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GENERALIZED CARDINAL PROPERTIES OF LATTICES AND LATTICE ORDERED GROUPS

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Abstract. We denote by K the class of all cardinals; put $K' = K \cup \{\infty\}$. Let $\mathscr{C}$ be a class of algebraic systems. A generalized cardinal property f on $\mathscr{C}$ is defined to be a rule which assings to each $A \in \mathscr{C}$ an element fA of K' such that, whenever $A_1, A_2 \in \mathscr{C}$ and $A_1 \simeq A_2$, then $fA_1 = fA_2$. In this paper we are interested mainly in the cases when (i) $\mathscr{C}$ is the class of all bounded lattices B having more than one element, or (ii) $\mathscr{C}$ is a class of lattice ordered groups.

Keywords: bounded lattice, lattice ordered group, generalized cardinal property, homogeneity

MSC 2000: 06F15, 06B05

1. Introduction

This paper can be considered a continuation of the author's article [6].

We denote by K the class of all cardinals. Put $K' = K \cup \{\infty\}$ and let $\mathscr{C}$ be a class of algebraic systems.

A generalized cardinal property f on $\mathscr{C}$ is defined to be a rule which assigns to each $A \in \mathscr{C}$ an element fA of K' such that, whenever A_1 and A_2 are isomorphic algebraic systems belonging to $\mathscr{C}$, then $fA_1 = fA_2$.

In particular, if $fA \in K$ for each $A \in \mathscr{C}$, then f is called a cardinal property on $\mathscr{C}$. This terminology is in accordance with Pierce [9], where $\mathscr{C}$ is the class of Boolean algebras; Monk [8] and van Douwen [3] apply the term "cardinal function on Boolean algebras".

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Analogous terminology can be used in the case when $\mathscr{C}$ is a class of topological spaces. For results and references on cardinal functions on Boolean spaces cf. the expository article by van Douwen [4].

Some cardinal properties on Boolean algebras introduced by Pierce [9] were applied for studying radical classes of generalized Boolean algebras [7].

2. Preliminaries

Let K' be as above. For each $\alpha \in K$ we put $\alpha < \infty$. The relation $<$ in K has the usual meaning.

We denote by $\mathscr{B}$ the class of all bounded lattices having more than one element.

Let f be a generalized cardinal property on the class $\mathscr{B}$. A lattice L is called f-homogeneous if $fL_1 = fL_2$ for any two convex sublattices L_1 and L_2 of L such that L_1 and L_2 belong to $\mathscr{B}$.

We say that f is increasing (decreasing) if $fL_1 \leqslant fL_2$ (or $fL_1 \geqslant fL_2$, respectively) for any pair $L_1, L_2 \in \mathscr{B}$ such that L_1 is a convex sublattice of L_2.

For lattice ordered groups we apply the notation as in [1] and [2]. Let $\mathscr{G}_1$ be a nonempty class of lattice ordered groups which is closed with respect to isomorphisms.

Let f be a generalized cardinal property on the class $\mathscr{G}_1$. We define f to be increasing (decreasing) if, whenever $G_1 \neq \{0\}$ is a convex ℓ-subgroup of a lattice ordered group G_2 and $G_1, G_2 \in \mathscr{G}_1$, then $fG_1 \leqslant fG_2$ (or $fG_1 \geqslant fG_2$, respectively).

A lattice ordered group G is called f-homogeneous if $fG_1 = fG_2$ whenever G_1 and G_2 are non-zero convex ℓ-subgroups of G such that $G_1, G_2 \in \mathscr{G}_1$.

The investigation made in [6] concerned mainly the case of increasing cardinal properties on $\mathscr{B}$ which are related to certain properties of lattice ordered groups. In the present paper we show that some results from [6] remain valid for decreasing cardinal properties on $\mathscr{B}$.

We denote by R the additive group of all reals with the natural linear order. If no misunderstanding can occur, then R denotes also the underlying lattice.

Examples.

2.1. For $L \in \mathscr{B}$ we put $f_1 L = \operatorname{card} L$. Then f_1 is an increasing cardinal property on the class $\mathscr{B}$. The lattice R is f_1-homogeneous.

2.2. Let $\mathscr{G}$ be the class of all lattice ordered groups. For $G \in \mathscr{G}$ we set $f_2 G = \operatorname{card} G$. Hence f_2 is an increasing cardinal property on $\mathscr{G}$. The lattice ordered group R is f_2-homogeneous.

Let α be a cardinal, $\alpha \neq 0$. For the notion of α-distributivity and complete distributivity of a lattice cf., e.g., [1]. We define each element of $\mathscr{B}$ to be 0-distributive.

2.3. Let $L \in \mathscr{B}$. If L is completely distributive, then we put $f_3L = \infty$. Otherwise there exists a cardinal β such that

(i) L is not β-distributive;

(ii) if β_1 is a cardinal with $\beta_1 < \beta$, then L is β_1-distributive.

We put $f_3L = \beta$. Then f_3 is a decreasing generalized cardinal property on $\mathscr{B}$.

Analogously we define the generalized cardinal property f'_3 on the class $\mathscr{G}$.

2.4. For $G \in \mathscr{G}$ we denote by $MC(G)$ the set of all maximal chains in G. Put

$$f_4G = \min\{\operatorname{card} X\colon\ X \in MC(G)\}.$$

Then f_1 is a cardinal property on $\mathscr{G}$ which is neither increasing nor decreasing.

2.5. Let $\beta \neq 0$ be a cardinal and let $L \in \mathscr{B}$. The lattice L is called β-complete if, whenever X is a nonempty subset of L with $\operatorname{card} X \leqslant \beta$, then $\sup X$ and $\inf X$ exist in L. We define f_5L as follows. If L is complete, then we put $f_5L = \infty$. Assume that L fails to be complete. Then there exists $\gamma \in K$ such that L is not γ-complete and L is γ_1 complete for $\gamma_1 < \gamma$. We put $f_5L = \gamma$.

3. The cardinal properties f^1 and f^2

Let $\mathscr{B}$ be as above and let $\{0\} \neq G \in \mathscr{G}$. Further, let f be a generalized cardinal property on $\mathscr{B}$. Then we consider the cardinals fL where L is an interval of G belonging to $\mathscr{B}$; the set of all these cardinals will be denoted by $\mathscr{A}$.

We will deal with the following conditions:

(c_1) If $t_i \in G$, $0 < t_i$ $(i = 1,2)$, $f[0,t_1] = f[0,t_2]$ and if the intervals $[0,t_1]$, $[0,t_2]$ are f-homogeneous, then $f[0,t_1+t_2] = f[0,t_1]$.

(c_2) If $t_n \in G$ $(n = 1,2,\ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $\vee t_n = t$, $f[0,t_1] = f[0,t_n]$ and if the intervals $[0,t_n]$ $(n = 1,2,\ldots)$ are f-homogeneous, then $f[0,t] = f[0,t_1]$.

The main results of § 1 in [6] are as follows.

(A) ([6, Theorem 1.6]) Let f be an increasing cardinal property and let (c_1) be valid. For any $\alpha \in \mathscr{A}$ and $g \in G$ let $G_\alpha(g)$ be the family of all convex sublattices L of G such that $g \in L$ and $f[t_1,t_2] = \alpha$ for each nontrivial interval of L. Let $G_\alpha(g)$ be partially ordered by the set-theoretical inclusion. Then (i) any family $G_\alpha(g)$ contains a greatest element (this will be denoted by $B_\alpha(g)$); (ii) $B_\alpha(0)$ is an ℓ-ideal of G and $B_\alpha(g) = B_\alpha(0) + g$; (iii) $B_\alpha(g) \cap B_\beta(g) = \{g\}$ for any $\beta \in \mathscr{A}$, $\beta \neq \alpha$.

(B) ([6, Theorem 1.21]) Let G be a complete lattice ordered group and let f be an increasing cardinal property satisfying (c_1) and (c_2). Then G is isomorphic to a complete subdirect product of f-homogeneous lattice ordered groups. If

G is also laterally complete, then it is isomorphic to a direct product of f-homogeneous lattice ordered groups.

We remark that the notion of complete subdirect product of lattice ordered groups was introduced by Šik [10].

By using the same steps as in [6] we can verify

3.1. Proposition. *The assertions of* (A) *and* (B) *remain valid if the assumption*

(i) *f is an increasing property on $\mathscr{B}$*

is replaced by the assumption

(ii) *f is an increasing generalized cardinal property on $\mathscr{B}$.*

For $L \in \mathscr{B}$ we denote by $C(L)$ the set of all elements α of K' having the property that there exists a convex sublattice L_1 of L such that $L_1 \in \mathscr{B}$ and $fL_1 = \alpha$. Put

$$f^1L = \sup C(L), \quad f^2L = \inf C(L).$$

Then we obviously have

3.2. Lemma. *Let f be a generalized cardinal property on B. Then*

(i) *f^1 is an increasing generalized cardinal property on $\mathscr{B}$;*

(ii) *f^2 is a decreasing generalized cardinal property on $\mathscr{B}$;*

(iii) *if L is a lattice such that L is f-homogeneous, then L is f^1-homogeneous and f^2-homogeneous; if, moreover, $L \in \mathscr{B}$, then $f^1L = f^2L = fL$.*

We will apply also the condition

(c_3) If $0 < t \in G$ and if the interval $[0,t]$ is f^1-homogeneous, then it is f-homogeneous.

The conditions (c_1) and (c_2) for a cardinal property f were investigated in [6].

3.3. Lemma. *Assume that* (c_1) *is valid. Let t_i $(i = 1,2)$ be as in the assumption of* (c_1). *Then the interval $[0, t_1 + t_2]$ is f-homogeneous.*

P r o o f. Let $[u,v] \subseteq [0, t_1 + t_2]$, $u < v$. It suffices to verify that $f[u,v] = f[0,t_1]$. Denote (cf. Fig. 1)

$$u_1 = t_1 \wedge u, \quad v_1 = t_1 \wedge v,$$
$$u_2 = t_1 \vee u, \quad v_2 = t_1 \vee v,$$
$$p = v_1 \wedge u, \quad q = u_2 \vee v, \quad r = v_1 \vee u.$$

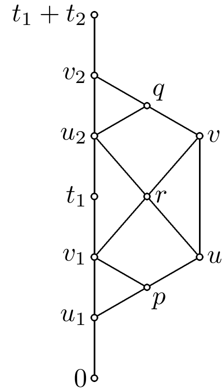

Fig. 1

Then we have

$$u_2 \wedge v = (t_1 \vee u) \wedge v = (t_1 \wedge v) \vee (u \wedge v) = v_1 \vee u = r.$$

Let us denote by $\sim$ the relation of projectivity of intervals of the lattice G. Since G is distributive, we get

$$[p, v_1] \sim [u, r], \quad [r, v] \sim [u_2, q].$$

a) Assume that $p = v_1$. Then $u = r$, whence $[u, v] = [r, v]$. Since the interval $[t_1, t_1 + t_2]$ is isomorphic to $[0, t_2]$, it is f-homogeneous and hence

$$f[u, v] = f[u, q] = f[t_1, t_1 + t_2] = f[0, t_2] = f[0, t_1].$$

b) If $u_2 = q$, then we proceed analogously as in the case a).

c) Suppose that $v_1 \neq p$ and $u_2 \neq q$. Denote

$$r' = r - u, \quad v' = v - u.$$

Then $0 < r'$ and $0 < v' - r'$. Further, we have:

(i) the interval $[0, r']$ is isomorphic to $[u, r]$, hence it is f-homogeneous and $f[0, r'] = f[0, t_1]$;

(ii) the interval $[0, v' - r']$ is isomorphic to $[r, v]$, thus it is f-homogeneous and $f[0, v' - r'] = f[0, t_1]$.

In view of (i) and (ii) we can apply the condition (c_1) to the elements $v' - r'$ and r'; we obtain

$$f[0, v'] = f[0, v' - r'] = f[0, t_1].$$

The intervals $[0,v']$ and $[u,v]$ are isomorphic; therefore

$$f[u,v]=f[0,t_1].$$

$\square$

3.4. Lemma. *Assume that* (c_2) *is valid. Let* t_i $(i=1,2,\dots)$ *and* t *be as in the assumption of* (c_2). *Then the interval* $[0,t]$ *is* f*-homogeneous.*

Proof. Let $[u,v]\subseteq[0,t]$, $u<v$. It suffices to verify that $f[u,v]=f[0,t_1]$. Put

$$t_i'=t_i\wedge v\quad (i=1,2,\dots).$$

Then

$$v=v\wedge t=v\wedge\bigvee t_i=\bigvee(v\wedge t_i)=\bigvee t_i'.$$

Hence there exists $i(0)\in\mathbb{N}$ such that $t'_{i(0)}>0$.

Let $i(0)$ have this property. Then $[0,t'_{i(0)}]\in\mathscr{B}$ and $[0,t'_{i(0)}]\subseteq[0,t_{i(0)}]$, whence $[0,t'_{i(0)}]$ is f-homogeneous and $f[0,t'_{i(0)}]=f[0,t_1]$.

Denote $t_i''=t_i'\vee u$ $(i\geqslant i(0))$. We have

$$v=u\vee v=u\vee\bigvee_{i\geqslant i(0)}t_i'=\bigvee_{i\geqslant i(0)}(u\vee t_i')=\bigvee_{i\geqslant i(0)}t_i''. \tag{1}$$

Hence there exists $i(1)\geqslant i(0)$ such that $t''_{i(1)}>u$.

Let $i\geqslant i(1)$. Put $q_i=t_i'\wedge u$. Then the interval $[u,t_i'']$ is projective to the interval $[q_i,t_i']$ and $[q_i,t_i']\subseteq[0,t_i']$. Hence $[q_i,t_i']$ is f-homogeneous and $f[q_i,t_i']=f[0,t_1]$. Then we obtain that $[u,t_i'']$ is f-homogeneous and $f[u,t_i'']=f[0,1]$.

Put $v^*=v-u$ and $t_i^*=t_i''-u$ for each $i\geqslant i(1)$. Thus the intervals $[0,t_i^*]$ and $[u,t_i'']$ are isomorphic, whence $[0,t_i^*]$ is f-homogeneous and $f[0,t_i^*]=f[0,t_1]$. Further, (1) yields

$$v=\bigvee_{i\geqslant i(1)}t_i''$$

and from this we infer

$$v^*=\bigvee_{i\geqslant i(1)}t_i^*.$$

Hence by applying (c_2) we obtain $f[0,v^*]=f[0,t_1]$. Since the intervals $[0,v^*]$ and $[u,v]$ are isomorphic we get $f[u,v]=f[0,t_1]$. $\square$

3.5. Lemma. *Assume that the conditions* (c_1) *and* (c_3) *are valid. Then* (c_1) *holds for* f^1.

Proof. Let $t_i \in G$, $0 < t_i$ $(i = 1, 2)$, $f^1[0, t_1] = f^1[0, t_2]$. Assume that $[0, t_1]$ and $[0, t_2]$ are f^1-homogeneous. Then in view of (c_3), $[0, t_1]$ and $[0, t_2]$ are f-homogeneous. Hence 3.2 (iii) yields that $f^1[0, t_i] = f[0, t_i]$ $(i = 1, 2)$. Thus f satisfies (c_1). Then 3.3 implies that $[0, t_1 + t_2]$ is f-homogeneous. By applying 3.2 (iii) again we get

$$f^1[0, t_1 + t_2] = f[0, t_1 + t_2] = f[0, t_1] = f^1[0, t_1].$$

□

Similarly we can prove

3.6. Lemma. *Assume that the conditions* (c_2) *and* (c_3) *are valid. Then* (c_2) *holds for* f^1.

From 3.1–3.6 we conclude:

3.7. Theorem. *Let* $G \in \mathscr{G}$ *and let* f *be a generalized cardinal property on* $\mathscr{B}$ *such that the conditions* (c_1) *and* (c_3) *are satisfied. Then the assertion of* (A) *is valid.*

3.8. Theorem. *Let* G *be a complete lattice ordered group and let* f *be a generalized cardinal property on* $\mathscr{B}$ *such that the conditions* (c_1), (c_2) *and* (c_3) *are satisfied. Then the assertion of* (B) *is valid.*

4. Decreasing generalized cardinal properties

In the present section we investigate the question whether in Theorem (A) and Theorem (B) the assumption that f is an increasing cardinal property on $\mathscr{B}$ can be replaced by the assumption that f is a decreasing generalized cardinal property on $\mathscr{B}$.

We show that the answer to this question is affirmative.

Let G be a lattice ordered group, $G \neq \{0\}$ and let$f$ be a decreasing generalized cardinal property on $\mathscr{B}$.

For $X \subseteq G$ we put

$$X^\delta = \{g \in\colon\ |g| \wedge |x| = 0 \ \text{ for each } \ x \in X\}.$$

Further, for $\emptyset \neq X_1 \subseteq G$ and $\emptyset \neq X_2 \subseteq G$ we write $X_1 \delta X_2$ if $X_1 \subseteq X_2^\delta$.

An interval $[u, v]$ of G is called nontrivial if $u < v$.

Analogously to the terminology applied in [6] we introduce the following notation. Let $\mathscr{A}$ be the set of all $\alpha \in K'$ such that there exist $a, b \in G$ with $a < b$, $f[a, b] = \alpha$. For each $\alpha \in \mathscr{A}$ we put

$$\begin{aligned} X_\alpha^1 &= \{x \in G\colon\ x > 0,\ \ f[0, u] \geqslant \alpha\} \cup \{0\}, \\ Y_\alpha^1 &= \{y \in G\colon\ y > 0,\ \ f[0, y] > \alpha\} \cup \{0\}, \\ Z_\alpha^1 &= (Y_\alpha^1)^\delta,\ \ A_\alpha^1 = X_\alpha^1 \cap Z_\alpha^1. \end{aligned}$$

We recall that in [6], the sets

$$X_\alpha,\ Y_\alpha,\ Z_\alpha \ \text{ and } \ A_\alpha \tag{1}$$

were defined in an analogous way with the following distinctions:

(i) in X_α, instead of $f[0, x] \geqslant \alpha$ the relation $f[0, x] \leqslant \alpha$ was used;

(ii) in Y_α, instead of $f[0, y] > \alpha$ the relation $f[0, y] < \alpha$ was applied.

4.1. Lemma. *Assume that* (c_1) *is valid. Let* $\alpha \in \mathscr{A}$. *Then*

(i) *the set* A_α *is an ideal of the lattice* G^+ *and a subsemigroup of* G^+*;*

(ii) $f[a, b] = \alpha$ *for each nontrivial interval* $[a, b]$ *of* A_α*;*

(iii) $\mathscr{A}_\alpha \delta A_\beta$ *for* $\beta \in \mathscr{A}$, $\beta \neq \alpha$.

P r o o f. If suffices to apply the same method as in the proof of 1.1 in [6] with the obvious modifications which are due to the above mentioned distinctions (i) and (ii). □

Now, 1.2–1.5 from [6] remain valid together with their proofs if the symbols from (1) are replaced by

$$X_\alpha^1,\ Y_\alpha^1,\ Z_\alpha^1 \ \text{ and } \ A_\alpha^1. \tag{2}$$

4.2. Theorem. *Let* f *be a decreasing generalized cardinal property on* $\mathscr{B}$ *and assume that* (c_1) *is satisfied. Then the assertion of* (A) *is valid.*

P r o o f. It suffices to apply the same argument as in the proof of 1.6 in [6]. □

For any $\alpha \in \mathscr{A}$ let $\overline{A}_\alpha$ be the set of all elements $t \in G$ that can be written in the form $t = \bigvee t_i$ where $\{t_i\} \subseteq A_\alpha$. Further, we denote by $\overline{B}_\alpha$ the set of all elements $t \in G$ such that there exist $t_1, t_2 \in \overline{A}_\alpha$ with $-t_1 \leqslant t \leqslant t_2$.

By the same arguments as in the proofs of 1.7 and 1.7.1 of [6] we obtain

4.3. Lemma. *Let $\alpha \in \mathscr{A}$ and assume that the condition* (c_1) *is satisfied. Then $\overline{B}_\alpha$ is an ℓ-ideal of G. If $\beta \in \mathscr{A}$ and $\beta \neq \alpha$, then $\overline{B}_\alpha \cap \overline{B}_\beta = \{0\}$.*

The same arguments which were used in proving Theorem 1.15 and Theorem 1.21 in [6] show (by an application of 4.3)

4.4. Theorem. *Let G be a complete lattice ordered group. Assume that f is a decreasing generalized cardinal property on $\mathscr{B}$ and that the condition* (c_1) *is satisfied.*

(i) *G is isomorphic to a complete subdirect product of the system $(\overline{B}_\alpha)_{\alpha \in \mathscr{A}}$.*

(ii) *If, moreover, G is laterally complete, then G is isomorphic to the direct product of the system $(\overline{B}_\alpha)_{\alpha \in \mathscr{A}}$.*

4.5. Theorem. *Let G be a complete lattice ordered group and let f be a decreasing generalized cardinal property on $\mathscr{B}$. Assume that the conditions* (c_1) *and* (c_2) *are satisfied. Then the assertion of* (B) *is valid.*

Let us consider the condition

(c_3') If $0 < t \in G$ and if the interval $[0, t]$ is f^2-homogeneous, then it is f-homogeneous.

By a method analogous to that used for proving 3.8 we obtain from 4.5

4.6. Corollary. *Let G be a complete lattice ordered group and let f be a generalized cardinal property on $\mathscr{B}$. Assume that the conditions* (c_1), (c_2) *and* (c_3) *be satisfied. Then the assertion* (B) *is valid.*

Again, let f be a generalized cardinal property on the class $\mathscr{B}$. Let $G \neq \{0\}$ be a lattice ordered group. We put

$$f^{01} = \sup\{\operatorname{card}[u, v] \colon [u, v] \text{ is a nontrivial interval of } G\}.$$

Further, let f^{02} be defined analogously with the distinction that sup is replaced by inf.

Let $\mathscr{G}_0$ be the class of all lattice ordered groups G with $G \neq \{0\}$. Then f^{01} (or, f^{02}, respectively) is an increasing (decreasing) generalized cardinal property on the class $\mathscr{G}_0$.

Let us denote by $\ell(G)$ the underlying lattice of G. If $G \in \mathscr{G}_0$ and the lattice $\ell(G)$ is f-homogeneous, then the lattice ordered group G is f^{01}-homogeneous and f^{02}-homogeneous.

Hence in view of 3.1 and according to (B) we have

4.7. Corollary. *Let f be an increasing generalized cardinal property on $\mathscr{B}$ and let $G \in \mathscr{G}_1$ be such that the conditions* (c_1) *and* (c_2) *are satisfied. If G is complete, then it is isomorphic to a complete subdirect product of lattice ordered groups which are f^{01}-homogeneous and f^{02}-homogeneous.*

According to 4.5, in 4.7 we can replace the assumption that f is increasing by the assumption that f is decreasing.

It is well-known that the system of all convex ℓ-subgroups of a lattice ordered group is a complete lattice (under the partial order defined by the set-theoretical inclusion).

Let T be a nonempty subclass of $\mathscr{G}$ such that

(i) if $G \in T$ and G_1 is a convex ℓ-subgroup of G, then $G_1 \in T$;

(ii) if $G \in \mathscr{G}$ and $\{G_i\}_{i \in I}$ is a nonempty system of convex ℓ-subgroups of G which belong to T, then $\bigvee_{i \in I} G_i \in T$.

The class with these properties is called a radical class.

Let f be a generalized cardinal property on the class $\mathscr{B}$ and let $\alpha \in K'$. We denote by $T(f, \alpha)$ the class of all lattice ordered groups G such that $f[u, v] = \alpha$ for each nontrivial interval of G.

The class $T(f, \alpha)$ is nonempty since the zero lattice ordered group belongs to $T(f, \alpha)$.

4.8. Proposition. *Let f be a generalized cardinal property on the class $\mathscr{B}$ such that*

(i) *f is either increasing or decreasing;*

(ii) *for each $G \in \mathscr{G}_0$, the condition* (c_1) *is satisfied.*

Then $T(f, \alpha)$ is a radical class.

P r o o f. If f is increasing, then the assertion is an easy consequence of (A) and of 3.1. If f is decreasing, then we have to apply 4.2 instead of (A). □

5. α-compactness

Let $\alpha \neq 0$ be a cardinal and let $\mathscr{L}_0$ be the class of all lattices having the least element.

The notion of α-compactness of subsets of a Boolean algebra was dealt with by Pierce [9].

We slightly modify the definition from [9] in order to have the possiblity to apply this notion for subsets of lattices belonging to $\mathscr{L}_0$.

Let $L \in \mathscr{L}_0$ and let x_0 be the least element of L. A nonempty subset D of L will be called α-compact if, whenever C is a nonempty subset of D such that

(i) $\operatorname{card} C < \alpha$,

(ii) $\inf F > x_0$ for each nonempty finite subset F of C,

then there exists a lower bound x of C with $x_0 < x$.

Let $L \in \mathscr{B}$. If L is α-compact for each cardinal $\alpha \neq 0$, then we put $f_c L = \infty$. Otherwise there exists a least cardinal β such that L fails to be β-compact; in this case we set $f_c L = \beta$. Then f_c is a decreasing generalized cardinal property on the class $\mathscr{B}$.

5.1. Lemma. *Let $G \in \mathscr{G}$, $a, b \in G$, $0 < a$, $0 < b$. Suppose that the intervals $[0, a]$ and $[0, b]$ are α-compact. Then the interval $[0, a + b]$ is α-compact as well.*

Proof. Let C be a nonempty subset of $[0, a + b]$ satisfying the conditions (i) and (ii) above. For each $c \in C$ we denote

$$c^1 = a \wedge c, \quad c^2 = a \vee c, \quad c^3 = -c^1 + c.$$

We have $-c^1 + c = -a + c^2$, hence

$$c^3 = -a + c^2.$$

We distinguish two cases.

a) There exists $c_1 \in C$ with $c_1^1 = 0$.

b) $c_1^1 > 0$ for each $c_1 \in C$.

Suppose that a) is valid. Then $c_2^3 > 0$ for each $c_2 \in C$. Indeed, assume that $c_1, c_2 \in C$ and $c_1^1 = 0 = c_2^3$. Then

$$c_2 = c_2^1 + c_2^3 = c_2^1 \in [0, a],$$

whence

$$0 \leqslant c_1 \wedge c_2 \leqslant c_1 \wedge a = c_1 = 0$$

and thus $c_1 \wedge c_2 = 0$, which contradicts the condition (ii). Thus $c_2^3 > 0$. Further, $c^2 > a$ for each $c \in C$.

Let c_1 be as above. Let F be a nonempty finite subset of C. Denote

$$C^2 = \{c^2 \colon\ c \in C\}, \quad F^2 = \{c^2 \colon\ c \in F\}.$$

Then we have $\operatorname{card} C^2 < \alpha$.

In view of (ii) there exists $f_0 \in G$ such that

$$0 < f_0 \leqslant c \quad \text{for each } c \in F. \tag{1}$$

In particular, we have

$$f_0 \leqslant c_1.$$

Then

$$0 \leqslant f_0 \wedge a \leqslant c_1 \wedge a = 0,$$

whence

$$f_0 \wedge a = 0. \tag{2}$$

If $f_0 \vee a = a$, then $f_0 \leqslant a$ and hence in view of (2) we obtain $f_0 = 0$, which is a contradiction. Thus $a < f_0 \vee a$ and according to (1) we have

$$f_0 \vee a \leqslant c^2 \quad \text{for each } c^2 \in F^2.$$

Therefore, the set C^2 satisfies the conditions (i) and (ii) with respect to the interval $[a, a+b]$. Since this interval is isomorphic to $[0, b]$ and $[0, b]$ is α-compact we infer that there exists $z \in [a, a+b]$ such that

$$a < z \leqslant c^2 \quad \text{for each } c^2 \in C^2.$$

Hence

$$0 < -a + z \leqslant -a + c^2 = c^3 \quad \text{for each } c \in C.$$

Because $c = c^1 + c^3$ we get

$$-a + z \leqslant c \quad \text{for each } c \in C,$$

which yields that the interval $[0, a+b]$ is α-compact.

Now suppose that b) holds. Again, let F be a nonempty finite subset of C. Let f_0 be as above, i.e., $0 < f_0 \leqslant c$ for each $c \in F$.

If (2) is valid, then we can apply the same steps as in the case a). Assume that (a) fails to be valid, thus $0 < f_0 \wedge a$. Then

$$f_0 \wedge a \leqslant c \wedge a \leqslant c \quad \text{for each } c \in C,$$

which completes the proof. $\square$

5.2. Lemma. *Let $G \in \mathscr{G}$, $t_n \in G$ $(n \in \mathbb{N})$, $0 < t_1 \leqslant t_2 \leqslant t_3 \ldots$, $t = \bigvee_{n \in \mathbb{N}} t_n$. Suppose that the intervals $[0, t_1]$ and $[t_n, t_{n+1}]$ $(n = 1, 2, \ldots)$ are α-compact. Then the interval $[0, t]$ is α-compact.*

Proof. Using induction we infer from 5.1 that for each $n \in \mathbb{N}$ the interval $[0, t_n]$ is α-compact.

Let C be a nonempty subset of $[0, t]$ such that the conditions (i) and (ii) above are satisfied.

For each $c \in C$ we have

$$c = c \wedge t = c \wedge \bigvee_{n \in \mathbb{N}} t_n = \bigvee_{n \in \mathbb{N}} (c \wedge t_n),$$

hence there exists $n(c) \in \mathbb{N}$ such that

$$c \wedge t_{n(c)} > 0.$$

Let F be a nonempty finite subset of C. Put

$$n(0) = \max\{n(c) \colon\ c \in F\}.$$

Thus

$$c \wedge t_{n(0)} > 0 \quad \text{for each } \ c \in F.$$

Now we can apply an argument analogous that in the part of the proof of 5.1 which was dealing with the case b) (let us remark that we consider the interval $[0, t_{n(0)}]$ instead of $[0, a]$). □

From 5.1 we immediately obtain

5.3. Proposition. *Let $G \in \mathscr{G}$, $a, b \in G$, $0 < a < b$. Suppose that $f_c[0, a] = f_c[0, b] = \beta$. Then $f_c[0, a + b] = \beta$.*

Similarly, 5.2 yields

5.4. Proposition. *Let $G \in \mathscr{G}$, $t_n \in G$ $(n = 1, 2, \ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $t = \bigvee_{n \in \mathbb{N}} t_n$. Suppose that $f_c[0, t_1] = \beta$. Further, suppose that $f_c[t_n, t_{n+1}] = \beta$ whenever $n \in \mathbb{N}$ and $t_n < t_{n+1}$. Then $f_0[0, t] = \beta$.*

In view of 5.3 and 5.4 we conclude that the assertions of (A) and (B) hold for the generalized cardinal property f_c.

6. The properties π_1' and π_2'

The properties π_1 and π_2 have been investigated by Pierce [9] for complete Boolean algebras and by the author [7] for generalized Boolean algebras.

We modify the corresponding definitions in order that these notions be applicable for lattices belonging to the class $\mathscr{B}$.

Let $L \in \mathscr{B}$ and suppose that x_0 is the least element of L.

A subset $Z \neq \emptyset$ of L is called disjoint if $z_1 \wedge z_2 = x_0$ whenever z_1 and z_2 are distinct elements of Z.

A subset X of L will be called strongly dense in L if for each element $y \in L$ with $y \neq x_0$ there exists $x \in X$ such that $x_0 < x \leqslant y$.

For each $L \in \mathscr{B}$ we denote

$$\begin{aligned}\pi_1 L &= \min\{\alpha \in K\colon\ D \subseteq L,\ \ D \text{ disjoint implies } \operatorname{card} D \leqslant \alpha\},\\ \pi_2 L &= \min\{\operatorname{card} D\colon\ D \text{ is strongly dense in } L\}.\end{aligned}$$

Further, we put

$$\pi_1' L = \max\{\pi_1 L, \aleph_0\}, \quad \pi_2' L = \max\{\pi_2 L, \aleph_0\}.$$

Then both π_1' and π_2' are increasing cardinal properties on the class $\mathscr{B}$.

6.1. Lemma. *Let $G \in \mathscr{G}$, $0 < t_i \in G$ $(i = 1, 2)$. Assume that*

$$\pi_1'[0, t_i] = \beta \quad \textit{for } i = 1, 2.$$

Then $\pi_1'[0, t_1 + t_2] = \beta$.

Proof. For $D \subseteq [0, t_1 + t_2]$ and $d \in D$ we denote

$$\begin{aligned} d^1 &= d \wedge t_1, \quad d^2 = d \vee t_1,\\ D^1 &= \{d^1\colon\ d \in D\}, \quad D^2 = \{d^2\colon\ d \in D\},\\ D^{10} &= \{d \in D\colon\ d^1 > 0\}, \quad D^{20} = \{d \in D\colon\ d^2 > t_1\}.\end{aligned}$$

Assume that the set D is disjoint in $[0, t_1 + t_2]$. Then D^1 is disjoint in $[0, t_1]$, thus $\operatorname{card} D^1 \leqslant \beta$. If d_1 and d_2 are distinct elements of D, then

$$d_1^2 \wedge d_2^2 = (d_1 \vee t_1) \wedge (d_2 \vee t_1) = (d_1 \wedge d_2) \vee t_1 = t_1,$$

whence D^2 is a disjoint subset of the set $[t_1, t_1 + t_2]$. Since the interval $[t_1, t_2 + t_2]$ is isomorphic to $[0, t_2]$, we infer that $\operatorname{card} D^2 \leqslant \beta$.

We have

$$D \subseteq D^{10} \cup D^{20} \cup \{0\}. \tag{1}$$

Let $d \in D$. If $d^1 = d^2 = 0$, then $d = 0$. If $d^1 > 0$ and d_1 is an element of D such that $d^1 = d_1^1$, then $d_1 = d$. Hence $\operatorname{card} D^{10} \leqslant \operatorname{card} D^1$. Similarly, $\operatorname{card} D^{20} \leqslant \operatorname{card} D^2$. Thus in view of (1) we obtain $\operatorname{card} D \leqslant \beta$. Hence

$$\pi_1'[0, t_1 + t_2] \leqslant \beta.$$

Since π_1' is increasing we conclude that $\pi_1'[0, t_1 + t_2] = \beta$. □

6.2. Lemma. *Let $G \in \mathscr{G}$, $t_n \in G$ $(n = 1, 2, \ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $\bigvee\limits_{n\in\mathbb{N}} t_n = t$. Suppose that $\pi_1'[0, t_1] = \beta$ and*

$$\pi_1'[t_n, t_{n+1}] = \beta \quad \textit{whenever } n \in \mathbb{N} \quad \textit{and } t_n < t_{n+1}.$$

Then $\pi_1'[0, t] = \beta$.

Proof. Let $\emptyset \neq D \subseteq [0, t]$ and $d \in D$. For $n \in \mathbb{N}$ we put $d^n = t \wedge t_n$. Then we have

$$d = d \wedge t = d \wedge \bigvee_{n\in\mathbb{N}} t_n = \bigvee_{n\in\mathbb{N}} (d \wedge t_n) = \bigvee_{n\in\mathbb{N}} d^n.$$

This yields that if $d^n = 0$ for each $n \in \mathbb{N}$, then $d = 0$.

Further, for each $n \in \mathbb{N}$ we put

$$D^n = \{d^n \colon d \in D\}, \quad D^{n0} = \{d \in D \colon d^n > 0\}.$$

By applying 6.1 and the induction we obtain that

$$\pi_1'[0, t_n] = \beta \quad \text{for each } n \in \mathbb{N}.$$

Assume that the set D is disjoint in $[0, t]$. Then each D^n is a disjoint subset of $[0, t_n]$. Hence $\operatorname{card} D^n \leqslant \beta$. Also, $\operatorname{card} D^{n0} \leqslant \operatorname{card} D^n$. Since

$$D \subseteq \{0\} \cup \bigcup_{n\in\mathbb{N}} D^{n0}$$

we obtain $\operatorname{card} D \leqslant \beta$. Since π_1' is increasing we conclude that the relation $\pi_1'[0, t] = \beta$ is valid. □

6.3. Lemma. *Let $G \in \mathscr{G}$, $0 < t_i \in G$, $\pi_2'[0,t_i] = \beta$ for $i = 1,2$. Then $\pi_2'[0, t_1 + t_2] = \beta$.*

P r o o f. Let $i \in \{1,2\}$. There exists a subset A_i of $[0,t_i]$ such that A_i is strongly dense in $[0,t_i]$ and $\operatorname{card} A_i = \beta$ $(i = 1,2)$.

Since the interval $[t_1, t_1 + t_2]$ is isomorphic to $[0,t_2]$ there exists a strongly dense subset A_2' of $[t_1, t_1 + t_2]$ with $\operatorname{card} A_2' = \beta$. Denote

$$A_3 = \{-t_1 + a_2' \colon\ a_2' \in A_2'\}, \quad A = A_1 \cup A_3.$$

Hence $A \subseteq [0, t_1 + t_2]$ and $\operatorname{card} A_3 = \beta$.

Let $0 < x \in [0, t_1 + t_2]$. Put

$$x^1 = t_1 \wedge x, \quad x^2 = t_1 \vee x, \quad x^3 = -t_1 + x_2.$$

Then either $x^1 > 0$ or $x^2 > t_1$. Further, $x = x^1 + x^3$.

a) Assume that $x^1 > 0$. Thus there exists $a_1 \in A_1$ with $0 < a_1 \leqslant x^1$. Therefore $a_1 \leqslant x$.

b) Suppose that $x^1 = 0$. Then $x^3 = x$, whence $x^3 > 0$ and $x^2 > t_1$. There exists $a_2' \in A_2'$ with $t_1 < a_2' \leqslant x^2$. Put $-t + a_2' = y$. Thus $y \in A_3$ and

$$0 < y \leqslant -t + x^2 = x^3 = x.$$

We conclude that A is strongly dense in $[0, t_1 + t_2]$ and then $\pi_2'[0, t_1 + t_2] \leqslant \beta$. Since π_2' is increasing we must have $\pi_2'[0, t_1 + t_2] = \beta$. □

6.4. Lemma. *Let $G \in \mathscr{G}$, $t_n \in G$ $(n = 1,2,\ldots)$, $0 < t_1 \leqslant t_2 \leqslant \ldots$, $\bigvee\limits_{n\in\mathbb{N}} t_n = t$. Let $\pi_2'[0,t_1] = \beta$ and $\pi_2'[t_n, t_{n+1}] = \beta$ whenever $n \in \mathbb{N}$ and $t_n < t_{n+1}$. Then $\pi_2'[0,t] = \beta$.*

P r o o f. By using 6.1 and the induction we infer that for each $n \in \mathbb{N}$ we have

$$\pi_2'[0,t_n] = \beta,$$

hence there exists a strongly dense subset A_n of $[0,t_n]$ with $\operatorname{card} A_n = \beta$. Denote

$$A = \bigcup_{n\in\mathbb{N}} A_n.$$

Let $0 < x \in [0,t]$. Then $x = \bigvee x^n$ $(n \in \mathbb{N})$, where $x^n = x \wedge t_n$. Thus there exists $n \in \mathbb{N}$ with $x^n > 0$. Since $x^n \in [0,t_n]$, there is $a_n \in A_n$ with $0 < a_n \leqslant x^n \leqslant x$. Therefore the set $A = \bigcup\limits_{n\in\mathbb{N}} A_n$ is a strongly dense subset of $[0, t_1 + t_2]$. We have $\operatorname{card} A = \beta$. Since π_2' is increasing, we conclude that $\pi_2'[0,t] = \beta$. □

Let $G \in \mathscr{G}$. In view of 6.1–6.4 the conditions (c_1) and (c_2) from Section 3 are satisfied for π_1' and π_2'. Therefore according to 4.2 and 4.4, the assertion of (A) and (B) hold for π_1' and π_2'.

Before returning to the cardinal properties π_1 and π_2 we need some auxiliary results.

6.5. Lemma. *Let G be an archimedean lattice ordered group and let $0 < x \in G$ be such that the interval $[0, x]$ is a chain. Then*

(i) *there exists a uniquely determined linearly ordered ℓ-subgroup H_x of G such that $[0, x] \subseteq H_x$;*

(ii) *H_x is a direct factor of the lattice ordered group G;*

(iii) *if $0 < y \in G$, $y \notin H_x$ and if $[0, y]$ is a chain, then $H_x \cap H_y = \{0\}$.*

Proof. These assertions have been obtained in [5]. □

It is well-known that each archimedean linearly ordered group is isomorphic to an ℓ-subgroup of R; hence each H_x has this property.

We denote by X the set of all elements $0 < x \in G$ such that $[0, x]$ is a chain. Put

$$A = X^{\delta\delta}, \quad B = X^{\delta}.$$

Suppose that the lattice ordered group G is complete. Then G is strongly projectable, whence

$$G = A \times B.$$

We remark that the set X can be empty; in such case we have $A = \{0\}$ and $B = G$.

6.6. Lemma. *Suppose that the lattice ordered group G is complete and that $X \neq \emptyset$. Then A is a complete subdirect product of lattice ordered groups H_x ($x \in X$).*

Proof. Let $g \in A$. We put $\varphi(g) = (g(H_x))_{x \in X}$, where $g(H_x)$ is the component of g in the direct factor H_x (cf. 6.5 (ii)); recall that each complete lattice ordered group is archimedean). Thus φ is a homomorphism of A into the direct product

$$\prod_{x \in X} H_x.$$

Suppose that $g_1 \in G$ and $\varphi(g_1) = 0$. Then $\varphi(|g_1|) = 0$ and hence $|g_1| \wedge x = 0$ for each $x \in X$. Thus $|g| \in X^{\delta} = B$, whence $|g| = 0$ and so $g = 0$. Therefore φ is an isomorphism of A into $\prod\limits_{x \in X} H_x$.

Let $x(0) \in X$ and $y \in H_{x(0)}$. Then we have

$$y(H_x) = \begin{cases} y & \text{if } x = x(0), \\ 0 & \text{otherwise.} \end{cases}$$

Thus φ yields a complete subdirect product decomposition of A. $\square$

Now let $0 < z \in G$ and suppose that $\pi_1[0, z]$ is finite. If $[0, z]$ fails to be a chain, then there are $0 < z_i \in G$ $(i = 1, 2)$ such that $z_1 \wedge z_2 = 0$ and $z_1 \vee z_2 \leqslant z$. We clearly have

$$\pi_1[0, z] \geqslant \pi_1[0, z_1] + \pi_1[0, z_2]$$

and $\pi_1[0, z_1] > 0$, $\pi_1[0, z_2] > 0$. By the obvious induction we get

6.7. Lemma. *Let $0 < z \in G$ such that $\pi_1[0, z]$ is finite. Then there exists $0 < z_1 \in G$ such that $z_1 \leqslant z$ and $[0, z_1]$ is a chain.*

If z_1 is as in 6.7, then $\pi_1[0, z_1] = 1$ and the lattice $\ell(H_{z_1})$ is π_1-homogeneous. Further, in view of the definition of B we infer that whenever $[u, v]$ is a nontrivial interval of B, then $[u, v]$ cannot be a chain. Hence

$$\pi_1[u_1, v_1] \geqslant \aleph_0$$

for each nontrivial interval $[u_1, v_1]$ of B; thus

$$\pi_1'[u_1, v_1] = \pi_1[u_1, v_1].$$

Summarizing, we have

6.8. Theorem. *Let $\{0\} \neq G \in \mathscr{G}$ and suppose that G is complete. Then G can be represented as a complete subdirect product of a system $(H_i)_{i \in I}$ of lattice ordered groups such that*

(i) *all H_i are π_1-homogeneous;*
(ii) *if $i \in I$, $u, v \in H_i$, $u < v$ and $\pi_1[u, v]$ is finite, then $\pi_1[u, v] = 1$ and H_i is isomorphic to an ℓ-subgroup of R.*

Now let us consider the cardinal property π_2. It is easy to verify that for each $[a, b] \in \mathscr{B}$ we have

$$\pi_1[a, b] \leqslant \pi_2[a, b].$$

Let G be a complete lattice ordered group and let A, B be as above. Hence if $[u, v]$ is a nontrivial interval of B, then $\pi_2[a, b]$ is infinite, whence $\pi_2'[a, b] = \pi_2[a, b]$.

Let $\mathbb{Z}$ have the usual meaning; i.e., $\mathbb{Z}$ is the additive group of all integers with the natural linear order.

If $x \in X$, then we have two possibilities:

(i) H_x is isomorphic to $\mathbb{Z}$; then H_x is π_2-homogeneous and $\pi_2[a, b] = 1$ for each nontrivial interval $[a, b]$ of H_x.

(ii) H_x fails to be isomorphic with $\mathbb{Z}$. Then H_x is π_2-homogeneous and $\pi_2[a, b]$ is infinite for each nontrivial interval $[a, b]$ of H_x; thus $\pi_2[a, b] = \pi'_2[a, b]$.

Therefore we conclude:

6.9. Theorem. *Let $\{0\} \neq G$ and suppose that G is complete. Then G can be represented as a complete subdirect product of a system $(H_j)_{j \in J}$ of lattice ordered groups such that*

(i) *all H_j are π_2-homogeneous;*

(ii) *if $j \in J$, $u, v \in H_j$, $u < v$ and if $\pi_2[u, v]$ is finite, then $\pi_2[u, v] = 1$ and H_j is isomorphic to $\mathbb{Z}$.*

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