X$. A class $\mathbb{D}$ of filters is called *embeddable* if the filter generated by $\mathscr{F}$ on $X$ is in $\mathbb{D}(X)$ whenever $Y \subset X$ and $\mathscr{F} \in \mathbb{D}(Y)$.

**Lemma 7.** *Let $\mathbb{J}$ and $\mathbb{D}$ be two classes of filters. Assume $\mathbb{D}$ is embeddable. If $\mathscr{H}\mathscr{F} \in \mathbb{J}(Y)$ whenever $\mathscr{H} \in \mathbb{D}(X \times Y)$ and $\mathscr{F} \in \mathbb{J}(X)$, then $\mathscr{F} \vee \mathscr{G} \in \mathbb{J}(X)$ whenever $\mathscr{F} \in \mathbb{J}(X)$ and $\mathscr{G} \in \mathbb{D}(X)$.*

Proof. If $\mathscr{H} \in \mathbb{D}(X)$ and $\mathscr{F} \in \mathbb{J}(X)$, then the filter $\widehat{\mathscr{H}}$ generated on $X \times X$ by $\{(x, x) \colon x \in H\}_{H \in \mathscr{H}}$ is in $\mathbb{D}(X \times X)$ so that $\widehat{\mathscr{H}}\mathscr{F} = \mathscr{H} \vee \mathscr{F}$ is in $\mathbb{J}(X)$. □

In particular, in view of Lemma 5, if $\mathscr{F}$ is a productively Fréchet filter and $A \# \mathscr{F}$, then $\mathscr{F} \vee A$ is productively Fréchet. Consequently,

**Proposition 8.** *A subspace of a productively Fréchet space is productively Fréchet.*

Notice that this is also true for Fréchet and strongly Fréchet (filters and) spaces.

**Theorem 9.** *$X$ is productively Fréchet if and only if $X \times Y$ is Fréchet (equivalently strongly Fréchet) for every strongly Fréchet space $Y$.*

Proof. Assume that $X$ is not productively Fréchet. Then there exists a strongly Fréchet filter $\mathscr{F}$ and a point $x_0$ such that $x_0 \in \operatorname{adh} \mathscr{F} \setminus \operatorname{adh}_{\text{First}} \mathscr{F}$. Let $Y$ be a copy of the underlying set of $X$ in which all points but $x_0$ are isolated and $\mathscr{N}_Y(x_0) = \mathscr{F} \wedge (x_0)$. Then $Y$ is strongly Fréchet, as each of its neighborhood filters is strongly Fréchet. There exists an ultrafilter $\mathscr{U}$ of $\mathscr{F}$ that converges to $x_0$ in $X$. Hence $\mathscr{U} \times \mathscr{U}$ converges to $(x_0, x_0)$ in $X \times Y$ so that $(x_0, x_0) \in \operatorname{cl}_{X \times Y}\{(x, x) \colon x \neq x_0\}$. But no countably based filter (a fortiori no sequence) on $\{(x, x) \colon x \neq x_0\}$ converges to $(x_0, x_0)$. Indeed, without loss of generality, such a filter can be assumed of the form $\mathscr{H} \times \mathscr{H}$ for a countably based filter $\mathscr{H}$. If $x_0 \in \lim_Y \mathscr{H}$, then $\mathscr{H} \geqslant \mathscr{F}$. But in this case, $x_0 \notin \lim_X \mathscr{H}$, as $x_0 \notin \operatorname{adh}_{\text{First}} \mathscr{F}$. Hence $X \times Y$ is not Fréchet.

Conversely, assume that $X$ is productively Fréchet and that $Y$ is strongly Fréchet. Let $\mathscr{H}$ be a countably based filter such that $(x, y) \in \operatorname{adh}_{X \times Y} \mathscr{H}$. In other words, $(\mathscr{N}_X(x) \times \mathscr{N}_Y(y)) \# \mathscr{H}$. As $Y$ is strongly Fréchet, $\mathscr{N}_Y(y)$ is a strongly Fréchet filter, so that, by Lemma 4, $\mathscr{H}^- \mathscr{N}_Y(y)$ is a strongly Fréchet filter that meshes with $\mathscr{N}_X(x)$. Hence, $x \in \operatorname{adh}_X \mathscr{H}^- \mathscr{N}_Y(y)$. As $X$ is productively Fréchet, there exists a countably based filter $\mathscr{G}$ such that $x \in \lim_X \mathscr{G}$ and $\mathscr{G} \# \mathscr{H}^- \mathscr{N}_Y(y)$. Hence $\mathscr{H}\mathscr{G}$ is a countably based filter that meshes with the strongly Fréchet filter $\mathscr{N}_Y(y)$. Therefore, there exists a countably based filter $\mathscr{L}$ finer than $\mathscr{H}\mathscr{G} \vee \mathscr{N}_Y(y)$. The filter $\mathscr{G} \times \mathscr{L}$ is countably based, meshes with $\mathscr{H}$, and converges to $(x, y)$ in $X \times Y$. Thus $X \times Y$ is strongly Fréchet. □

**Corollary 10.** *A product of finitely many productively Fréchet spaces is productively Fréchet.*

P r o o f. Let $X$ and $Y$ be two productively Fréchet spaces. In view of Theorem 9 it suffices to show that $(X \times Y) \times W$ is strongly Fréchet for every strongly Fréchet space $W$ to conclude that $X \times Y$ is productively Fréchet. As $Y$ is productively Fréchet, by Theorem 9, $Y \times W$ is strongly Fréchet. Applying the same argument to the product of the productively Fréchet space $X$ with the strongly Fréchet space $Y \times W$, we get that $X \times Y \times W$ is strongly Fréchet. □

As a direct consequence of Theorems 9, 1 and 2, $\aleph_0$-bisequential and regular $q$ Fréchet $\beta_3$ spaces are productively Fréchet. However, we provide a direct proof of the latter, while a direct proof of the former will be published in a forthcoming paper, because it requires the introduction of more machinery.

**Proposition 11.** *A regular $q$ and $\beta_3$ Fréchet space is productively Fréchet.*

P r o o f. Let $\mathscr{F}$ be a strongly Fréchet filter and let $x \in \operatorname{adh} \mathscr{F}$. As $x$ is a $q$-point, for every closed neighborhood $W$ of $x$, there exists a sequence $(Q_n \cap W)$ of neighborhoods of $x$ witnessing the definition of a $q$-point. As $(Q_n \cap W) \# \mathscr{F}$ and $\mathscr{F}$ is strongly Fréchet, there exists a sequence $(x_n^W) \geqslant \mathscr{F} \vee (Q_n \cap W)$, so that $\operatorname{adh}(x_n^W) \cap W \neq \emptyset$. As under the present assumptions, $X$ is strongly Fréchet, there exists a sequence $(y_n^W) \geqslant \mathscr{F}$ such that $\lim(y_n^W) \cap W \neq \emptyset$. By regularity and Fréchetness of $X$, there exists a sequence $(y_p)_{p\in\omega}$ on $\bigcup_{W=\operatorname{cl} W \in \mathscr{N}(x)} \lim(y_n^W) \cap W$ which converges to $x$. For each $p$, pick $W_p$ such that $y_p \in \lim(y_n^{W_p})$. By the property $\beta_3$, there exists a compact metrizable subset $C$ of $\{x\} \cup \{y_p \colon p \in \omega\} \cup \{y_n^{W_p} \colon n \in \omega, p \in \omega\}$ such that $|\{p \colon |C \cap \{y_n^{W_p} \colon n \in \omega\}| = \omega\}| = \omega$. Since each sequence $(y_n^W)_n \geqslant \mathscr{F}$, the countably based filter generated by $\{\{y_n^{W_p} \in C \colon n \in \omega, p \geqslant k\} \colon k \in \omega\}$ meshes with the strongly Fréchet filter $\mathscr{F}$. Therefore, there exists a sequence $(w_n)_\omega \geqslant \mathscr{F} \vee \{\{y_n^{W_p} \in C \colon n \in \omega, p \geqslant k\} \colon k \in \omega\}$. As a sequence on a compact metrizable set, $(w_n)_\omega$ has a convergent subsequence $(v_n)_{n\in\omega}$. Moreover $x \in \lim(v_n)_{n\in\omega}$ because $(v_n)_{n\in\omega} \geqslant \{\{y_n^{W_p} \in C \colon n \in \omega, p \geqslant k\} \colon k \in \omega\}$. Hence there exists a sequence finer that $\mathscr{F}$ converging to $x$. □

On the other hand, a regular productively Fréchet space need not be a $q$-space, nor a bisequential space. Hence, Theorem 9 generalizes both Theorem 2 and [13, Proposition 4.D.4].

**Example 12.** (A regular productively Fréchet space which is neither bisequential nor a $q$-space).

The $\Sigma$-product of uncountably many copies of $\mathbb{R}$ is normal and $\aleph_0$-bisequential, hence productively Fréchet, but neither bisequential [1, Example 6.24] nor $q$ [6, Proposition 11].

We consider a space $X$ constructed by Nyikos [15, Remark 3.12]. The space $X$ is productively Fréchet and not $q$. If we make some set-theoretical assumptions we can have that $X$ is not $\alpha_3$. Recall that a topological space is $\alpha_3$ if for every stationary bisequence

$$x_{n,k} \underset{k}{\to} x,$$

there exists a sequence $(x_{n_p,k_p})_{p\in\omega}$ converging to $x$ such that

$$|\{n\colon\ |\{k_p\colon\ n_p = n\}| = \omega\}| = \omega.$$

In particular, $X$ is not $\aleph_0$-bisequential [1, Theorem 6.15], so that Theorem 9 also generalizes Theorem 1. We construct this example again here because we will have to work to show that it is productively Fréchet.

Given two functions $f, g \in \omega$ we write $f <^* g$ if $\{n \in \omega\colon\ g(n) \leqslant f(n)\}$ is finite. We let $\mathfrak{b}$ denote the bounding number of $\omega$, i.e., the smallest cardinality of a collection $F \subseteq \omega^\omega$ such that for every $g \in \omega^\omega$ there is an $f \in F$ such that $f \not<^* g$. It is well known that one can construct a $<^*$-increasing family $\{f_\xi\}_{\xi\in\mathfrak{b}}$ of strictly increasing functions in $\omega^\omega$ witnessing the definition of $\mathfrak{b}$. Given an $f \in \omega^\omega$, we let $f^{\downarrow\downarrow} = \{(n,k)\colon\ k < f(n)\}$. We let $\mathfrak{c}$ denote the cardinality of the real numbers.

**Example 13.** (A countable productively Fréchet space $X$ which is not a $q$-space. Moreover, if we assume $\mathfrak{b} = \mathfrak{c}$, then $X$ is not $\alpha_3$.)

Let $K = \{f_\xi\}_{\xi\in\mathfrak{b}}$ be as above. Define a topology on $X_0 = (\omega\times\omega)\cup K$ so that: for each $f \in K$ the neighborhoods of $f$ will be sets of the form $\{f\}\cup\{(k, f(k))\colon\ k \geqslant n\}$ where $n \in \omega$ and all other points are isolated. Notice that $X_0$ is locally compact. We let $X_1$ be the one point compactification of $X_0$. We denote the compactifying point by $\infty$. Our space $X$ will be $(\omega\times\omega)\cup\{\infty\}$ with the subspace topology from $X_1$.

We show that $X$ is productively Fréchet. In view of Proposition 8, it is enough to show that $X_1$ is productively Fréchet. As every point in $X_1$ is first countable except the point $\infty$, it is enough to show that the neighborhood filter $\mathscr{N}$ of $\infty$ is productively Fréchet. Let $\mathscr{F}$ be a strongly Fréchet filter such that $\mathscr{F} \# \mathscr{N}$.

We may assume that $\mathscr{F} \# \{\omega\times\omega\}$. Otherwise, $\mathscr{F}$ has a base in $K\cup\{\infty\}$ which is a one point compactification of a discrete set which is $\aleph_0$-bisequential, by [1, Proposition 6.20]. By Lemma 4, $\mathscr{F} \vee (K\cup\{\infty\})$ is strongly Fréchet and is easily seen to mesh with $\mathscr{N}$. By the productive Fréchetness of $K\cup\{\infty\}$ we can find a countably based filter $\mathscr{L} \geqslant \mathscr{F}\vee\mathscr{N}$. So, we assume that $\mathscr{F} \# \{\omega\times\omega\}$.

Notice that the filter $\mathscr{F}\vee\{\omega\times\omega\}$ is strongly Fréchet, by Lemma 4. Moreover, we may assume that $(\mathscr{F}\vee\{\omega\times\omega\}) \# \mathscr{N}$. Otherwise, we have $\mathscr{F} \# (K\cup\{\infty\})$ and the previous argument applies.

We may also assume that $\mathscr{F} \geqslant \mathscr{C}$, where $\mathscr{C}$ is the filter generated by sets of the form $C_n = \{(k,l)\colon\ n \leqslant k\}$. Indeed, $\mathscr{F} \# \mathscr{C}$ and by Lemmas 4 and 7, $\mathscr{F}\vee\mathscr{C}$ is a strongly Fréchet filter meshing $\mathscr{N}$, that can be taken as our new $\mathscr{F}$.

Let $T = \{f \in K\colon f \# \mathscr{F}\}$. We consider two cases, when $T$ is finite and when it is infinite.

Suppose $T$ is finite. Since $\mathscr{F} \# \mathscr{N}$, we have that $\mathscr{F}^* = \mathscr{F} \vee ((\omega \times \omega) \setminus \bigcup T)$ also meshes with $\mathscr{N}$. By Lemmas 4 and 7, we also have that $\mathscr{F}^*$ is strongly Fréchet. Since $\mathscr{F}^*$ is strongly Fréchet, there is a sequence $(x_n)_{n\in\omega}$ such that $(x_n)_{n\in\omega} \geqslant \mathscr{F}^*$. By way of contradiction, assume that $\lim\limits_{n\in\omega} x_n \neq \infty$. In this case, there is an $f \in K \setminus T$ such that $f \# (x_n)_{n\in\omega}$. Since $(x_n)_{n\in\omega} \geqslant \mathscr{F}^* \geqslant \mathscr{F}$ and $\mathscr{F} \geqslant \mathscr{C}$, we have $f \# \mathscr{F}$. However, $f \notin T$ so we have a contradiction.

Suppose now that $T$ is infinite. Let $\{f_{\xi_n}\}_{n\in\omega}$ be the first $\omega$-many elements of $T$ and $\xi = \sup\{\xi_n\colon n \in \omega\}$.

For every $n \in \omega$ let $D_n = \{(k,l)\colon k,l \in \omega\colon f_{\xi_n}(k) \leqslant l < f_\xi(k)\}$. Let $\mathscr{D}$ be the filter generated by the collection $\{D_n\colon n \in \omega\}$.

Let $F \in \mathscr{F}$ and $n \in \omega$. Since $f_{\xi_{n+1}} \# \mathscr{F}$ and $\mathscr{F} \geqslant \mathscr{C}$, we have $|f_{\xi_{n+1}} \cap F| = \omega$. Since $f_{\xi_n} <^* f_{\xi_{n+1}} <^* f_\xi$, we have that $F \# D_n$. Hence, $\mathscr{F} \# \mathscr{D}$.

Since $\mathscr{F}$ is strongly Fréchet, there exists a sequence $(x_n)_{n\in\omega} \geqslant \mathscr{F} \vee \mathscr{D}$. We claim that $\lim x_n = \infty$. If not, then there is $\beta \in \mathfrak{b}$ such that $\{x_n\colon n \in \omega\} \cap f_\beta$ is infinite. Since $\mathscr{D}$ has a base consisting of subsets of $f_\xi^{\downarrow\downarrow}$, we have $\beta < \xi$. Since, $(x_n)_{n\in\omega} \geqslant \mathscr{F}$, we have that $f_\beta \# \mathscr{F}$. So, $f_\beta = f_{\xi_n}$ for some $n \in \omega$. Since $(x_n)_{n\in\omega} \geqslant \mathscr{D}$, we have that $\{x_n\colon n \in \omega\} \cap f_\beta$ is finite, a contradiction. Thus, $\lim x_n = \infty$.

Therefore, $X_1$ is productively Fréchet.

We show that $X$ is not $q$. Let $\{U_n\}_{n\in\omega}$ be a nested sequence of neighborhoods of $\infty$. Without loss of generality, we may assume that $X \setminus U_n$ is contained in the union of a finite collection $L$ of graphs of functions from $K$ together with some finite collection of points. Pick $f \in K$ such that $g <^* f$ for all $g \in L$. Consider $\{(k, f(k))\}_{k\in\omega}$ as a sequence. Since $g <^* f$ for all $g \in L$, there is for every $n \in \omega$ a $k_n \in \omega$ such that $(k_n, f(k_n)) \in U_n$. Clearly, $\{(k_n, f(k_n)\}_{n\in\omega}$ has no point of adherence in $X$. Thus, $X$ is not $q$.

The simple argument that $\mathfrak{b} = \mathfrak{c}$ implies that $X$ is not $\alpha_3$ is given in [15, Remark 3.12].

Fréchet spaces are exactly the hereditarily quotient images of metrizable spaces; strongly Fréchet spaces are the countably biquotient images of metrizable spaces and bisequential spaces are biquotient images of metrizable spaces (see [13] and [3] for details). Analagously, productively Fréchet spaces can be characterized as images of metrizable spaces under a nice class of quotient maps, intermediate between countably biquotient and biquotient maps.

A continuous surjection $f\colon X \to Y$ is called $\varphi^1_\omega$*-biquotient* if for every strongly Fréchet filter $\mathscr{H}$ on $Y$

$$y \in \mathrm{adh_Y}\, \mathscr{H} \Longrightarrow f^- y \cap \mathrm{adh_X}\, f^- \mathscr{H} \neq \emptyset.$$

In view of the convergence-theoretic interpretation [3, Theorem 1.2] of quotient maps, of [14, Theorem 4.8] and of [14, Theorem 7.1], we have:

**Proposition 14.** *Let $f\colon X \to Y$ be a continuous surjection. The following are equivalent*

1. *$f$ is $\varphi^1_\omega$-biquotient;*
2. *$y \in \operatorname{adh}_Y \mathscr{H} \Longrightarrow f^- y \cap \operatorname{adh}_X f^- \mathscr{H} \neq \emptyset$, for every strongly Fréchet filter $\mathscr{H}$;*
3. *for every $\varphi^1_\omega$-cover [5] $\mathscr{S}$ of $f^- y$, there exists a finite subfamily $\mathscr{R} \subset \mathscr{S}$ such that $y \in \operatorname{int}\Big(\bigcup_{R\in\mathscr{R}} f(R)\Big)$;*
4. *$f \times \mathrm{Id}_Z$ is hereditarily quotient for every atomic[6] strongly Fréchet topological space $Z$;*
5. *$f \times g$ is countably biquotient for every biquotient map with strongly Fréchet range.*

Since productively Fréchet spaces are exactly topological spaces that satisfy (1) for every strongly Fréchet filter, the following is an immediate consequence of [3, Theorems 5.2 and 1.2].

**Theorem 15.** *A space $Y$ is productively Fréchet if and only if there exists a $\varphi^1_\omega$-biquotient map $f\colon X \to Y$ with metrizable domain $X$.*

**Problem 16.** Is the product of countably many productively Fréchet spaces productively Fréchet?

**Problem 17.** Is there a ZFC example of a countable productively Fréchet space that is not bisequential?

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[5] An open cover $\mathscr{S}$ is called *$\varphi^1_\omega$-cover*, if the family of complements of elements of the ideal generated by $\mathscr{S}$ form a strongly Fréchet filter.

[6] A topology is called *atomic* if all but one point are isolated.

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*Authors' address*: Department of Mathematical Sciences, Georgia Southern University, 203 Georgia Ave., Room 3008, Statesboro, GA 30458, U.S.A., e-mails: `fmynard@georgiasouthern.edu`, `fjordan@georgiasouthern.edu`.