Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 55 (2005), No. 1, 25–40

Persistent URL: `http://dml.cz/dmlcz/127957`

Czechoslovak Mathematical Journal, 55 (130) (2005), 25–40

GRACEFUL SIGNED GRAPHS: II. THE CASE OF SIGNED CYCLES WITH CONNECTED NEGATIVE SECTIONS

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(Received January 29, 2002)

Abstract. In our earlier paper [9], generalizing the well known notion of graceful graphs, a (p,m,n)-*signed graph* S of order p, with m *positive* edges and n *negative* edges, is called *graceful* if there exists an injective function f that assigns to its p vertices integers $0,1,\dots,q=m+n$ such that when to each edge uv of S one assigns the absolute difference $|f(u)-f(v)|$ the set of integers received by the positive edges of S is $\{1,2,\dots,m\}$ and the set of integers received by the negative edges of S is $\{1,2,\dots,n\}$. Considering the conjecture therein that all signed cycles Z_k, of admissible length $k\geqslant 3$ and signed structures, are graceful, we establish in this paper its truth for all possible signed cycles of lengths $0,2$ or $3 \pmod 4$ in which the set of negative edges forms a connected subsigraph.

Keywords: graceful signed graphs, signed cycles

MSC 2000: 05C78

0. Introduction

For terminology in graph theory we follow [18] and for that of signed graphs (henceforth abbreviated as *sigraphs*) we refer the reader to [12], [13], [17], [21], [26]. Additional terms will be defined as and when necessary.

An ordered pair $S=(S^u,s)$ where $S^u=(V,E)$ is a graph called the *underlying graph* of S and $s\colon E\to\{+,-\}$ is a function from the edge set E into $\{+,-\}$, is called a *signed graph* (or *sigraph* in short). We let $E^+(S)=\{e\in E\colon s(e)=+\}$ and $E^-(S)=E-E^+(S)$. Then the set $E(S)=E^+(S)\cup E^-(S)$ is called the *edge set* of S. The elements of $E^+(S)$ (respectively, $E^-(S)$) are called *positive* (*negative*) edges of S. We shall regard graphs as sigraphs in which all the edges are positive (or, *all-positive* sigraphs; *all-negative* sigraphs are defined similarly). A sigraph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. Given a subsigraph H of S, by a *negative* (*positive*) *section* of H we mean a maximal connected all-negative (all-positive) subsigraph of H.

Sigraphs were invoked by F. Harary [16] as appropriate prototype models to represent structures of cognitively dichotomic interrelationships in a social group. Ever since, sigraphs have received much attention in social psychology (where they are called *sociograms*) because of their extensive use in modelling a variety of cognition-based social processes (e.g., see [1], [5], [13], [17], [21]).

Further intensive study of the topic has been due to their subsequently discovered strong connections with many classical mathematical systems [2], [5], [10], [20], [21], [25], [26] used in solving a variety of problems of theoretical and practical interest (e.g., see [27]).

The notion of *graceful graphs* in graph theory (see [3], [4], [6], [7], [11], [14], [15], [19], [22]–[24]) was recently extended to the class of sigraphs (see [8], [9]) as follows:

By a (p,q)-*graph* we mean a graph with p vertices and q edges. By a (p,m,n)-*sigraph* we mean a sigraph $S = (S^u, s)$ where $S^u = (V, E)$ is a (p,q)-graph, $|E^+(S)| = m$ and $|E^-(S)| = n$ so that $m + n = q$. If f is a function assigning distinct labels to the vertices of S from the set $\{0, 1, 2, \ldots, q\}$ such that when each edge $uv \in E$ is assigned $g_f(uv) = s(uv)|f(u) - f(v)|$ the q edges receive all the integers from the set $\{1, 2, \ldots, m, -1, -2, \ldots, -n\}$; such a labelling f is called a *graceful labelling* of S. A sigraph which admits such a labelling is called a *graceful sigraph* (see [9]). If $E^-(S) = \emptyset$ in the above definition one obtains the standard notion of graceful graphs and graceful numberings of a graph (see [14], [15], [22]). In Fig. 1 we depict some examples of graceful graphs and sigraphs.

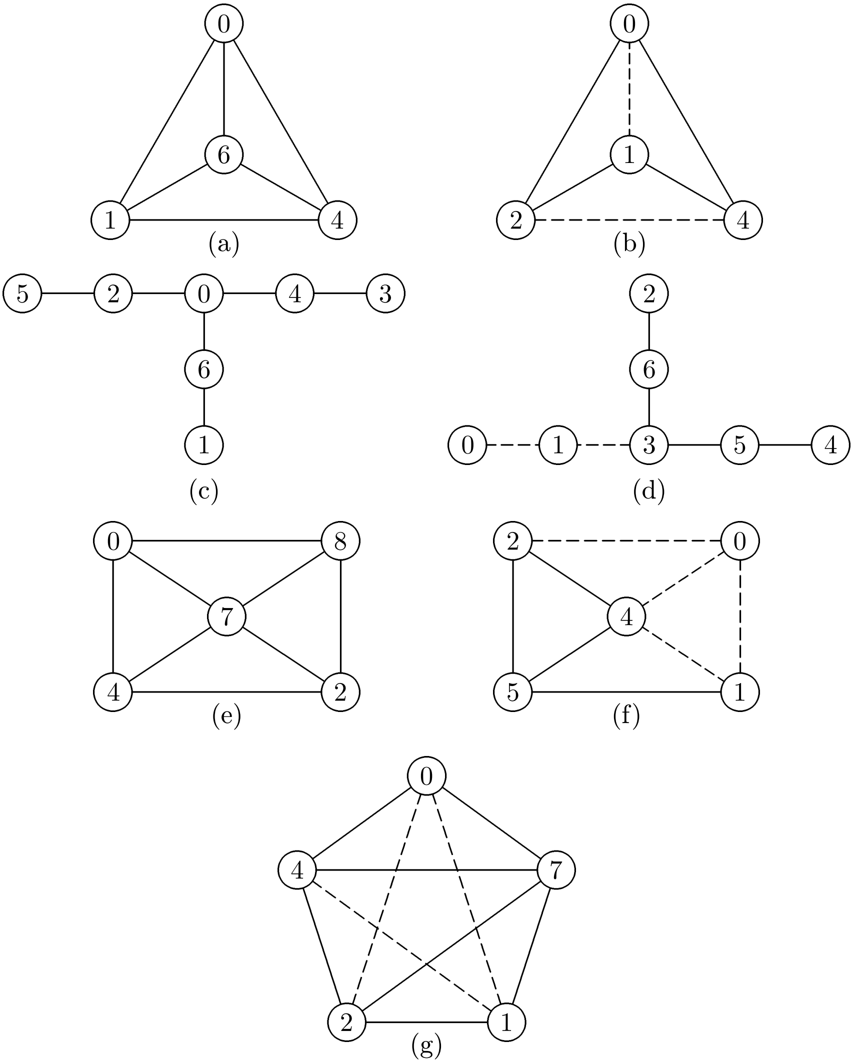
Theorem 1 [9]. *Let $S = (S^u, s)$ be any (p,m,n)-sigraph such that S^u is an eulerian graph. If S is graceful, then $m^2 + n^2 + m + n \equiv 0 \pmod 4$.*

Corollary 1.1 [9]. *If a signed cycle Z_k, $m + n = k \geqslant 3$, is graceful then $k \equiv 0, 2$ or $3 \pmod 4$.*

It was conjectured in [9] that the converse of Corollary 1.1 must also hold for all $k \geqslant 3$. Further, the following result was obtained.

Theorem 2A [9]. *If a heterogeneous signed cycle Z_k of length $k \equiv 0 \pmod 4$ is graceful then the number of negative sections of odd lengths in Z_k is even.*

In this paper, not only we shall establish the sufficiency part of Theorem 2A when Z_k contains exactly one negative section of any even length but also show that Z_k, $3 \leqslant k \equiv 2 \pmod 4$ (or $\equiv 3 \pmod 4$), is graceful when it has exactly one negative section of odd length (respectively, of any length), thus settling the



(a) (b) (c) (d) (e) (f) (g)

Figure 1.

above mentioned original conjecture in the special case when cycle contains exactly one negative section. The case of determining graceful heterogeneous signed cycles with more than one negative section appears rather involved and will be attempted elsewhere.

1. RESULTS

Everywhere in what follows and whenever mentioned, a signed cycle Z_k of length $k \geqslant 3$ is assumed to be imbedded in the Euclidean plane as a polygon without any of its edges crossing another at any point including its corners that represent vertices of the sigraph. Hence, first of all, we shall establish the following partial result towards the sufficiency part of Theorem 2A.

Theorem 2B. *If* Z_k, $4 \leqslant k \equiv 0 \pmod 4$, *is a signed cycle consisting of just one negative section of even length then* Z_k *is graceful.*

Proof. It is enough to provide a graceful labelling of Z_k whose *sign structure* is as laid down in the hypothesis, with m and n denoting respectively the lengths of positive and negative sections in Z_k. To this aim, we define a graceful labelling ψ of Z_k as follows: Let the vertices of Z_k be labelled consecutively as $u_1, u_2, \ldots, u_k$ along any one of the two directions of traversing its edges (i.e., clockwise or anti-clockwise), with u_1 appearing as the second vertex of the negative section along the chosen direction of traversing the edges of Z_k.

Case 1: $n = \frac{1}{2}k$. In this case, we let

$\psi(u_i) = \frac{1}{2}(i-1)$ for $i \in \{1, 3, 5, \ldots, n-1\}$;

$\psi(u_i) = n - \frac{1}{2}i$ for $i \in \{2, 4, \ldots, n\}$;

$\psi(u_i) = k - \frac{1}{2}m + 1 + \lfloor \frac{1}{2}(i-n-3) \rfloor$ for odd integers $i \in \{n+1, n+3, \ldots, k-1\}$;

$\psi(u_i) = k - \frac{1}{2}m + 1 - \lfloor \frac{1}{2}(i-n) + 1 \rfloor$ for even integers $i \in \{n+2, n+4, \ldots, k-2\}$,

and

$\psi(u_k) = n.$

Then, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| \\
&= s(u_iu_{i+1})|n-i| \colon\ i \in \{1, 2, \ldots, n-1\}\} = \{-1, -2, \ldots, -(n-1)\} \\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})\bigl|\psi(u_i) - \psi(u_{i+1})\bigr| \\
&= s(u_iu_{i+1})\bigl|\lfloor \tfrac{1}{2}(i-n-2) \rfloor + \lfloor \tfrac{1}{2}(i-n) + 1 \rfloor\bigr| \\
&= s(u_iu_{i+1})|i-n| \colon\ i \in \{n+1, n+2, \ldots, k-2\}\} = \{1, 2, \ldots, \tfrac{1}{2}k - 2\}; \\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n) - \psi(u_{n+1})| \\
&= s(u_nu_{n+1})\bigl|k + 1 - \tfrac{1}{2}m + \lfloor \tfrac{1}{2}(n-n-2) \rfloor - n + \tfrac{1}{2}n\bigr| = \tfrac{1}{2}k; \\
g_\psi(u_ku_{k-1}) &= s(u_ku_{k-1})|\psi(u_k) - \psi(u_{k-1})| \\
&= s(u_ku_{k-1})\bigl|n - (k + 1 - \tfrac{1}{2}m + \lfloor \tfrac{1}{2}(k-n-4) \rfloor)\bigr| = \tfrac{1}{2}k - 1,
\end{aligned}$$

and

$$g_\psi(u_1u_k) = -n.$$

Case 2: n is even and $2 \leqslant n < \frac{1}{2}k$. In this case, let

$$\begin{aligned}
&\psi(u_i) = \tfrac{1}{2}(i-1) \text{ for } i \in \{1,3,5,\dots,n-1\};\\
&\psi(u_i) = n - \tfrac{1}{2}i \text{ for } i \in \{2,4,\dots,n\};\\
&\psi(u_i) = k - \tfrac{1}{2}m + 1 + \lfloor \tfrac{1}{2}(i-n-1)\rfloor \text{ for odd integers } i \in \{n+1,n+3,\dots,k-1\};\\
&\psi(u_i) = k - \tfrac{1}{2}m + 1 - \lfloor \tfrac{1}{2}(i-n)\rfloor\\
&\qquad\qquad \text{for even integers } i \in \{n+2,n+4,\dots,k-t\} \text{ where } m-n=2t;\\
&\psi(u_i) = k - \tfrac{1}{2}m + 1 - \lfloor \tfrac{1}{2}(i-n)+1\rfloor\\
&\qquad\qquad \text{for even integers } i \in \{(k-t)+2,(k-t)+4,\dots,k-2\},
\end{aligned}$$

and

$$\psi(u_k) = n.$$

Then, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})|\\
&= s(u_iu_{i+1})|\tfrac{1}{2}(i-1)-n+\tfrac{1}{2}(i+1)|\\
&= s(u_iu_{i+1})|n-i|\colon\ i \in \{1,2,\dots,n-1\}\} = \{-1,-2,\dots,-(n-1)\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})| = s(u_iu_{i+1})|\lfloor\tfrac{1}{2}(i-n)\rfloor + \lfloor\tfrac{1}{2}(i-n)\rfloor|\\
&= s(u_iu_{i+1})|i-n|\colon\ i \in \{n+1,n+2,\dots,k-t\}\} = \{1,2,\dots,m-t\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})| = s(u_iu_{i+1})\big|\lfloor\tfrac{1}{2}(i-n)+1\rfloor + \lfloor\tfrac{1}{2}(i-n)\rfloor\big|\\
&= s(u_iu_{i+1})|i-n+1|\colon\ i \in \{k-t+1,k-t+2,\dots,k-2\}\}\\
&= \{m-t+2,m-t+3,\dots,m-1\};\\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n)-\psi(u_{n+1})|\\
&= s(u_nu_{n+1})|k+1-\tfrac{1}{2}m-\tfrac{1}{2}n| = \tfrac{1}{2}k+1;\\
g_\psi(u_{k-1}u_k) &= s(u_{k-1}u_k)|\psi(u_{k-1})-\psi(u_k)|\\
&= s(u_ku_{k-1})|k+1-\tfrac{1}{2}m+\lfloor\tfrac{1}{2}(i-n-1)\rfloor - n|\\
&= s(u_{k-1}u_k)|k-n| = m,
\end{aligned}$$

and

$$g_\psi(u_1u_k) = -n.$$

In each of the above cases, the injectivity of ψ is straightforward to see by its very definition. Also, in each case, we have seen separately above that the *induced edge labelling* g_ψ is also injective, thus completing the proof. □

The above theorem is illustrated in Fig. 2.

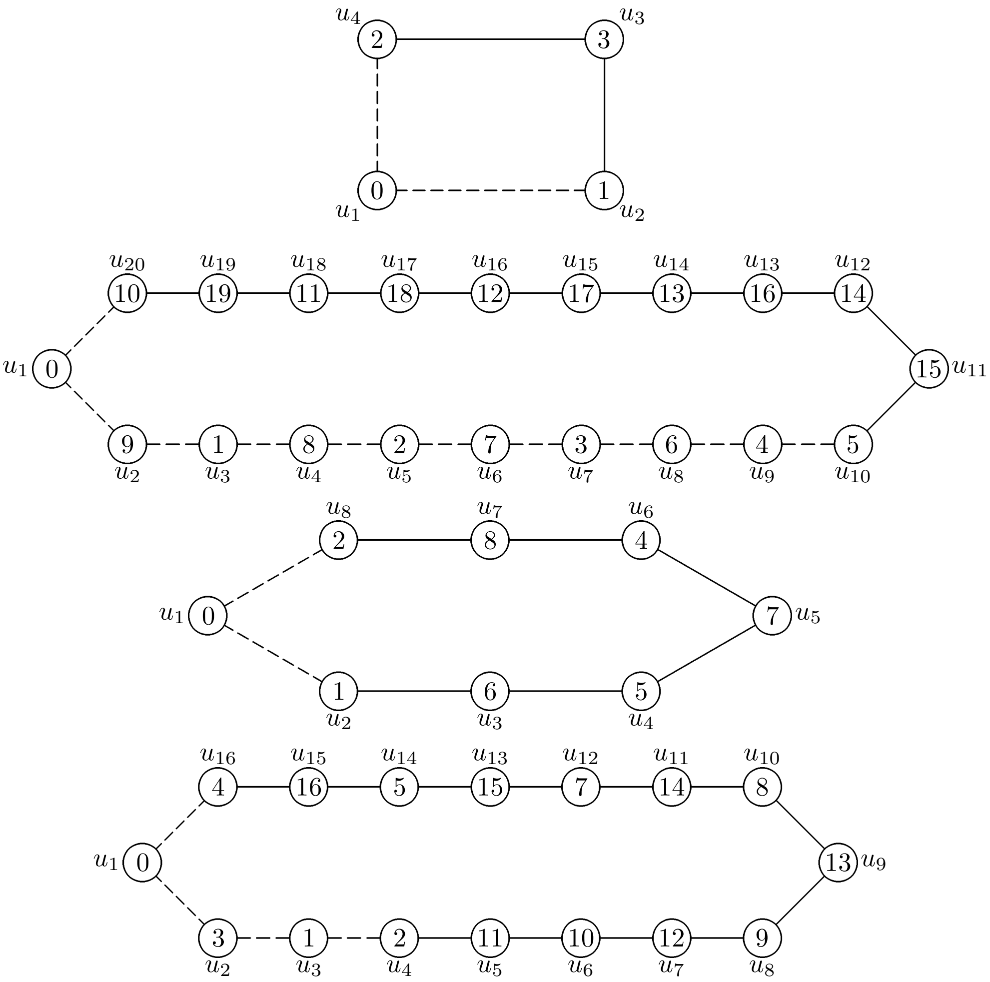


Figure 2.

Theorem 3. *Let* Z_k *be a heterogeneous signed cycle of length* $k \equiv 2 \pmod 4$*. If* Z_k *is graceful then the number of negative sections of odd lengths in* Z_k *is odd.*

Proof. Let Z_k be any heterogeneous signed cycle of length $k > 2$, $k \equiv 2 \pmod 4$ possessing a graceful numbering f and let $l_1, l_2, \ldots, l_r$ be the lengths of the negative sections, $r \geqslant 1$. Suppose that the number of the negative sections of odd lengths in Z_k is even, say $2x$ for some positive integer x. Without loss of generality, we may assume $l_1, l_2, \ldots, l_{2x}$ to be the odd ones. Let $l_i = 2a_i+1$ for $i \in \{1, 2, \ldots, 2x\}$ and $l_i = 2b_i$ for $i \in \{2x+1, 2x+2, \ldots, r\}$ where a_i is nonnegative integer and b_i is a positive integer. Then

$$n = \sum_{i\in\{1,2,\ldots,2x\}} l_i + \sum_{i\in\{2x+1,2x+2,\ldots,r\}} l_i = 2u$$

and hence $m = k - n = (4a+2) - 2u = 2(2a - u + 1)$ where a and u are positive integers. Then

$$\begin{aligned} m^2 + n^2 + m + n &= (k-n)^2 + n^2 + (k-n) + n \\ &= ((4a+2) - 2u)^2 + 4u^2 + 4a + 2 \\ &= (4a+2)^2 + 4u^2 - 8u(2a+1) + 4u^2 + 4a + 2 \\ &= 16a^2 + 20a + 6 + 8u^2 - 16au - 8u \equiv 2 \pmod{4}, \end{aligned}$$

a contradiction to the hypothesis. Therefore, the number of negative sections of odd lengths in Z_k must be odd as claimed. ☐

Since we are considering heterogeneous signed cycles having exactly one negative section in this paper, in the case of signed cycles whose lengths are congruent to 2 (mod 4) integers, Theorem 3 implies that if such a signed cycle is graceful then its only negative section must have an odd length. The following result shows that every such signed cycle is indeed graceful.

Theorem 4. *Let Z_k be a heterogeneous signed cycle of length $k \equiv 2 \pmod 4$ having exactly one negative section of odd length. Then, Z_k is graceful.*

Proof. It is enough to provide a graceful labelling of Z_k whose *sign structure* is as laid down in the hypothesis, with m and n denoting respectively the lengths of the positive and negative sections in Z_k. To this aim, we define a graceful labelling ψ of Z_k as follows: Let the vertices of Z_k be labelled consecutively as $u_1, u_2, \ldots, u_k$ along any one of the two directions of traversing its edges (i.e., clockwise or anti-clockwise), with u_1u_2 as the first edge of the negative section.

Case 1: $n = 1$. In this case, we let

$\psi(u_1) = 0;$

$\psi(u_i) = \frac{1}{2}i$ for even integers $i \in \{2, 4, \ldots, \frac{1}{2}k - 1\};$

$\psi(u_i) = \frac{1}{2}(i+2)$ for even integers $i \in \{\frac{1}{2}k + 1, \frac{1}{2}k + 3, \ldots, k\};$

$\psi(u_i) = k - \frac{1}{2}(i-3)$ for odd integers $i \in \{3, 5, \ldots, k-1\}.$

Then, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
g_\psi(u_1u_2) &= s(u_1u_2)|\psi(u_1) - \psi(u_2)| = -1;\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|k - \tfrac{1}{2}(i-2) - \tfrac{1}{2}i|\\
&= s(u_iu_{i+1})|k - i + 1| \colon\ i \in \{2, 3, \ldots, \tfrac{1}{2}k - 1\}\}\\
&= \{\tfrac{1}{2}k + 2, \tfrac{1}{2}k + 3, \ldots, k - 1\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|k - \tfrac{1}{2}(i-2) - \tfrac{1}{2}(i+2)|\\
&= s(u_iu_{i+1})|k - i| \colon\ i \in \{\tfrac{1}{2}k, \tfrac{1}{2}k + 1, \ldots, k - 1\}\} = \{1, 2, \ldots, \tfrac{1}{2}k\},
\end{aligned}$$

and

$$g_\psi(u_1u_k) = s(u_1u_k)|\psi(u_k) - \psi(u_1)| = \tfrac{1}{2}k + 1.$$

Case 2: $n = \frac{1}{2}k$. In this case, without loss of generality we assume u_1u_k as the first negative edge of the negative section and we let

$\psi(u_i) = \frac{1}{2}(i-1)$ for $i \in \{1, 3, \ldots, n\}$;
$\psi(u_i) = n - \frac{1}{2}i$ for $i \in \{2, 4, \ldots, n-1\}$;
$\psi(u_i) = k - \frac{1}{2}(m-1) - \frac{1}{2}(i - n + 1)$ for even integers $i \in \{n+1, n+3, \ldots, k\}$,

and

$\psi(u_i) = k - \frac{1}{2}(m-1) + \frac{1}{2}(i - n - 2)$ for odd integers $i \in \{n+2, n+4, \ldots, k-1\}$.

Then, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})|\\
&= s(u_iu_{i+1})|n - i| \colon\ i \in \{1, 2, \ldots, n-1\}\} = \{-1, -2, \ldots, -(n-1)\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| = s(u_iu_{i+1})|\tfrac{1}{2}(i - n - 2) + \tfrac{1}{2}(i - n + 2)|\\
&= s(u_iu_{i+1})|i - n| \colon\ i \in \{n+1, n+2, \ldots, k-1\}\} = \{1, 2, \ldots, (m-1)\};\\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n) - \psi(u_{n+1})|\\
&= s(u_nu_{n+1})|\tfrac{1}{2}(n-1) - k + \tfrac{1}{2}(m-1) + 1| = \tfrac{1}{2}k,
\end{aligned}$$

and

$$g_\psi(u_1u_k) = s(u_1u_k)|\psi(u_1) - \psi(u_k)| = s(u_1u_k)|k - m| = -n.$$

Case 3: n is odd and $1 < n < \frac{1}{2}k$. In this case also without loss of generality, we assume u_1u_k as the first negative edge of the negative section and we let

$\psi(u_i) = \frac{1}{2}(i-1)$ for $i \in \{1, 3, \ldots, n\}$;

$$\begin{aligned}
\psi(u_i) &= n - \tfrac{1}{2}i \text{ for } i \in \{2, 4, \dots, n-1\};\\
\psi(u_i) &= k - \tfrac{1}{2}(m-1) - \tfrac{1}{2}(i-n-1)\\
&\qquad \text{for even integers } i \in \{n+1, n+3, \dots, k-t\}, \text{ where } m-n = 2t;\\
\psi(u_i) &= k - \tfrac{1}{2}(m-1) - \tfrac{1}{2}(i-n+1)\\
&\qquad \text{for even integers } i \in \{(k-t)+2, (k-t)+4, \dots, k\},
\end{aligned}$$

and

$$\begin{aligned}
\psi(u_i) &= k - \tfrac{1}{2}(m-1) + \tfrac{1}{2}(i-n)\\
&\qquad \text{for odd integers } i \in \{n+2, n+4, \dots, k-1\}.
\end{aligned}$$

Then, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})|\\
&= s(u_iu_{i+1})|n-i| \colon i \in \{1, 2, \dots, n-1\}\} = \{-1, -2, \dots, -(n-1)\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})|\\
&= s(u_iu_{i+1})|i-n| \colon i \in \{n+1, n+2, \dots, k-t\}\} = \{1, 2, \dots, m-t\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})|\\
&= s(u_iu_{i+1})|i-n+1| \colon i \in \{k-t+1, k-t+2, \dots, k-1\}\}\\
&= \{m-t+2, m-t+3, \dots, m\};\\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n) - \psi(u_{n+1})|\\
&= s(u_nu_{n+1})|\tfrac{1}{2}(n-1) - k + \tfrac{1}{2}(m-1)| = \tfrac{1}{2}k + 1,
\end{aligned}$$

and

$$g_\psi(u_1u_k) = s(u_1u_k)|\psi(u_1) - \psi(u_k)| = s(u_1u_k)|k-m| = -n.$$

The injectivity of ψ is straightforward to see by its very definition of the above in each case. Also, in each case, we have seen separately above that the *induced edge labelling* g_ψ is also injective, thus completing the proof. □

Fig. 3 illustrates the findings of Theorem 4.

In the theory of graceful graphs, it is well known (e.g., see [14]) that the cycle C_k is graceful for all values of $k \equiv 3 \pmod 4$. This conclusion can be extended to certain signed graphs on C_k as found in our next result.

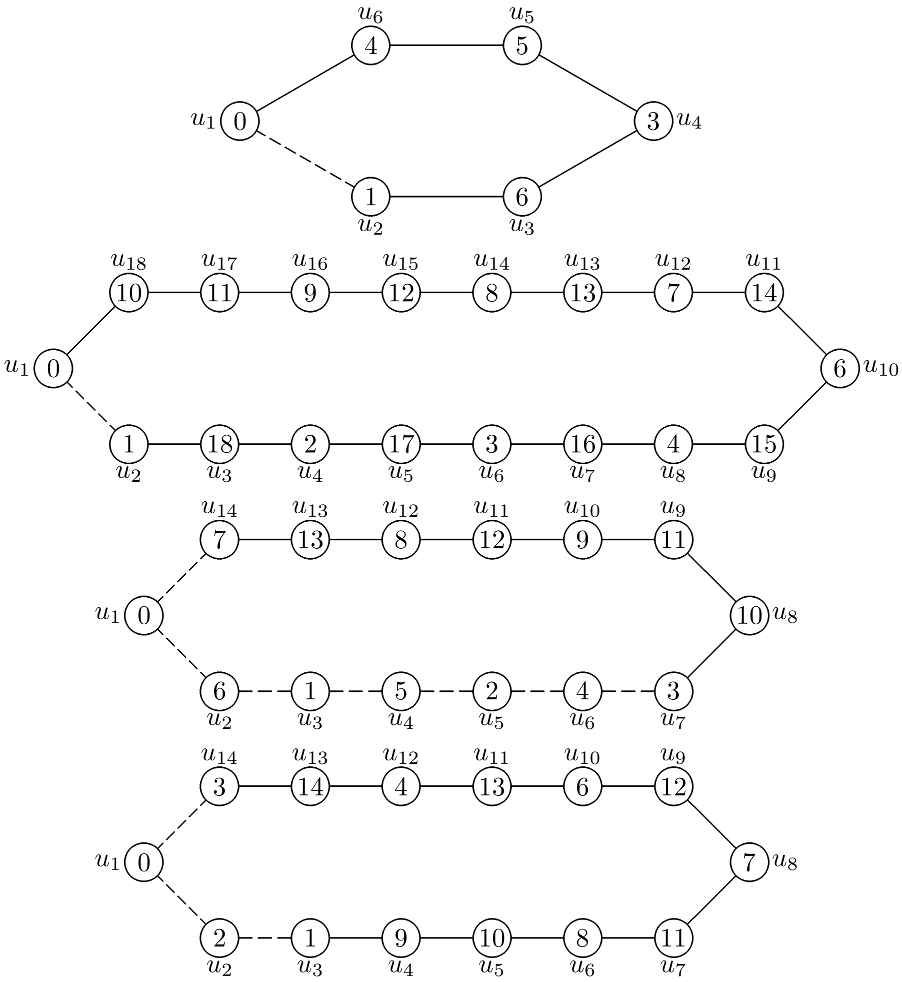


Figure 3.

Theorem 5. *If a signed cycle Z_k, $k \equiv 3 \pmod 4$, contains exactly one negative section then it is graceful.*

Proof. It is enough to provide a graceful labelling of Z_k whose *sign structure* is as laid down in the hypothesis, with m and n denoting respectively the lengths of the positive and negative sections in Z_k. To this aim, we define a graceful labelling ψ of Z_k as follows: Let the vertices of Z_k be labelled consecutively as $u_1, u_2, \ldots, u_k$ along any one of the two directions of traversing its edges (i.e., clockwise or anti-clockwise), with u_1 appearing as the second vertex of the negative section along the chosen direction of traversing the edges of Z_k so that the edge u_1u_k is negative.

Case 1: $k \geqslant 7$ and $n = 1$. In this case, we let

$$\begin{aligned}
&\psi(u_1) = 0 \ \text{ and } \ \psi(u_k) = 1;\\
&\psi(u_i) = k - \lfloor \tfrac{1}{2}(m-1) \rfloor + \tfrac{1}{2}(i-2) \text{ for even integers } i \in \{2,4,\ldots,k-1\};\\
&\psi(u_i) = k - \lfloor \tfrac{1}{2}(m-1) \rfloor - \tfrac{1}{2}(i-1) \text{ for odd integers } i \in \{3,5,\ldots,\tfrac{1}{2}(k+3)\},
\end{aligned}$$

and

$$\begin{aligned}
&\psi(u_i) = k - \lfloor \tfrac{1}{2}(m-1) \rfloor - \tfrac{1}{2}(i-2) - 1\\
&\qquad\qquad \text{for odd integers } i \in \{\tfrac{1}{2}(k+7), \tfrac{1}{2}(k+11), \ldots, k-2\}.
\end{aligned}$$

Then the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
g_\psi(u_1u_k) &= s(u_1u_k)|\psi(u_1) - \psi(u_k)| = s(u_1u_k)|0-1| = -1;\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})|\\
&= s(u_iu_{i+1})|\tfrac{1}{2}(i-1) + \tfrac{1}{2}(i-1)|\\
&= s(u_iu_{i+1})|i-1| \colon\ i \in \{2,3,\ldots,\tfrac{1}{2}(k+3)\}\}\\
&= \{1,2,3,\ldots,\tfrac{1}{2}(k+1)\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| = s(u_iu_{i+1})|\tfrac{1}{2}(i-2) + \tfrac{1}{2}(i+2)|\\
&= s(u_iu_{i+1})|i| \colon\ i \in \{\tfrac{1}{2}(k+5), \tfrac{1}{2}(k+7), \ldots, k-2\}\}\\
&= \{\tfrac{1}{2}(k+5), \tfrac{1}{2}(k+7), \ldots, k-2\};\\
g_\psi(u_1u_2) &= s(u_1u_2)|\psi(u_1) - \psi(u_2)| = s(u_1u_2)|k - \lfloor \tfrac{1}{2}(m-1) \rfloor|,
\end{aligned}$$

and

$$\begin{aligned}
g_\psi(u_{k-1}u_k) &= s(u_{k-1}u_k)|\psi(u_{k-1}) - \psi(u_k)|\\
&= s(u_{k-1}u_k)|k - \lfloor \tfrac{1}{2}(m-1) \rfloor + \tfrac{1}{2}(k-3) - 1| = k-1 = m.
\end{aligned}$$

Case 2: $k \geqslant 3$ and $n = \lfloor \tfrac{1}{2}k \rfloor$. In this case, we let

$$\begin{aligned}
\psi(u_i) &= \tfrac{1}{2}(i-1) \text{ for } i \in \{1,3,\ldots,n\};\\
\psi(u_i) &= n - \tfrac{1}{2}i \text{ for } i \in \{2,4,\ldots,n-1\};\\
\psi(u_i) &= k - \lfloor \tfrac{1}{2}(m-1) \rfloor + \lfloor \tfrac{1}{2}(i-n-3) \rfloor\\
&\qquad \text{for even integers } i \in \{n+1, n+3, \ldots, k-1\};\\
\psi(u_i) &= k - \lfloor \tfrac{1}{2}(m-1) \rfloor - \lfloor \tfrac{1}{2}(i-n+2) \rfloor\\
&\qquad \text{for odd integers } i \in \{n+2, n+4, \ldots, k-2\},
\end{aligned}$$

and

$$\psi(u_k) = n.$$

Then the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| \\
&= s(u_iu_{i+1})|n-i|\colon\ i \in \{1,2,\ldots,n-1\}\} = \{-1,-2,\ldots,-(n-1)\}; \\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| \\
&= s(u_iu_{i+1})|\lfloor \tfrac{1}{2}(i-n+2)\rfloor + \lfloor \tfrac{1}{2}(i-n-2)\rfloor| \\
&= s(u_iu_{i+1})|i-n|\colon\ i \in \{n+1,n+2,\ldots,k-2\}\} \\
&= \{1,2,3,\ldots,m-2\}; \\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n) - \psi(u_{n+1})| \\
&= s(u_nu_{n+1})|\tfrac{1}{2}(n-1) - k + \lfloor \tfrac{1}{2}(m-1)\rfloor + 1| = m; \\
g_\psi(u_{k-1}u_k) &= s(u_{k-1}u_k)|\psi(u_{k-1}) - \psi(u_k)| \\
&= s(u_{k-1}u_k)|(k - \lfloor \tfrac{1}{2}(m-1)\rfloor + \lfloor \tfrac{1}{2}(k-n-4)\rfloor) - n| = m-1,
\end{aligned}$$

and

$$g_\psi(u_1u_k) = \ -n.$$

Case 3: n is odd and $1 < n < \lfloor \frac{1}{2}k\rfloor$. Then, we let

$$\begin{aligned}
&\psi(u_i) = \tfrac{1}{2}(i-1) \text{ for } i \in \{1,3,\ldots,n\}; \\
&\psi(u_i) = n - \tfrac{1}{2}i \text{ for } i \in \{2,4,\ldots,n-1\}; \\
&\psi(u_i) = k - (\tfrac{1}{2}m - 1) + \lfloor \tfrac{1}{2}(i-n-1)\rfloor \\
&\qquad\qquad \text{for even integers } i \in \{n+1,n+3,\ldots,k-1\}; \\
&\psi(u_i) = k - (\tfrac{1}{2}m - 1) - \tfrac{1}{2}(i-n) \text{ for odd integers } i \in \{n+2,n+4,\ldots,k-t\},
\end{aligned}$$

and

$$\begin{aligned}
&\psi(u_i) = k - (\tfrac{1}{2}m - 1) - \tfrac{1}{2}(i-n+2) \\
&\qquad\qquad \text{for odd integers } i \in \{k+2-t,k+4-t,\ldots,k\}, \text{ where } t = \tfrac{1}{2}(m-n-1).
\end{aligned}$$

In this case, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i) - \psi(u_{i+1})| \\
&= s(u_iu_{i+1})|n-i|\colon\ i \in \{1,2,\ldots,n-1\}\} = \{-1,-2,\ldots,-(n-1)\};
\end{aligned}$$

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})| = s(u_iu_{i+1})|\tfrac{1}{2}(i-n)+\tfrac{1}{2}(i-n)|\\
&= s(u_iu_{i+1})|i-n|\colon\ i\in\{n+1,n+2,\dots,k-t\}\} = \{1,2,3,\dots,m-t\};\\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n)-\psi(u_{n+1})|\\
&= s(u_nu_{n+1}|\tfrac{1}{2}(n-1)-(k-\tfrac{1}{2}m+1)| = m-t+1;\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})|\\
&= s(u_iu_{i+1})|i-n+1|\colon\ i\in\{k-t+1,k-t+2,\dots,k-1\}\}\\
&= \{m-t+2,m-t+3,\dots,m\},
\end{aligned}$$

and

$$g_\psi(u_1u_k) = s(u_1u_k)|k-m| = -n.$$

Case 4: n is even and $2\leqslant n<\lfloor\frac{1}{2}k\rfloor$. In this case, we let

$$\begin{aligned}
&\psi(u_i) = \tfrac{1}{2}(i-1) \text{ for } i\in\{1,3,\dots,n-1\};\\
&\psi(u_i) = n-\tfrac{1}{2}i \text{ for } i\in\{2,4,\dots,n\};\\
&\psi(u_i) = k-\tfrac{1}{2}(m-1)-\tfrac{1}{2}(i-n-1)\\
&\qquad\qquad \text{for odd integers } i\in\{n+1,n+3,\dots,k-t-2\};\\
&\psi(u_i) = k-\tfrac{1}{2}(m-1)-\tfrac{1}{2}(i-n+1)\\
&\qquad\qquad \text{for odd integers } i\in\{k-t,k+2-t,\dots,k\}, \text{ where } t=\tfrac{1}{2}(m-n-3),
\end{aligned}$$

and

$$\psi(u_i) = k-\tfrac{1}{2}(m-1)+\tfrac{1}{2}(i-n) \text{ for even integers } i\in\{n+2,n+4,\dots,k-1\}.$$

In this case, the induced edge function g_ψ yields the edge labels

$$\begin{aligned}
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})|\\
&= s(u_iu_{i+1})|n-i|\colon\ i\in\{1,2,\dots,n-1\}\} = \{-1,-2,\dots,-(n-1)\};\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})|\\
&= s(u_iu_{i+1})|i-n|\colon\ i\in\{n+1,n+2,\dots,k-t-2\}\}\\
&= \{1,2,3,\dots,m-t-2\};\\
g_\psi(u_nu_{n+1}) &= s(u_nu_{n+1})|\psi(u_n)-\psi(u_{n+1})|\\
&= s(u_nu_{n+1}|k-\tfrac{1}{2}(m-1)-\tfrac{1}{2}n| = \tfrac{1}{2}(k+1);\\
\{g_\psi(u_iu_{i+1}) &= s(u_iu_{i+1})|\psi(u_i)-\psi(u_{i+1})|\\
&= s(u_iu_{i+1})|i-n+1|\colon\ i\in\{k-t-1,k-t,\dots,k-1\}\}\\
&= \{m-t,m-t+1,\dots,m\},
\end{aligned}$$

and

$$g_\psi(u_1u_k) = s(u_1u_k)|k-m| = -n.$$

The injectivity of ψ can be seen straightforwardly by its very definition, in each of the above cases. Also, in each case, the *induced edge labelling* g_ψ has been verified to be injective, which completes the proof. ☐

We illustrate the findings of Theorem 5 in Fig. 4.

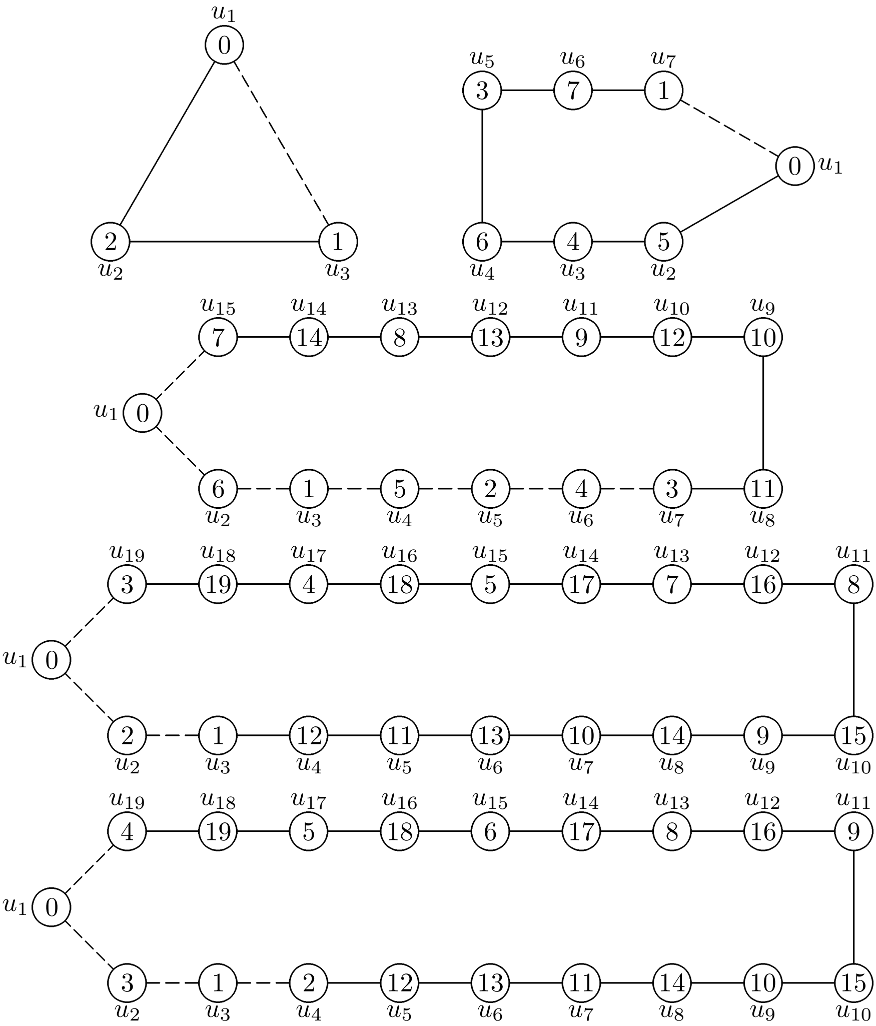


Figure 4.

2. CONCLUDING REMARKS

Thus, we have determined all the possible graceful signed cycles Z_k for all integers $k \geqslant 3$, $k \equiv 0,\ 2$ or $3 \pmod 4$, each consisting of just one negative section. However, the problem is open for such cycles containing more than one negative sections.

In general, determining graceful sigraphs in which more than one negative sections exist seems to be a hard problem. Graceful labelling of signed graphs provide an insight into more general problem of finding a *unified* model for automatic continuous coding of monochromatic factors in an edge-packing of a graph as mentioned in [8].

Acknowledgements. The authors are thankful to the anonymous referees for their valuable suggestions and comments and also thankful to Dr. B. D. Acharya for sparing his valuable time for rigorous discussion and suggestions which were useful in the preparation of the paper.

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