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ON COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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Abstract. Some strong convergence theorems of common fixed points of asymptotically nonexpansive mappings in the intermediate sense are obtained. The results presented in this paper improve and extend the corresponding results in Huang, Khan and Takahashi, Chang, Schu, and Rhoades.

Keywords: common fixed point, iteration, asymptotically nonexpansive mapping in the intermediate sense, uniformly convex Banach space

MSC 2000: 47H10

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E. Let T be a mapping of C into itself and let $F(T)$ denote the set of fixed points of T. Then T is said to be *asymptotically nonexpansive with a sequence* $\{k_n\}_{n=1}^{\infty}$ [3] if $k_n \in [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ are such that

$$\|T^n x - T^n y\| \leqslant k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The weaker definition (cf. [5]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leqslant 0 \tag{1.2}$$

for every $x \in C$, and that T^N be continuous for some $N \geqslant 1$. Consider a definition somewhere between these two: T is called *asymptotically nonexpansive in the*

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intermediate sense [1] if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leqslant 0. \tag{1.3}$$

It is known [5] that if E is a uniformly convex Banach space and T is a selfmapping of a bounded closed convex subset C of E which is asymptotically nonexpansive in the intermeditae sense, then $F(T) \neq \emptyset$.

Recently, for each given mapping $T_j\colon C \to C$, $j = 1, 2, \ldots, r$, Huang [4] considered the following iteration scheme with errors in the sense of [12] generated by $T_1, T_2, \ldots, T_r$ as follows: let $U_{n(0)} = I$, where I is the identity mapping,

$$\begin{aligned} U_{n(1)} &= a_{n(1)} I + b_{n(1)} T_1^n U_{n(0)} + c_{n(1)} u_{n(1)}, \\ U_{n(2)} &= a_{n(2)} I + b_{n(2)} T_2^n U_{n(1)} + c_{n(2)} u_{n(2)}, \\ &\vdots \\ U_{n(r)} &= a_{n(r)} I + b_{n(r)} T_r^n U_{n(r-1)} + c_{n(r)} u_{n(r)}, \\ x_1 \in C, \quad x_{n+1} &= a_{n(r)} x_n + b_{n(r)} T_r^n U_{n(r-1)} x_n + c_{n(r)} u_{n(r)}, \quad n \geqslant 1. \end{aligned} \tag{1.4}$$

Here, $\{u_{n(j)}\}_{n=1}^{\infty}$ is a bounded sequence in C for each $j = 1, 2, \ldots, r$, and $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$ and $\{c_{n(j)}\}_{n=1}^{\infty}$ are three sequences in $[0,1]$ satisfying the condition

$$a_{n(j)} + b_{n(j)} + c_{n(j)} = 1 \tag{1.5}$$

for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, r$. This scheme contains the modified Mann and Ishikawa iteration methods with errors in the sense of [12] (cf. [7]): for $r = 1$, our scheme reduces to a Mann-Xu type iteration and for $r = 2$, $T_1 = T_2$ to an Ishikawa-Xu type iteration.

Using the iteration scheme (1.4), Huang [4, Theorem 2.1] obtained the following result. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_j\colon C \to C$ an asymptotically nonexpansive mapping with a sequence $\{k_{n(j)}\}_{n=1}^{\infty}$ for each $j = 1, 2, \ldots, r$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $k_n := \max_{1\leqslant j\leqslant r} \{k_{n(j)}\} \geqslant 1$ and $\bigcap_{j=1}^{r} F(T_j) \neq \emptyset$. Let $\{u_{n(j)}\}_{n=1}^{\infty}$ be a bounded sequence in C for each $j = 1, 2, \ldots, r$ and let $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$ and $\{c_{n(j)}\}_{n=1}^{\infty}$ be three sequences in $[0,1]$ satisfying the following conditions:

(i) $a_{n(j)} + b_{n(j)} + c_{n(j)} = 1$ for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, r$;

(ii) $\sum_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1, 2, \ldots, r$;

(iii) $0 < a \leqslant \alpha_{n(j)} \leqslant b < 1$ for all $n \in \mathbb{N}$, each $j = 1, 2, \ldots, r$, and some constants a, b, where $\alpha_{n(j)} := b_{n(j)} + c_{n(j)}$.

Suppose that $\{x_n\}$ is given by (1.4). Then $\lim\limits_{n\to\infty} \|T_j x_n - x_n\| = 0$ for each $j = 1, 2, \ldots, r$. So Huang extended [10, Lemma 1.5] and [9, Theorem 1].

In this paper, we first consider the behaviour of an iteration scheme (see Theorem 2.1 below) which improves [4, Theorem 2.1]. Then we generalize [6, Theorem 2], [2, Theorem 1.2], [10, Theorem 2.2 and 2.4], and [9, Theorem 2 and 3].

In the sequel, we shall need the following results.

Lemma 1.1 (see [11, Lemma 1] and [8, Lemma 1]). *Let $\{\varrho_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$\varrho_{n+1} \leqslant \varrho_n + \sigma_n, \quad n \geqslant 1. \tag{1.6}$$

If $\sum\limits_{n=1}^{\infty} \sigma_n < \infty$, then $\lim\limits_{n\to\infty} \varrho_n$ exists. In particular, if $\{\varrho_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim\limits_{n\to\infty} \varrho_n = 0$.

Lemma 1.2 (see [10, Lemma 1.3]). *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in a uniformly convex Banach space E and let $\{t_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $0 < a \leqslant t_n \leqslant b < 1$ for all $n \in \mathbb{N}$. Suppose that $\lim\limits_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = \varepsilon$ exists. If $\limsup\limits_{n\to\infty} \|x_n\| \leqslant \varepsilon$ and $\limsup\limits_{n\to\infty} \|y_n\| \leqslant \varepsilon$, then $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$.*

2. Main results

We now prove the following results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T_1, T_2, \ldots, T_r$ be asymptotically nonexpansive mappings in the intermediate sense of C into itself such that $\bigcap\limits_{j=1}^{r} F(T_j) \neq \emptyset$. Put*

$$d_n = \max\Big\{\max_{1\leqslant j\leqslant r} \sup_{x,y\in C} (\|T_j^n x - T_j^n y\| - \|x - y\|), 0\Big\} \tag{2.1}$$

so that $\sum\limits_{n=1}^{\infty} d_n < \infty$. Let $\{u_{n(j)}\}_{n=1}^{\infty}$ be a bounded sequence in C for each $j = 1, 2, \ldots, r$ and let $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$ and $\{c_{n(j)}\}_{n=1}^{\infty}$ be three sequences in $[0, 1]$ satisfying the following conditions:

(i) *$a_{n(j)} + b_{n(j)} + c_{n(j)} = 1$ for all $n \in \mathbb{N}$ and each $j = 1, 2, \ldots, r$;*

(ii) *$\sum\limits_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1, 2, \ldots, r$;*

(iii) $0 < a \leqslant b_{n(j)} \leqslant b < 1$ *for all* $n \in \mathbb{N}$*, each* $j = 1, 2, \ldots, r$ *and some constants* $a, b \in (0, 1)$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ *is given by* (1.4). *Then* $\lim\limits_{n\to\infty} \|T_j x_n - x_n\| = 0$ *for each* $j = 1, 2, \ldots, r$*. Moreover, if* $\{x_n\}$ *has a subsequence which converges strongly to a point* z*, then* $\{x_n\}$ *converges strongly to* $z \in \bigcap\limits_{j=1}^{r} F(T_j)$.

Proof. Let $p \in \bigcap\limits_{j=1}^{r} F(T_j)$. Since $\{u_{n(j)}\}_{n=1}^{\infty}$ is bounded for each $j = 1, 2, \ldots, r$, there exists a constant $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \{\|u_{n(j)} - p\| \colon\ j = 1, 2, \ldots, r\} \leqslant M.$$

Then we have

$$\begin{aligned}
(2.2)\quad \|x_{n+1} - p\| &= \|U_{n(r)} x_n - p\| \\
&= \|a_{n(r)} x_n + b_{n(r)} T_r^n U_{n(r-1)} x_n + c_{n(r)} u_{n(r)} - p\| \\
&\leqslant a_{n(r)} \|x_n - p\| + b_{n(r)} \|T_r^n U_{n(r-1)} x_n - p\| + c_{n(r)} \|u_{n(r)} - p\| \\
&\leqslant (1 - b_{n(r)}) \|x_n - p\| + b_{n(r)} \|U_{n(r-1)} x_n - p\| \\
&\quad + b_{n(r)} d_n + c_{n(r)} M \\
&\ \vdots \\
&\leqslant (1 - b_{n(r)} b_{n(r-1)} \ldots b_{n(1)}) \|x_n - p\| \\
&\quad + b_{n(r)} b_{n(r-1)} \ldots b_{n(1)} \|U_{n(0)} x_n - p\| \\
&\quad + (b_{n(r)} + b_{n(r)} b_{n(r-1)} + \ldots + b_{n(r)} b_{n(r-1)} \ldots b_{n(1)}) d_n \\
&\quad + (c_{n(r)} + b_{n(r)} c_{n(r-1)} + \ldots + b_{n(r)} b_{n(r-1)} \ldots b_{n(2)} c_{n(1)}) M \\
&\leqslant \|x_n - p\| + r d_n + M \sum_{j=1}^{r} c_{n(j)}.
\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty} d_n < \infty$ and $\sum\limits_{n=1}^{\infty} \sum\limits_{j=1}^{r} c_{n(j)} < \infty$, thus, by Lemma 1.1, we conclude that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists. Put $\varepsilon = \lim\limits_{n\to\infty} \|x_n - p\|$ and consider a fixed j with $1 \leqslant j \leqslant r-1$. Then we obtain

$$\|U_{n(j)} x_n - p\| \leqslant \|x_n - p\| + j d_n + M \sum_{i=1}^{j} c_{n(i)},$$

and hence

$$(2.3)\qquad \limsup_{n\to\infty} \|U_{n(j)} x_n - p\| \leqslant \varepsilon.$$

Further, since

$$\begin{aligned}\|x_{n+1}-p\| \leqslant (1-b_{n(r)}b_{n(r-1)}\dots b_{n(j+1)})\|x_n-p\| \\ + b_{n(r)}b_{n(r-1)}\dots b_{n(j+1)}\|U_{n(j)}x_n-p\| \\ + (r-j)d_n + M\sum_{i=1}^{r-j} c_{n(i)},\end{aligned}$$

we have

$$\begin{aligned}\|x_n-p\| &\leqslant \|U_{n(j)}x_n-p\| + \frac{\|x_n-p\|-\|x_{n+1}-p\|+(r-j)d_n+M\sum_{i=1}^{r-j}c_{n(i)}}{b_{n(r)}b_{n(r-1)}\dots b_{n(j+1)}} \\ &\leqslant \|U_{n(j)}x_n-p\| + \frac{\|x_n-p\|-\|x_{n+1}-p\|+(r-j)d_n+M\sum_{i=1}^{r-j}c_{n(i)}}{a^{r-j}},\end{aligned}$$

and hence

$$\varepsilon \leqslant \liminf_{n\to\infty}\|U_{n(j)}x_n-p\|. \tag{2.4}$$

From (2.3) and (2.4) we obtain $\varepsilon = \lim_{n\to\infty}\|U_{n(j)}x_n-p\|$ for each $j=1,2,\dots,r$. So, we have

$$\begin{aligned}\varepsilon &= \lim_{n\to\infty}\|a_{n(j)}x_n + b_{n(j)}T_j^nU_{n(j-1)}x_n + c_{n(j)}u_{n(j)} - p\| \\ &= \lim_{n\to\infty}\Big\|b_{n(j)}(T_j^nU_{n(j-1)}x_n-p) + (1-b_{n(j)})\Big(\frac{a_{n(j)}x_n+c_{n(j)}u_{n(j)}}{1-b_{n(j)}}-p\Big)\Big\|.\end{aligned}$$

Since

$$\limsup_{n\to\infty}\|T_j^nU_{n(j-1)}-p\| \leqslant \limsup_{n\to\infty}(\|U_{n(j-1)}x_n-p\|+d_n) = \varepsilon$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\Big\|\frac{a_{n(j)}x_n+c_{n(j)}u_{n(j)}}{1-b_{n(j)}}-p\Big\| \\ &\qquad\leqslant \limsup_{n\to\infty}\Big(\frac{a_{n(j)}}{1-b_{n(j)}}\|x_n-p\| + \frac{c_{n(j)}}{1-b_{n(j)}}\|u_{n(j)}-p\|\Big) \\ &\qquad\leqslant \limsup_{n\to\infty}\Big(\|x_n-p\| + \frac{M}{1-b}c_{n(j)}\Big) = \varepsilon,\end{aligned}$$

thus, by Lemma 1.2, we get

$$\lim_{n\to\infty}\Big\|T_j^nU_{n(j-1)}x_n - \frac{a_{n(j)}x_n+c_{n(j)}u_{n(j)}}{1-b_{n(j)}}\Big\| = 0. \tag{2.5}$$

Let $N = \sup\limits_{n\in\mathbb{N}}\{\|u_{n(j)} - x_n\|\colon\ j = 1,2,\dots,r\} < \infty$. Since

$$\left\|\frac{a_{n(j)} + c_{n(j)}u_{n(j)}}{1-b_{n(j)}} - x_n\right\| \leqslant \frac{c_{n(j)}}{1-b_{n(j)}}\|u_{n(j)} - x_n\| \leqslant \frac{N}{1-b}c_{n(j)},$$

and hence

$$\lim_{n\to\infty}\left\|\frac{a_{n(j)}x_n + c_{n(j)}u_{n(j)}}{1-b_{n(j)}} - x_n\right\| = 0, \tag{2.6}$$

thus, from (2.5), (2.6) and the triangle inequality we obtain

$$\lim_{n\to\infty}\|T_j^nU_{n(j-1)}x_n - x_n\| = 0. \tag{2.7}$$

If $j=1$, we have $\lim\limits_{n\to\infty}\|T_1^nx_n - x_n\| = 0$. For any j with $2\leqslant j\leqslant r$, from

$$\begin{aligned}
\|T_j^nx_n - x_n\| &\leqslant \|T_j^nx_n - T_j^nU_{n(j-1)}x_n\| + \|T_j^nU_{n(j-1)}x_n - x_n\| \\
&\leqslant \|x_n - U_{n(j-1)}x_n\| + d_n + \|T_j^nU_{n(j-1)}x_n - x_n\| \\
&\leqslant b_{n(j-1)}\|T_{j-1}^nU_{n(j-2)}x_n - x_n\| + c_{n(j-1)}\|u_{n(j-1)} - x_n\| \\
&\quad + d_n + \|T_j^nU_{n(j-1)}x_n - x_n\| \\
&\leqslant b\|T_{j-1}^nU_{n(j-2)}x_n - x_n\| + Nc_{n(j-1)} + d_n + \|T_j^nU_{n(j-1)}x_n - x_n\|
\end{aligned}$$

we obtain

$$\lim_{n\to\infty}\|T_j^nx_n - x_n\| = 0. \tag{2.8}$$

Since

$$\begin{aligned}
\|x_n - T_j^nx_n\| &\leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - T_j^{n+1}x_{n+1}\| \\
&\quad + \|T_j^{n+1}x_{n+1} - T_j^{n+1}x_n\| + \|T_j^{n+1}x_n - T_jx_n\| \\
&\leqslant 2\|x_n - x_{n+1}\| + d_{n+1} + \|x_{n+1} - T_j^{n+1}x_{n+1}\| + \|T_j^{n+1}x_n - T_jx_n\| \\
&\leqslant 2(b\|T_r^nU_{n(r-1)}x_n - x_n\| + Nc_{n(r)}) + d_{n+1} + \|x_{n+1} - T_j^{n+1}x_{n+1}\| \\
&\quad + \|T_j^{n+1}x_n - T_jx_n\|,
\end{aligned}$$

the uniform continuity of T_j together with (2.7), (2.8) and the relations $c_{n(r)}\to 0$ as $n\to\infty$ and $d_{n+1}\to 0$ as $n\to\infty$ yields

$$\lim_{n\to\infty}\|x_n - T_jx_n\| = 0 \tag{2.9}$$

for each $j = 1,2,\dots,r$.

Suppose $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $\lim_{i\to\infty} \|x_{n_i} - z\| = 0$. Then $z \in C$ and by the continuity of T_j, $\lim_{i\to\infty} \|T_j x_{n_i} - T_j z\| = 0$ for each $j = 1, 2, \ldots, r$. Hence, $\lim_{i\to\infty} \|x_{n_i} - T_j x_{n_i}\| = \|z - T_j z\| = 0$ for each $j = 1, 2, \ldots, r$, so that $z \in \bigcap_{j=1}^{r} F(T_j)$. Thus, by (2.2), we have $\|x_{n+1} - z\| \leqslant \|x_n - z\| + rd_n + M \sum_{j=1}^{r} c_{n(j)}$, so that it follows from Lemma 1.1 that $\lim_{n\to\infty} \|x_n - z\| = 0$, i.e., $\{x_n\}$ converges strongly to $z \in \bigcap_{j=1}^{r} F(T_j)$. The proof is complete. □

For our next results, we shall need the following definition.

Definition. Let C be a nonempty closed subset of a Banach space E. A mapping $T\colon C \to C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.

Theorem 2.2. *Under the hypotheses of Theorem* 2.1, *assume that one of the following conditions is satisfied:*

(1) *C is compact;*
(2) *T_1 is semi-compact;*
(3) *T_1^m is compact for some $m \in \mathbb{N}$;*
(4) *there exists a nonempty compact convex subset K of E and some $\alpha \in (0, 1)$ such that*

$$d(T_j x, K) \leqslant \alpha d(x, K)$$

for all $x \in C$ and each $j = 1, 2, \ldots, r$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$.

Proof of Theorem 2.2. Proof of (1) and (2): They follows immediately from Theorem 2.1.

Proof of (3): Using the uniform continuity of T_1 and $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$, we can show that for each $l \geqslant 1$

$$\lim_{n\to\infty} \|T_1^l x_n - x_n\| = 0. \tag{2.10}$$

Since $\{x_n\}$ is bounded and T_1^m is compact, $\{T_1^m x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{T_1^m x_{n_i}\}_{i=1}^{\infty}$. Suppose $\lim_{i\to\infty} T_1^m x_{n_i} = z$. Then the inequality

$$\|x_{n_i} - z\| \leqslant \|x_{n_i} - T_1^m x_{n_i}\| + \|T_1^m x_{n_i} - z\| \tag{2.11}$$

yields $\lim_{i\to\infty} \|x_{n_i} - z\| = 0$, which implies by Theorem 2.1 that $\{x_n\}$ converges strongly to $z \in \bigcap_{j=1}^{r} F(T_j)$.

Proof of (4): As in the proof of [4, Theorem 2.10] by using Theorem 2.1, $\{x_n\}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$. □

Since every asymptotically nonexpansive mapping is uniformly continuous, we have the following result:

Corollary 2.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T_j \colon C \to C$ an asymptotically nonexpansive mapping with a sequence $\{k_{n(j)}\}_{n=1}^{\infty}$ for each $j = 1, 2, \ldots, r$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, where $k_n := \max_{1\leqslant j\leqslant r} \{k_{n(j)}\} \geqslant 1$ and $\bigcap_{j=1}^{r} F(T_j) \neq \emptyset$. Let $\{u_{n(j)}\}$, $\{a_{n(j)}\}$, $\{b_{n(j)}\}$, $\{c_{n(j)}\}$ and $\{x_n\}$ be as in Theorem 2.1. Assume that one of the following conditions is satisfied:*

(1) *C is compact;*
(2) *T_1 is semi-compact;*
(3) *T_1^m is compact for some $m \in \mathbb{N}$;*
(4) *there exists a nonempty compact convex subset K of E and some $\alpha \in (0, 1)$ such that*

$$d(T_j x, K) \leqslant \alpha d(x, K)$$

for all $x \in C$ and each $j = 1, 2, \ldots, r$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$.

P r o o f. Observe that

$$\sum_{n=1}^{\infty} d_n \leqslant \sum_{n=1}^{\infty} (k_n - 1) \operatorname{diam}(C) < \infty,$$

where $\operatorname{diam}(C) = \sup_{x,y\in C} \|x - y\| < \infty$. The conclusion now follows easily from Theorem 2.2. □

Remark 2.4. Corollary 2.3 extends the corresponding results of [6, Theorem 2], [2, Theorem 1.2], [10, Theorems 2.2 and 2.4], and [9, Theorems 2 and 3].

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