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A NONTRIVIAL SOLUTION FOR NEUMANN NONCOERCIVE HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper we consider Neumann noncoercive hemivariational inequalities, focusing on nontrivial solutions. We use the critical point theory for locally Lipschitz functionals.

Keywords: noncoercive hemivariational inequality, critical point theory, nontrivial solution, locally Lipschitz functionals

MSC 2000: 35J20, 35J85

1. Introduction

The problem under consideration is a hemivariational inequality of Neumann type. Let $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^1-boundary $\partial\Omega$,

$$(1) \qquad \begin{cases} -\operatorname{div}(\|Du(x)\|^{p-2}Du(x)) \in \partial j(x, u(x)) & \text{a.e. on } \Omega, \\ -\dfrac{\partial u}{\partial n_p} = 0 \quad \text{a.e. on } \partial\Omega, \quad 2 \leqslant p < \infty. \end{cases}$$

The study of hemivariational inequalities has been initiated and developed by P. D. Panagiotopoulos [10]. Such inequalities arise in physics when we have nonconvex, nonsmooth energy functionals. For applications one can see [11].

Many authors studied Dirichlet hemivariational inequalities. See for example [5], [6] and others. Here we are interested in finding nontrivial solutions for

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Neumann hemivariational inequalities. So our result seems to be the first in this direction.

In the next section we recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the subdifferential of Clarke.

2. Preliminaries

Let X be a Banach space and let Y be a subset of X. A function $f\colon Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$|f(y) - f(x)| \leqslant K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x; v)$, is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^o(x; v)$ is finite, positively homogeneous, subadditive and satisfies $|f^o(x; v)| \leqslant K\|v\|$. In addition, f^o satisfies $f^o(x; -v) = (-f)^o(x; v)$. Now we are ready to introduce the generalized gradient which is denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \{w \in X^*\colon\ f^o(x; v) \geqslant \langle w, v\rangle \ \text{ for all } \ v \in X\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following ones:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leqslant K$ for every w in $\partial f(x)$.

(b) For every v in X one has

$$f^o(x; v) = \max\{\langle w, v\rangle :\ w \in \partial f(x)\}.$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (PS)-condition introduced by Chang.

Definition. We say that a Lipschitz function f satisfies the Palais-Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and

$$\lambda(x_n) = \min_{w \in \partial f(x_n)} \|w\|_{X^*} \to 0$$

possesses a convergent subsequence.

The (PS)-condition can also be formulated as follows (see [3]).

$(\mathrm{PS})^*_{c,+}$: Whenever $(x_n) \subseteq X$, $(\varepsilon_n), (\delta_n) \subseteq \mathbb{R}_+$ are sequences with $\varepsilon_n \to 0$, $\delta_n \to 0$, and such that

$$\begin{aligned} &f(x_n) \to c, \\ &f(x_n) \leqslant f(x) + \varepsilon_n \|x - x_n\| \quad \text{if} \quad \|x - x_n\| \leqslant \delta_n, \end{aligned}$$

then (x_n) possesses a convergent subsequence: $x_{n'} \to \hat{x}$.

Similarly, we define the $(\mathrm{PS})^*_c$ condition from below, $(\mathrm{PS})^*_{c,-}$, by interchanging x and x_n in the above inequality. And finally, we say that f satisfies $(\mathrm{PS})^*_c$ provided it satisfies $(\mathrm{PS})^*_{c,+}$ and $(\mathrm{PS})^*_{c,-}$.

Note that these two definitions are equivalent when f is a locally Lipschitz functional.

The next theorem is a Mountain-Pass theorem for locally Lipschitz functionals.

Theorem 1. *If a locally Lipschitz functional $f\colon X \to \mathbb{R}$ on the reflexive Banach space X satisfies the* (PS)*-condition and the hypotheses*

(i) *there exist positive constants ϱ and a such that*

$$f(u) \geqslant a \quad \textit{for all } u \in X \quad \textit{with } \|u\| = \varrho;$$

(ii) $f(0) = 0$ *and there is a point $e \in X$ such that*

$$\|e\| > \varrho \quad \textit{and } f(e) \leqslant 0,$$

then there exists a critical value $c \geqslant a$ of f determined by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f(g(t))$$

where

$$G = \{g \in C([0,1], X)\colon\ g(0) = 0,\ g(1) = e\}.$$

In what follows we will use the well-known inequality

$$(2) \qquad \sum_{j=1}^{N}(a_j(\eta)-a_j(\eta'))(\eta_j-\eta'_j)\geqslant C|\eta-\eta'|^p$$

for $\eta,\eta'\in\mathbb{R}^N$, with $a_j(\eta)=|\eta|^{p-2}\eta_j$.

3. Existence theorem

Let $X=W^{1,p}(\Omega)$. Our hypotheses on j are as follows:

H(j): $j\colon\ \Omega\times\mathbb{R}\to\mathbb{R}$ is such that $x\to j(x,u)$ is measurable and $u\to j(x,u)$ is locally Lipschitz;

(i) for almost all $x\in\Omega$, all $u\in\mathbb{R}$ and all $v\in\partial j(x,u)$ we have $|v(x)|\leqslant a(x)$ with $a\in L^\infty(\Omega)$;

(ii) uniformly for almost all $x\in\Omega$ we have that for all $v\in\partial j(x,u)$ we have $v(x)u/|u|\to f_+(x)$ as $u\to\pm\infty$ where $f_+\in L^1(\Omega)$, $f_+\geqslant 0$ with strict inequality on a set of positive Lebesgue measure;

(iii) uniformly for almost all $x\in\Omega$ we have that

$$\limsup_{u\to 0}\frac{j(x,u)}{|u|^p}\leqslant\theta(x)$$

with $\theta(x)\in L^\infty(\Omega)$ and $\theta(x)\leqslant 0$ with strict inequality in a set of positive measure.

Remark 1. Note that hypothesis H(j) (iii) is crucial to use the mountain-pass theorem and moreover, mountain-pass theorem is crucial to prove the existence of a nontrivial solution.

Theorem 2. *If hypotheses* H(j) *hold, then problem* (1) *has a nontrivial solution* $u\in W^{1,p}(\Omega)$.

Proof. Let $\Phi\colon\ W^{1,p}(\Omega)\to\mathbb{R}$ and $\psi\colon\ W^{1,p}(\Omega)\to\mathbb{R}_+$ be defined by

$$\Phi(u)=-\int_\Omega j(x,u(x))\,\mathrm{d}x\quad\text{and}\quad\psi(u)=\frac{1}{p}\|Du\|_p^p.$$

Clearly Φ is locally Lipschitz (see [1]), and we can check that ψ is locally Lipschitz, too. Set $R=\Phi+\psi$.

Claim 1. $R(\cdot)$ satisfies the (PS)-condition (in the sense of Costa and Goncalves). We start with $(\mathrm{PS})_{c,+}$ first.

Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be such that $R(u_n) \to c$ when $n \to \infty$ and

$$R(u_n) \leqslant R(u) + \varepsilon_n \|u - u_n\| \quad \text{with } \|u - u_n\| \leqslant \delta_n.$$

The above inequality is equivalent to

$$R(u) - R(u_n) \geqslant -\varepsilon_n \|u - u_n\| \quad \text{with } \|u - u_n\| \leqslant \delta_n$$

with $\varepsilon_n, \delta_n \to 0$. Choose $u = u_n + \delta u_n$ with $\delta\|u_n\| \leqslant \delta_n$. Divide by δ. So, if $\delta \to 0$ we have

$$\lim_{\delta\to 0} \frac{R(u_n + \delta u_n) - R(u_n)}{\delta} \leqslant R^o(u_n; u_n).$$

Then we obtain

$$R^o(u_n; u_n) \geqslant -\varepsilon_n \|u_n\|. \tag{3}$$

For $(\mathrm{PS})_{c,-}$ we have the following assertion: Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be such that $R(u_n) \to c$ when $n \to \infty$ and

$$R(u) \leqslant R(u_n) + \varepsilon_n \|u - u_n\| \quad \text{with } \|u - u_n\| \leqslant \delta_n.$$

The above inequality is equivalent to

$$0 \leqslant (-R)(u) - (-R)(u_n) + \varepsilon_n \|u - u_n\| \quad \text{with } \|u - u_n\| \leqslant \delta_n.$$

Choose here $u = u_n - \delta u_n$ with $\delta\|u_n\| \leqslant \delta_n$. We obtain

$$0 \leqslant (-R)(u_n + \delta(-u_n)) - (-R)(u_n) + \varepsilon_n \delta \|u_n\|.$$

Divide this by δ. In the limit we have

$$0 \leqslant \lim_{\delta\to 0} \frac{(-R)(u_n + \delta(-u_n)) - (-R)(u_n)}{\delta} + \varepsilon_n \|u_n\|.$$

Note that $\lim\limits_{\delta\to 0} \delta^{-1}((-R)(u_n+\delta(-u_n))-(-R)(u_n)) \leqslant (-R)^o(u_n; -u_n) = R^o(u_n; u_n)$. So finally we obtain again (3).

Also, $p^{-1}\|D(u_n + \delta u_n)\|_p^p - p^{-1}\|Du_n\| = p^{-1}\|Du_n\|_p^p(1 - (1 + \delta)^p)$. So if we divide this by δ and let $\delta \to 0$ we obtain that the result is equal to $\|Du_n\|_p^p$. Finally, there exists $v_n(x) \in \partial\Phi(u_n)$ such that $\langle v_n, u_n\rangle = \Phi^o(u_n; u_n)$. Note that $v_n \in \partial(-\int_\Omega j(x, u_n(x))\,\mathrm{d}x) = -\partial \int_\Omega j(x, u_n(x)))\,\mathrm{d}x$. So, it follows from (3) that

$$\int_\Omega v_n u_n(x)\,\mathrm{d}x - \|Du_n\|_p^p \leqslant \varepsilon_n \|u_n\|,$$

for some $v_n \in \partial(\int_\Omega j(x, u_n(x))\,\mathrm{d}x)$.

Suppose that $\{u_n\} \subseteq W^{1,p}(\Omega)$ is unbounded. Then (at least for a subsequence) we may assume that $\|u_n\| \to \infty$. Let $y_n = u_n/\|u_n\|$, $n \geqslant 1$; it is easy to see that $\|y_n\| = 1$. By passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega), \quad y_n \to y \text{ in } L^p(\Omega), \quad y_n(x) \to y(x) \text{ a.e. on } \Omega \text{ as } n \to \infty$$

and $|y_n(x)| \leqslant k(x)$ a.e. on Ω with $k \in L^p(\Omega)$.

Recall that from the choice of the sequence $\{u_n\}$ we have $|R(u_n)| \leqslant M_1$ for some $M_1 > 0$ and all $n \geqslant 1$,

$$\frac{1}{p}\|Du_n\|_p^p - \int_\Omega j(x, u_n(x))\,\mathrm{d}x \leqslant M_1.$$

Divide by $\|u_n\|^p$. We obtain

$$\frac{1}{p}\|Dy_n\|n_p^p - \int_\Omega \frac{j(x, u_n(x))}{\|u_n\|^p}\,\mathrm{d}x \leqslant \frac{M_1}{\|u_n\|^p}. \tag{4}$$

We must show now that $\int_\Omega j(x, u_n(x))/\|u_n\|^p\,\mathrm{d}x \to 0$ as $n \to \infty$.

Using the Lebourg mean value theorem (see [2, p. 41, Theorem 2.3.7]) we obtain that for almost all $x \in \Omega$, all $u \in \mathbb{R}$ and for some $v \in \partial j(x, s)$ with $s \in (0, u)$ we have

$$|j(x, u) - j(x, 0)| \leqslant |\langle v, u\rangle| \leqslant a(x)|u|;$$

here we have used hypothesis H(j) (i).

So we obtain

$$|j(x, u)| \leqslant c_1 + c_2|u|$$

for some $c_1, c_2 > 0$. Note that $j(x, 0) \in L^\infty$.

So we can establish the estimate,

$$\begin{aligned}\left|\int_\Omega \frac{j(x, u_n(x))}{\|u_n\|^p}\,\mathrm{d}x\right| &\leqslant \int_\Omega \frac{|j(x, u_n(x))|}{\|u_n\|^p}\,\mathrm{d}x \leqslant \int_\Omega \frac{c_1 + c_2|u_n(x)|}{\|u_n\|n^p}\,\mathrm{d}x \\ &\leqslant \frac{c_3}{\|u_n\|^p} + \frac{c_4}{\|u_n\|^{p-1}},\end{aligned}$$

where we have used the fact that $W^{1,p}(\Omega)$ embeds continuously in $L^1(\Omega)$.

Thus, we have

$$\left|\int_\Omega \frac{j(x, u_n(x))}{\|u_n\|^p}\,\mathrm{d}x\right| \to 0.$$

So by passing to the limit as $n \to \infty$ in (4), we obtain

$$
\begin{aligned}
&\lim \frac{1}{p}\|Dy_n\|_p^p = 0\\
&\quad \Rightarrow \|Dy\|_p = 0 \quad \text{(recall that } Dy_n \xrightarrow{w} Dy \text{ in } L^p(\Omega,\mathbb{R}^{\mathbb{N}}) \text{ as } n \to \infty)\\
&\quad \Rightarrow y = \xi \in \mathbb{R}.
\end{aligned}
$$

Note that $y_n \to \xi$ in $W^{1,p}(\Omega)$ and since $\|y_n\| = 1$, $n \geqslant 1$ we infer that $\xi \neq 0$. We deduce that $|u_n(x)| \to +\infty$ a.e. on Ω as $n \to \infty$.

From the choice of the sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ we have

$$
\int_Z v_n(x)u_n(x)\,\mathrm{d}x - \|Du_n\|_p^p \leqslant \varepsilon_n\|u_n\| \tag{5}
$$

and

$$
\|Du_n\|_p^p - p\int_\Omega j(x,u_n(x))\,\mathrm{d}x \leqslant pM_1. \tag{6}
$$

Adding (5) and (6), we obtain

$$
\int_\Omega (v_n(x)u_n(x) - pj(x,u_n(x)))\,\mathrm{d}x \leqslant pM_1 + \varepsilon_n\|u_n\|.
$$

Divide this inequality by $\|u_n\|$. We have

$$
\int_\Omega \frac{v_n(x)}{\|u_n\|}u_n(x)\,\mathrm{d}x - \int_\Omega \frac{pj_1(x,u_n(x))}{\|u_n\|}\,\mathrm{d}x \leqslant \frac{1}{\|u_n\|}pM_1 - \varepsilon_n. \tag{7}
$$

From the Lebourg mean value theorem, for any $0 < \varepsilon < 1$ we have

$$
j(x,u_n(x)) = j(x,\varepsilon u_n(x)) + s_n(x)(1-\varepsilon)u_n(x)
$$

with $s_n(x) \in \partial j(x,\varphi_n(x))$ where $\varphi_n(x) = (1-c_n)u_n(x) + c_n\varepsilon u_n(x)$ with $0 < c_n < 1$. Note that

$$
|\varphi_n(x)| = |u_n(x)|(1 - c_n(1-\varepsilon)).
$$

Since $|u_n(x)| \to \infty$ we have that $|\varphi_n(x)| \to \infty$. Suppose now that $\xi > 0$. Then $u_n(x) \to \infty$ and $\varphi_n(x) \to \infty$.

From H(j) (ii) we can say that there exists some $M > 0$ such that for $u > M$ and for all $v \in \partial j(x,u)$ we have

$$
(f_+(x) - \varepsilon) \leqslant v(x) \leqslant (f_+(x) + \varepsilon).
$$

So for almost every $x \in \Omega$ we can find $n_0 = n_0(x)$ such that for $n \geqslant n_0$ we have

$$p(1-\varepsilon)u_n(x)(f_+(x)-\varepsilon) \leqslant ps_n(x)(1-\varepsilon)u_n(x) \\ \leqslant p(1-\varepsilon)u_n(x)(f_+(x)+\varepsilon).$$

Dividing this by $\|u_n\|$ we arrive at

$$p(1-\varepsilon)y_n(x)(f_+(x)-\varepsilon) \leqslant ps_n(x)(1-\varepsilon)y_n(x) \leqslant p(1-\varepsilon)y_n(x)(f_+(x)+\varepsilon).$$

Thus, for any $x \in \Omega$ we have

$$ps_n(x)(1-\varepsilon)y_n(x) \to p\xi f_+(x)$$

as $n \to \infty$ and $\varepsilon \to 0$.

Recall that $|j(x, \varepsilon u_n(x))| \leqslant c_1 + c_2\varepsilon|u_n(x)|$. So $|j(x, \varepsilon u_n(x))|/\|u_n\| \leqslant c_1/\|u_n\| + c_2\varepsilon|y_n(x)|$. Therefore, we obtain that $\lim |j(x, \varepsilon u_n(x))|/\|u_n\| \to 0$ as $n \to \infty$ and $\varepsilon \to 0$.

So, from the above we can say that

$$\int_\Omega \frac{pj(x, u_n(x))}{\|u_n\|}\,\mathrm{d}x \to p\xi \int_\Omega f_+(x)\,\mathrm{d}x.$$

But we already know that

$$(f_+(x)-\varepsilon)y_n(x) \leqslant v(x)\frac{u_n(x)}{\|u_n\|} \leqslant (f_+(x)+\varepsilon)y_n(x).$$

So, $\int_\Omega v(x)u_n(x)/\|u_n\|\,\mathrm{d}x \to \xi \int_\Omega f_+(x)\,\mathrm{d}x$.

Thus by passing to the limit in (7) we obtain

$$(1-p)\xi \int_\Omega f_+(x) \geqslant 0,$$

a contradiction to hypothesis H(j) (ii). The same argument holds when $\xi < 0$.

Therefore it follows that $\{u_n\} \subseteq W^{1,p}(\Omega)$ is bounded. Hence we may assume that $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, $u_n(x) \to u(x)$ a.e. on Ω as $n \to \infty$ and $|u_n(x)| \leqslant k(x)$ a.e. on Ω with $k \in L^p(\Omega)$.

From the properties of the subdifferential of Clarke we have

$$\partial R(u_n) \subseteq \partial\Phi(u_n) + \partial\psi(u_n) \subseteq \partial\Phi(u_n) + \partial\Big(\frac{1}{p}\|Du_n\|_p^p\Big) \quad \text{(see [2, p. 83])}.$$

So we have

$$\langle w_n, y\rangle = \langle Au_n, y\rangle - \int_\Omega v_n(x)y(x)\,\mathrm{d}x$$

with $v_n(x) \in \partial j(x, u_n(x))$, w_n the element with the minimal norm of the subdifferential of R and $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ such that

$$\langle Au, y\rangle = \int_\Omega (\|Du(x)\|^{p-2}(Du(x), Dy(x))\,\mathrm{d}x.$$

But $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$, so $u_n \to u$ in $L^p(\Omega)$ and $u_n(x) \to u(x)$ a.e. on Ω by virtue of the compact embedding $W^{1,p}(\Omega) \subseteq L^p(\Omega)$. Thus, v_n is bounded in $L^q(\Omega)$ (see [1, p. 104, Proposition 2]), i.e. $v_n \xrightarrow{w} v$ in $L^q(\Omega)$. Choose $y = u_n - u$. Then in the limit we have that $\limsup \langle Au_n, u_n - u\rangle = 0$. By virtue of the inequality (2) we have that $Du_n \to Du$ in $L^p(\Omega)$. So we have $u_n \to u$ in $W^{1,p}(\Omega)$. The claim is proved.

For every $\xi \in \mathbb{R}$, $\xi \neq 0$ we have

$$R(\xi) = -\int_\Omega j(x,\xi)\,\mathrm{d}x \Rightarrow \frac{1}{|\xi|}R(\xi) \leqslant -\frac{1}{|\xi|}\int_\Omega j(x,\xi)\,\mathrm{d}x.$$

As before we can show that $-|\xi|^{-1}\int_\Omega j(x,\xi)\,\mathrm{d}x \to -\int_\Omega f_+(x)\,\mathrm{d}x < 0$.

Thus we obtain that $R(\xi) \to -\infty$ as $|\xi| \to \infty$.

In order to be able to use the mountain-pass theorem it remains to show that there exists $\varrho > 0$ such that for $\|u\| = \varrho$ we have $R(u) \geqslant a > 0$. In fact, we will show that for every sequence $\{u_n\} \subseteq W^{1,p}(\Omega)$ with $\|u_n\| = \varrho_n \downarrow 0$ we have $R(u_n) > 0$. Indeed, suppose that this is not the case. Then there exists a sequence $\{u_n\}$ such that $R(u_n) \leqslant 0$. Thus, we have

$$\frac{1}{p}\|Du_n\|_p^p \leqslant \int_\Omega j(x, u_n(x))\,\mathrm{d}x.$$

Divide this inequality by $\|u_n\|^p$. Let $y_n(z) = u_n(x))/\|u_n\|$. Then we have

$$\|Dy_n\|_p^p \leqslant \int_\Omega p\frac{j(x,u_n(x))}{\|u_n\|^p}\,\mathrm{d}x.$$

From H(j) (iii) we have that for almost all $x \in \Omega$ and any $\varepsilon > 0$ we can find $\delta > 0$ such that for $|u| \leqslant \delta$ we have

$$pj(x,u) \leqslant (\theta(x) + \varepsilon)|u|^p.$$

On the other hand, as before we have that for almost all $x \in \Omega$ and all $|u| \geqslant \delta$ we have

$$p|j(z,u)| \leqslant c_1 + c_2|u|.$$

Thus we can always find $\gamma > 0$ such that $p|j(x,u)| \leqslant (\theta(x)+\varepsilon)|u|^p + \gamma |u|^{p^*}$ for all $u \in \mathbb{R}$. Indeed, choose $\gamma \geqslant (c_1 + c_2\delta)\delta^{-p^*} - (\theta(x)+\varepsilon)\delta^{p-p^*}$.

Therefore, we obtain

$$\begin{aligned} \|Dy_n\|_p^p &\leqslant \int_\Omega (\theta(x)+\varepsilon)|y_n(x)|^p \,\mathrm{d}x + \gamma \int_\Omega \frac{|u_n(x)|^{p^*}}{\|u_n\|^p} \,\mathrm{d}x \\ &\leqslant \int_\Omega (\theta(x)+\varepsilon)|y_n(x)|^p \,\mathrm{d}x + \gamma_1 \|u_n\|^{p^*-p}. \end{aligned} \tag{8}$$

Here we have used the fact that $W^{1,p}(\Omega)$ embeds continuously in $L^{p^*}(\Omega)$.

So we obtain

$$0 \leqslant \|Dy_n\|_p^p \leqslant \varepsilon \|y_n\|_p^p + \gamma_1 \|u_n\|^{p^*-p}.$$

Therefore in the limit we have that $\|Dy_n\|_p \to 0$. Recall that $y_n \to y$ weakly in $W^{1,p}(\Omega)$. So $\|Dy\|_p \leqslant \liminf \|Dy_n\|_p \leqslant \limsup \|Dy_n\|_p \to 0$. So $\|Dy\|_p = 0$, thus $y = \xi \in \mathbb{R}$. Note that $Dy_n \to Dy$ weakly in $L^p(\Omega)$ and $\|Dy_n\|_p \to \|Dy\|_p$, so $y_n \to y$ in $W^{1,p}(\Omega)$. Since $\|y_n\| = 1$ we have that $\|y\| = 1$, so $\xi \neq 0$. Suppose that $\xi > 0$. Going back to (8) we obtain

$$0 \leqslant \int_\Omega (\theta(x)+\varepsilon) y_n^p(x) \,\mathrm{d}x + \gamma_1 \|u_n\|^{p^*-p}.$$

In the limit we have

$$0 \leqslant \int_\Omega (\theta(x)+\varepsilon)\xi^p \,\mathrm{d}x \leqslant \varepsilon \xi^p |\Omega| \quad \text{(recall that } \theta(x) \leqslant 0).$$

Thus we obtain that $\int_\Omega \theta(x)\xi^p \,\mathrm{d}x = 0$. But this is a contradiction. So there exists $\varrho > 0$ such that for $\|u\| = \varrho$ we have $R(u) \geqslant a > 0$.

So by Theorem 1 we have that there exists $x \in W^{1,p}(\Omega)$ such that $0 \in \partial R(u)$. That is, $0 \in \partial\Phi(u) + \partial\psi(u)$.

So, we can say that

$$\int_\Omega w(x)y(x) \,\mathrm{d}x = \int_\Omega \|Du(x)\|^{p-2}(Du(x), Dy(x)) \,\mathrm{d}x \tag{9}$$

for some $w \in L^q(\Omega)$ such that $w(x) \in \partial j(x, u(x))$ for every $y \in W^{1,p}(\Omega)$. $\square$

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