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IDEALS OF NONCOMMUTATIVE $DR\ell$-MONOIDS

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Abstract. In this paper, we introduce the concept of an ideal of a noncommutative dually residuated lattice ordered monoid and we show that congruence relations and certain ideals are in a one-to-one correspondence.

Keywords: dually residuated lattice ordered monoid, ideal, normal ideal

MSC 2000: 06F05, 06D35

1. Introduction

Commutative $DR\ell$-monoids (called $DR\ell$-semigroups) were introduced by K. L. N. Swamy in [11] as a common generalization of commutative ℓ-groups and Brouwerian algebras. A noncommutative extension of $DR\ell$-semigroups is mentioned in [12], but the present definition, due to [8], is more general. In fact, Swamy's noncommutative $DR\ell$-semigroup was considered as an algebra $(A,+,0,\vee,\wedge,-)$, where "$-$" agrees with "$\rightharpoonup$".

Definition. An algebra $\mathfrak{A}=(A,+,0,\vee,\wedge,\rightharpoonup,\leftharpoondown)$ is a *dually residuated lattice ordered monoid*, or simply a *$DR\ell$-monoid*, iff

(1) $(A,+,0,\vee,\wedge)$ is an ℓ-monoid, that is, $(A,+,0)$ is a monoid, $(A,\vee,\wedge)$ is a lattice and, for any $x,y,s,t\in A$, the following distributive laws are satisfied:

$$s+(x\vee y)+t=(s+x+t)\vee(s+y+t),$$
$$s+(x\wedge y)+t=(s+x+t)\wedge(s+y+t);$$

(2) for any $x,y\in A$, $x\rightharpoonup y$ is the least $s\in A$ such that $s+y\geqslant x$, and $x\leftharpoondown y$ is the least $t\in A$ such that $y+t\geqslant x$;

(3) $\mathfrak{A}$ fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leqslant x \vee y, \quad y + ((x \leftharpoondown y) \vee 0) \leqslant x \vee y,$$
$$x \rightharpoonup x \geqslant 0, \quad x \leftharpoondown x \geqslant 0.$$

Note that the condition (2) is equivalent to the following system of identities (see [10]):

$$(x \rightharpoonup y) + y \geqslant x, \quad y + (x \leftharpoondown y) \geqslant x,$$
$$x \rightharpoonup y \leqslant (x \vee z) \rightharpoonup y, \quad x \leftharpoondown y \leqslant (x \vee z) \leftharpoondown y,$$
$$(x + y) \rightharpoonup y \leqslant x, \quad (y + x) \leftharpoondown y \leqslant x.$$

Also, Swamy indroduced the notion of an *ideal* of a commutative $DR\ell$-monoid as a nonempty subset closed under "+" containing with any x also all y such that $y * 0 \leqslant x * 0$ (where $a * b = (a - b) \vee (b - a)$ is the *symmetric difference* of a and b). In addition, ideals and congruence relations are in a one-to-one correspondence; for any ideal I of a commutative $DR\ell$-monoid $\mathfrak{A}$, the corresponding congruence relation $\Theta(I)$ is defined by $\langle x, y \rangle \in \Theta(I)$ iff $x * y \in I$.

We generalize the notion of an ideal and, in an attempt to describe congruence kernels of noncommutative $DR\ell$-monoids, we introduce normal ideals which in the case that a $DR\ell$-monoid is an ℓ-group coincide with ℓ-ideals.

The concepts of distance functions and normal ideals are motivated by GMV-algebras (pseudo MV-algebras) which are included among $DR\ell$-monoids (see [10]).

Recall that GMV-algebras were introduced by J. Rachůnek in [10] (and independently by G. Georgescu and A. Iorgulescu in [4] under the name pseudo MV-algebras) to be a noncommutative generalization of MV-algebras. As shown in [10], if $(A, \oplus, \neg, \sim, 0, 1)$ is a GMV-algebra with the additional binary operation "$\odot$" defined by $x \odot y = \sim(\neg x \oplus \neg y)$ and if we put $x \vee y = (\neg x \odot y) \oplus x$, $x \wedge y = (\neg x \oplus y) \odot x$, $x \rightharpoonup y = \neg y \odot x$, and $x \leftharpoondown y = x \odot \sim y$, then $(A, \oplus, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a bounded $DR\ell$-monoid whose greatest element is 1.

2. Distance functions, absolute value

Definition. Let $\mathfrak{A}$ be a $DR\ell$-monoid. We define the *distance functions* by

$$d_1(x, y) := (x \rightharpoonup y) \vee (y \rightharpoonup x),$$
$$d_2(x, y) := (x \leftharpoondown y) \vee (y \leftharpoondown x),$$

for any $x, y \in A$.

Further, for each $x \in A$, $|x| := d_1(x, 0)$ is the *absolute value of* x, and $x^+ := x \vee 0$ is the *positive part of* x.

Before stating some results concerning the above notions, it is useful to mention basic properties of $DR\ell$-monoids.

Lemma 1 [8, Lemmas 1.1.7, 1.1.5, 1.1.8, 1.1.12]. *In any $DR\ell$-monoid we have*

(1) $x \vee y = (x \rightharpoonup y)^+ + y = y + (x \leftharpoondown y)^+$;

(2) $x \rightharpoonup x = x \leftharpoondown x = 0$;

(3) $x \geqslant y \Longrightarrow x \rightharpoonup z \geqslant y \rightharpoonup z$, $x \leftharpoondown z \geqslant y \leftharpoondown z$, $z \rightharpoonup x \leqslant z \rightharpoonup y$, *and* $z \leftharpoondown x \leqslant z \leftharpoondown y$;

(4) $x \rightharpoonup (y + z) = (x \rightharpoonup z) \rightharpoonup y$, $x \leftharpoondown (y + z) = (x \leftharpoondown y) \leftharpoondown z$.

Lemma 2. *Suppose that all joins and meets on the left-hand side exist. Then the following is valid:*

(1) $x + \bigwedge\limits_{\lambda \in \Lambda} y_\lambda = \bigwedge\limits_{\lambda \in \Lambda} (x + y_\lambda)$, $\bigwedge\limits_{\lambda \in \Lambda} y_\lambda + x = \bigwedge\limits_{\lambda \in \Lambda} (y_\lambda + x)$;

(2) $x \rightharpoonup \bigwedge\limits_{\lambda \in \Lambda} y_\lambda = \bigvee\limits_{\lambda \in \Lambda} (x \rightharpoonup y_\lambda)$, $x \leftharpoondown \bigwedge\limits_{\lambda \in \Lambda} y_\lambda = \bigvee\limits_{\lambda \in \Lambda} (x \leftharpoondown y_\lambda)$;

(3) $\bigvee\limits_{\lambda \in \Lambda} x_\lambda \rightharpoonup y = \bigvee\limits_{\lambda \in \Lambda} (x_\lambda \rightharpoonup y)$, $\bigvee\limits_{\lambda \in \Lambda} x_\lambda \leftharpoondown y = \bigvee\limits_{\lambda \in \Lambda} (x_\lambda \leftharpoondown y)$;

(4) $x \vee \bigwedge\limits_{\lambda \in \Lambda} y_\lambda = \bigwedge\limits_{\lambda \in \Lambda} (x \vee y_\lambda)$.

Remark. (2) and (3) extend [8, Lemma 1.1.9] for the arbitrary existing joins and meets, respectively.

P r o o f. (1) From $y_\lambda \geqslant \bigwedge\limits_{\lambda \in \Lambda} y_\lambda$ it follows that $x + y_\lambda \geqslant x + \bigwedge\limits_{\lambda \in \Lambda} y_\lambda$, for any $\lambda \in \Lambda$. Conversely, if there is $z \in A$ with $x + y_\lambda \geqslant z$, for all $\lambda \in \Lambda$, then $y_\lambda \geqslant z \leftharpoondown x$, for all $\lambda \in \Lambda$, and so $\bigwedge\limits_{\lambda \in \Lambda} y_\lambda \geqslant z \leftharpoondown x$, whence $x + \bigwedge\limits_{\lambda \in \Lambda} y_\lambda \geqslant z$, proving the first identity in (1). The rest of (1), and (2) and (3) have a similar proof.

(4) Obviously, $x \vee \bigwedge\limits_{\lambda \in \Lambda} y_\lambda \leqslant x \vee y_\lambda$ for all $\lambda \in \Lambda$. Choose $z \in A$ such that $z \leqslant x \vee y_\lambda = (x \rightharpoonup y_\lambda)^+ + y_\lambda$ for each $\lambda \in \Lambda$. Then $y_\lambda \geqslant z \leftharpoondown (x \rightharpoonup y_\lambda)^+$, for all $\lambda \in \Lambda$, and therefore $\bigwedge\limits_{\lambda \in \Lambda} y_\lambda \geqslant z \leftharpoondown (x \rightharpoonup \bigwedge\limits_{\lambda \in \Lambda} y_\lambda)^+$ which gives $z \leqslant (x \rightharpoonup \bigwedge\limits_{\lambda \in \Lambda} y_\lambda)^+ + \bigwedge\limits_{\lambda \in \Lambda} y_\lambda = x \vee \bigwedge\limits_{\lambda \in \Lambda} y_\lambda$. □

Corollary 3 [8, Theorem 1.1.23]. *For any $DR\ell$-monoid $\mathfrak{A}$, the lattice $\mathfrak{L}(\mathfrak{A}) = (A, \vee, \wedge)$ is distributive.*

Lemma 4 [8, Lemma 1.1.11]**.** *For all x, y of any DRℓ-monoid, it holds*

$$(x \rightharpoonup y) \vee (y \rightharpoonup x) = (x \vee y) \rightharpoonup (x \wedge y),$$
$$(x \leftharpoondown y) \vee (y \leftharpoondown x) = (x \vee y) \leftharpoondown (x \wedge y).$$

P r o o f. Using Lemma 2, (2) and (3), we obtain $(x \vee y) \rightharpoonup (x \wedge y) = (x \rightharpoonup y) \vee (y \rightharpoonup x) \vee 0$. However, $(x \rightharpoonup y) \vee (y \rightharpoonup x) \geqslant (x \rightharpoonup (x \vee y)) \vee (y \rightharpoonup (x \vee y)) = (x \vee y) \rightharpoonup (x \vee y) = 0$, again by Lemma 2. $\square$

Lemma 5 [8, Lemma 1.1.15]**.** *If $x \geqslant y \geqslant z$ then*

$$(x \rightharpoonup y) + (y \rightharpoonup z) = x \rightharpoonup z \quad \textit{and} \quad (y \leftharpoondown z) + (x \leftharpoondown y) = x \leftharpoondown z.$$

P r o o f. If $y \geqslant z$ then $y \rightharpoonup z \geqslant 0$ and $y = y \vee z = (y \rightharpoonup z)^+ + z = (y \rightharpoonup z) + z$. Hence $x \rightharpoonup y = x \rightharpoonup ((y \rightharpoonup z) + z) = (x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)$. Similarly, $x \geqslant y$ entails $x \rightharpoonup z \geqslant y \rightharpoonup z$ which yields $x \rightharpoonup z = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) + (y \rightharpoonup z)$. Summarizing, $x \rightharpoonup z = (x \rightharpoonup y) + (y \rightharpoonup z)$. $\square$

Lemma 6 [8, Lemmas 1.1.5, 1.1.13]**.** *The following holds in any DRℓ-monoid:*
(1) $0 \rightharpoonup x = 0 \leftharpoondown x$,
(2) $(x \rightharpoonup y) + (y \rightharpoonup z) \geqslant x \rightharpoonup z$,
(3) $(y \leftharpoondown z) + (x \leftharpoondown y) \geqslant x \leftharpoondown z$.

P r o o f. (1) From $(x + (0 \rightharpoonup x)) + x = x + ((0 \rightharpoonup x) + x) \geqslant x + 0 = x$ it follows that $x + (0 \rightharpoonup x) \geqslant x \rightharpoonup x = 0$, whence $0 \rightharpoonup x \geqslant 0 \leftharpoondown x$. Similarly, $0 \leftharpoondown x \geqslant 0 \rightharpoonup x$.
(2) and similarly (3) $(x \rightharpoonup y) + (y \rightharpoonup z) + z \geqslant (x \rightharpoonup y) + y \geqslant x$ implies $(x \rightharpoonup y) + (y \rightharpoonup z) \geqslant x \rightharpoonup z$. $\square$

Applying (2) and (3), we immediately get

Lemma 7. *In every DRℓ-monoid we have*
(1) $y \rightharpoonup x \geqslant (z \rightharpoonup x) \leftharpoondown (z \rightharpoonup y)$,
(2) $y \leftharpoondown x \geqslant (z \leftharpoondown x) \rightharpoonup (z \leftharpoondown y)$,
(3) $y \rightharpoonup x \geqslant (y \rightharpoonup z) \rightharpoonup (x \rightharpoonup z)$,
(4) $y \leftharpoondown x \geqslant (y \leftharpoondown z) \leftharpoondown (x \leftharpoondown z)$.

Proposition 8. *The distance functions have the following properties:*

(1) $d_1(x,y) = d_1(y,x)$,
(2) $d_2(x,y) = d_2(y,x)$,
(3) $d_1(x,0) = d_2(x,0)$,
(4) $d_1(x,y) = (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+$,
(5) $d_2(x,y) = (y \leftharpoondown x)^+ + (x \leftharpoondown y)^+$,
(6) $d_1(x,y) \geqslant 0$ *with* $d_1(x,y) = 0$ *iff* $x = y$,
(7) $d_2(x,y) \geqslant 0$ *with* $d_2(x,y) = 0$ *iff* $x = y$,
(8) $d_1(x,y) \leqslant d_1(x,z) + d_1(z,y) + d_1(x,z)$,
(9) $d_1(x,y) \leqslant d_1(z,y) + d_1(x,z) + d_1(z,y)$,
(10) $d_2(x,y) \leqslant d_2(x,z) + d_2(z,y) + d_2(x,z)$,
(11) $d_2(x,y) \leqslant d_2(z,y) + d_2(x,z) + d_2(z,y)$,
(12) $d_1(x,y) \vee d_1(s,t) \geqslant d_1(x \vee s, y \vee t), d_1(x \wedge s, y \wedge t)$,
(13) $d_2(x,y) \vee d_2(s,t) \geqslant d_2(x \vee s, y \vee t), d_2(x \wedge s, y \wedge t)$,
(14) $d_2(z \rightharpoonup x, z \rightharpoonup y) \leqslant d_1(x,y)$,
(15) $d_1(z \leftharpoondown x, z \leftharpoondown y) \leqslant d_2(x,y)$,
(16) $d_1(x \rightharpoonup z, y \rightharpoonup z) \leqslant d_1(x,y)$,
(17) $d_2(x \leftharpoondown z, y \leftharpoondown z) \leqslant d_2(x,y)$.

P r o o f. Obviously, (1) and (2) hold; (3) follows by Lemma 6 (1). To check (4), and similarly (5), we compute

$$\begin{aligned} d_1(x,y) &= (x \rightharpoonup y) \vee (y \rightharpoonup x) = (x \vee y) \rightharpoonup (x \wedge y) && \text{by Lemma 4} \\ &= [(x \vee y) \rightharpoonup y] + [y \rightharpoonup (x \wedge y)] && \text{by Lemma 5} \\ &= [(x \rightharpoonup y) \vee (y \rightharpoonup y)] + [(y \rightharpoonup x) \vee (y \rightharpoonup y)] && \text{by Lemma 2} \\ &= [(x \rightharpoonup y) \vee 0] + [(y \rightharpoonup x) \vee 0] \\ &= (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+. \end{aligned}$$

Further, (6) follows from (4) and (7) from (5), respectively, since

$$d_1(x,y) = (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+ \geqslant 0.$$

It is clear that $x = y$ entails $d_1(x,y) = 0$. Conversely, if

$$d_1(x,y) = (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+ = 0$$

then $(x \rightharpoonup y)^+ = (y \rightharpoonup x)^+ = 0$. Hence $x \rightharpoonup y \leqslant 0$ and $y \rightharpoonup x \leqslant 0$, and so $x \leqslant y$ and $y \leqslant x$, thus $x = y$.

Now, we will prove (8) (similarly (9), (10) and (11)):

$$
\begin{aligned}
&d_1(x,z)+d_1(z,y)+d_1(x,z)\\
&\quad=[(x\rightharpoonup z)\vee(z\rightharpoonup x)]+[(z\rightharpoonup y)\vee(y\rightharpoonup z)]+[(x\rightharpoonup z)\vee(z\rightharpoonup x)]\\
&\quad=[(x\rightharpoonup z)+(z\rightharpoonup y)+(x\rightharpoonup z)]\vee[(x\rightharpoonup z)+(z\rightharpoonup y)+(z\rightharpoonup x)]\\
&\qquad\vee[(x\rightharpoonup z)+(y\rightharpoonup z)+(x\rightharpoonup z)]\vee[(x\rightharpoonup z)+(y\rightharpoonup z)+(z\rightharpoonup x)]\\
&\qquad\vee[(z\rightharpoonup x)+(z\rightharpoonup y)+(x\rightharpoonup z)]\vee[(z\rightharpoonup x)+(z\rightharpoonup y)+(z\rightharpoonup x)]\\
&\qquad\vee[(z\rightharpoonup x)+(y\rightharpoonup z)+(x\rightharpoonup z)]\vee[(z\rightharpoonup x)+(y\rightharpoonup z)+(z\rightharpoonup x)]\\
&\quad\geqslant[(x\rightharpoonup z)+(z\rightharpoonup y)+(x\rightharpoonup z)]\vee[(x\rightharpoonup z)+(z\rightharpoonup y)+(z\rightharpoonup x)]\\
&\qquad\vee[(x\rightharpoonup z)+(y\rightharpoonup z)+(z\rightharpoonup x)]\vee[(z\rightharpoonup x)+(y\rightharpoonup z)+(z\rightharpoonup x)]\\
&\quad=[((x\rightharpoonup z)+(z\rightharpoonup y))+((x\rightharpoonup z)\vee(z\rightharpoonup x))]\\
&\qquad\vee[((x\rightharpoonup z)\vee(z\rightharpoonup x))+((y\rightharpoonup z)+(z\rightharpoonup y))]\\
&\qquad\text{(using } (x\rightharpoonup z)\vee(z\rightharpoonup x)\geqslant 0\text{, by (4))}\\
&\quad\geqslant[(x\rightharpoonup z)+(z\rightharpoonup y)]\vee[(y\rightharpoonup z)+(z\rightharpoonup x)]\\
&\quad\geqslant(x\rightharpoonup y)\vee(y\rightharpoonup x)=d_1(x,y).
\end{aligned}
$$

Let us verify (12):

$$
\begin{aligned}
d_1(x,y)\vee d_1(s,t)&=(x\rightharpoonup y)\vee(y\rightharpoonup x)\vee(s\rightharpoonup t)\vee(t\rightharpoonup s)\\
&=(x\rightharpoonup y)\vee(s\rightharpoonup t)\vee(y\rightharpoonup x)\vee(t\rightharpoonup s)\\
&\geqslant[x\rightharpoonup(y\vee t)]\vee[s\rightharpoonup(y\vee t)]\vee[y\rightharpoonup(x\vee s)]\vee[t\rightharpoonup(x\vee s)]\\
&\quad\text{(by Lemma 2)}\\
&=[(x\vee s)\rightharpoonup(y\vee t)]\vee[(y\vee t)\rightharpoonup(x\vee s)]=d_1(x\vee s,y\vee t).
\end{aligned}
$$

The rest of (12) and (13) is analogous. Finally, (14)–(17) are consequences of Lemma 7. □

Proposition 9. *The following holds in any DRℓ-monoid:*
(1) $|x|\geqslant 0$ *with* $|x|=0$ *iff* $x=0$,
(2) $|x|=x$ *iff* $x\geqslant 0$,
(3) $|x+y|\leqslant|x|+|y|+|x|$, $|x+y|\leqslant|y|+|x|+|y|$,
(4) $|x\vee y|\leqslant|x|\vee|y|$.

P r o o f. (1) follows immediately by Proposition 8 (6), (7); (4) is a consequence of Proposition 8 (12).

(2) If $x\geqslant 0$ then $x\geqslant 0\geqslant 0\rightharpoonup x$, whence $|x|=x\vee(0\rightharpoonup x)=x$. Obviously, $x=|x|$ entails $x\geqslant 0$.

(3) Since

$$\begin{aligned} d_1(x+y,y) &= [(x+y) \rightharpoonup y] \vee [y \rightharpoonup (x+y)] \\ &= [(x+y) \rightharpoonup y] \vee [(y \rightharpoonup y) \rightharpoonup x] \\ &= [(x+y) \rightharpoonup y] \vee (0 \rightharpoonup x) \\ &\leqslant x \vee (0 \rightharpoonup x) = |x|, \end{aligned}$$

it follows that

$$|x+y| = d_1(x+y,0) \leqslant d_1(x+y,y) + d_1(y,0) + d_1(x+y,y) \leqslant |x| + |y| + |x|.$$

$\square$

3. Ideals

Definition. Let $\mathfrak{A}$ be a $DR\ell$-monoid. A subset $I \subseteq A$ is said to be an *ideal of* $\mathfrak{A}$ if the following conditions are fulfilled:
(I1) $0 \in I$;
(I2) if $x, y \in I$ then $x + y \in I$;
(I3) if $x \in I, y \in A$ and $|y| \leqslant |x|$ then $y \in I$.

It can be easily seen that the intersection of any family of ideals of $\mathfrak{A}$ is still an ideal. For any $M \subseteq A$, the smallest ideal containing M, i.e., the intersection of all ideals I such that $M \subseteq I$, is called the *ideal generated by* M. It will be denoted by $I(M)$.

Proposition 10. *Let $\mathfrak{A}$ be a $DR\ell$-monoid. Then for any $\emptyset \neq M \subseteq A$, for any $a \in A$, and for any ideal J we have*
(1) $I(M) = \{x \in A;\ |x| \leqslant |a_1| + \ldots + |a_n|$ *for some* $a_1, \ldots, a_n \in M, n \geqslant 1\}$;
(2) $I(a) = \{x \in A;\ |x| \leqslant n|a|$ *for some* $n \geqslant 1\}$;
(3) $I(J \cup \{a\}) = \Big\{x \in A;\ |x| \leqslant \sum\limits_{i=1}^{k} (a_i + n_i|a|)$, *for some* $a_1, \ldots, a_k \in J$, $n_1, \ldots, n_k \geqslant 0, k \geqslant 1\Big\}$.

Proof. (1) Suppose that $x, y \in I(M)$, i.e., $|x| \leqslant |a_1| + \ldots + |a_n|$, $|y| \leqslant |b_1| + \ldots + |b_m|$ for some $a_1, \ldots, a_n$, $b_1, \ldots, b_m \in M$ and $n, m \geqslant 1$. Then

$$\begin{aligned} |x+y| &\leqslant |x| + |y| + |x| \\ &\leqslant |a_1| + \ldots + |a_n| + |b_1| + \ldots + |b_m| + |a_1| + \ldots + |a_n|. \end{aligned}$$

Hence $x + y \in I(M)$. It is easy to see that $|y| \leqslant |x|$, $x \in I(M)$, implies $y \in I(M)$. Thus $I(M)$ is an ideal. Finally, if I is an ideal such that $M \subseteq I$ then $I(M) \subseteq I$.

(2) and (3) follow by (1); note only that $a_i \in J$ iff $|a_i| \in J$ since J is an ideal. $\square$

Lemma 11. *For each* $0 \leqslant x, y, z \in A$, *it holds* $x \wedge (y + z) \leqslant (x \wedge y) + (x \wedge z)$.

P r o o f. We compute $(x \wedge y) + (x \wedge z) = (x + x) \wedge (x + z) \wedge (y + x) \wedge (y + z) \geqslant x \wedge x \wedge x \wedge (y + z) = x \wedge (y + z)$. $\square$

Proposition 12. *If* $\mathfrak{A}$ *is a DRℓ-monoid then for all* $x, y \in A$ *we have*

$$I(x) \cap I(y) = I(|x| \wedge |y|) \quad \textit{and} \quad I(x) \vee I(y) = I(|x| \vee |y|) = I(|x| + |y|).$$

P r o o f. Since $\|x\| = |x|$ it is obvious that $I(x) = I(|x|)$. Further, $|x| \wedge |y| \leqslant |x|, |y|$ implies $|x| \wedge |y| \in I(x) \cap I(y)$. Thus $I(|x| \wedge |y|) \subseteq I(x) \cap I(y)$. Conversely, $z \in I(x) \cap I(y)$ iff $|z| \leqslant n|x|$ and $|z| \leqslant m|y|$ for some $n, m \in \mathbb{N}$. Hence $|z| \leqslant n|x| \wedge m|y| \leqslant nm(|x| \wedge |y|)$, by Lemma 11. Therefore, $z \in I(|x| \wedge |y|)$, and so $I(x) \cap I(y) \subseteq I(|x| \wedge |y|)$.

It is easy to see that $I(x) \vee I(y) \subseteq I(|x| \vee |y|) \subseteq I(|x| + |y|)$. Suppose that J is an ideal such that $I(x), I(y) \subseteq J$ and $z \in I(|x| + |y|)$. Then $|z| \leqslant n(|x| + |y|)$ for some $n \in \mathbb{N}$. But $|x|, |y| \in J$, thus $|x| + |y| \in J$ and $z \in J$. This yields $I(x) \vee I(y) = I(|x| \vee |y|) = I(|x| + |y|)$. $\square$

Theorem 13. *If* $\mathfrak{A}$ *is a DRℓ-monoid then any ideal* I *is a convex subalgebra in* $\mathfrak{A}$. *Conversely, if* C *is a convex subalgebra of* $\mathfrak{A}$ *such that, for each* $x \in A$, $|x| \in C$ *iff* $x \in C$, *then* C *is an ideal of* $\mathfrak{A}$.

P r o o f. If $x, y \in I$ then, by Proposition 8,

$$|d_1(x, y)| = d_1(x, y) \leqslant d_1(0, y) + d_1(x, 0) + d_1(0, y) = |y| + |x| + |y| \in I.$$

Hence $d_1(x, y) \in I$. Further,

$$|x \rightharpoonup y| = (x \rightharpoonup y) \vee (0 \rightharpoonup (x \rightharpoonup y)) \leqslant (x \rightharpoonup y) \vee (y \rightharpoonup x) = d_1(x, y) \in I$$

since $y \rightharpoonup x \geqslant 0 \rightharpoonup (x \rightharpoonup y)$. Thus $x \rightharpoonup y \in I$. Similarly, $d_2(x, y) \in I$, $|x \leftharpoondown y| \leqslant d_2(x, y) \in I$ and hence $x \leftharpoondown y \in I$.

To prove that I is a convex subset, suppose $a, b \in I$ and $a \wedge b \leqslant x \leqslant a \vee b$ for some $x \in A$. Then

$$\begin{aligned} |x| = x \vee (0 \rightharpoonup x) &\leqslant (a \vee b) \vee (0 \rightharpoonup (a \wedge b)) = a \vee b \vee (0 \rightharpoonup a) \vee (0 \rightharpoonup b) \\ &= a \vee (0 \rightharpoonup a) \vee b \vee (0 \rightharpoonup b) = |a| \vee |b| \leqslant |a| + |b| \in I. \end{aligned}$$

Hence $x \in I$.

The proof of the second statement is straightforward. $\square$

As argued at the beginning of this section, it is obvious that the set of all ideals of an arbitrary $DR\ell$-monoid, ordered by set inclusion, is a complete lattice.

Theorem 14. *For any $DR\ell$-monoid $\mathfrak{A}$, the lattice $\mathrm{Id}(\mathfrak{A})$ of all its ideals is algebraic and Brouwerian.*

P r o o f. It suffices to show that $\mathrm{Id}(\mathfrak{A})$ is distributive and algebraic. (It is well-known that every algebraic distributive lattice satisfies the join infinite distributive identity and any such a lattice is Brouwerian.)

Let $I, J, K \in \mathrm{Id}(\mathfrak{A})$ and suppose that $x \in I \cap (J \vee K)$. Then $|x| \leqslant a_1 + \ldots + a_n$, for some $0 \leqslant a_1, \ldots, a_n \in J \cup K$. Hence $|x| = |x| \wedge (a_1 + \ldots + a_n) \leqslant (|x| \wedge a_1) + \ldots + (|x| \wedge a_n)$. But $|x| \wedge a_i \in (I \cap J) \cup (I \cap K) \subseteq (I \cap J) \vee (I \cap K)$, for all $i = 1, \ldots, n$, and so $I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K)$, proving the distributivity of $\mathrm{Id}(\mathfrak{A})$.

Let $\emptyset \neq M \subseteq A$. For any $x \in A$, $x \in I(M)$ iff there are $a_1, \ldots, a_n \in M$ such that $|x| \leqslant |a_1| + \ldots + |a_n|$. Hence $x \in I(\{a_1, \ldots, a_n\})$ and therefore

$$I(M) = \bigcup\{I(X);\ X \subseteq M, |X| < \aleph_0\}.$$

Thus $M \mapsto I(M)$ is an algebraic closure operator and, consequently, $\mathrm{Id}(\mathfrak{A})$ is an algebraic lattice. □

The following result describes relative pseudocomplements in the lattice $\mathrm{Id}(\mathfrak{A})$.

Proposition 15. *For any ideals J, K of $\mathfrak{A}$, the relative pseudocomplement of J with respect to K is given by*

$$J * K = \{x \in A;\ |x| \wedge |a| \in K \text{ for any } a \in J\}.$$

P r o o f. Let us denote by H the set on the right-hand side. First, we will prove that H is an ideal. (I1) $0 \in H$, because $|0| \wedge |a| = 0 \in K$ for all $a \in J$. (I2) If $x, y \in H$ then, for each $a \in J$,

$$|x + y| \wedge |a| \leqslant (|x| + |y| + |x|) \wedge |a| \leqslant (|x| \wedge |a|) + (|y| \wedge |a|) + (|x| \wedge |a|) \in K;$$

so that $x + y \in H$. (I3) If $x \in H$ and $|y| \leqslant |x|$ then $|y| \wedge |a| \leqslant |x| \wedge |a| \in K$, for any $a \in J$, whence $y \in H$.

Now, we have to prove that $H = J * K$. If $x \in J \cap H$ then $|x| \wedge |x| \in K$, thus $x \in K$ and therefore $J \cap H \subseteq K$. In addition, from

$$J * K = \bigvee\{I \in \mathrm{Id}(\mathfrak{A});\ I \cap J \subseteq K\}$$

it follows that $H \subseteq J * K$. Conversely, if $x \in J * K$ then, for each $a \in J$, $|x| \wedge |a| \in J \cap (J * K) \subseteq K$ since $|x| \wedge |a| \leqslant |a| \in J$ and $|x| \wedge |a| \leqslant |x| \in J * K$. Hence $x \in H$. So $H = J * K$. □

The *pseudocomplement* of an ideal I is $I^* := I * \{0\}$.

Corollary 16. $I^* = \{x \in A;\ |x| \wedge |a| = 0 \text{ for each } a \in I\}$.

Let $\mathfrak{A}$ be a $DR\ell$-monoid and $I \in \mathrm{Id}(\mathfrak{A})$. Let us define two binary relations on A by

$$\langle x, y\rangle \in \Theta_1(I) \Longleftrightarrow d_1(x,y) \in I,$$
$$\langle x, y\rangle \in \Theta_2(I) \Longleftrightarrow d_2(x,y) \in I,$$

for each $x, y \in A$.

Lemma 17. *For any ideal I, $\Theta_1(I)$ and $\Theta_2(I)$ are equivalence relations.*

Proof. It is obvious that $\Theta_1(I)$ is reflexive and symmetric. The transitivity follows from Proposition 8. Indeed, if $\langle x, y\rangle, \langle y, z\rangle \in \Theta_1(I)$ then $d_1(x,z) \leqslant d_1(x,y) + d_1(y,z) + d_1(x,y) \in I$, hence $d_1(x,z) \in I$. Similarly for $\Theta_2(I)$. $\square$

Theorem 18. *For any ideal I of $\mathfrak{A}$, the relations $\Theta_1(I)$ and $\Theta_2(I)$ are congruence relations on the lattice $\mathfrak{L}(\mathfrak{A})$. Moreover, $I = [0]_{\Theta_1(I)} = [0]_{\Theta_2(I)}$.*

Proof. Let $\langle x, y\rangle, \langle s, t\rangle \in \Theta_1(I)$, i.e., $d_1(x,y), d_1(s,t) \in I$. Then, by Proposition 8,

$$d_1(x \vee s, y \vee t) \leqslant d_1(x,y) \vee d_1(s,t) \leqslant d_1(x,y) + d_1(s,t) \in I,$$
$$d_1(x \wedge s, y \wedge t) \leqslant d_1(x,y) \vee d_1(s,t) \leqslant d_1(x,y) + d_1(s,t) \in I.$$

Hence $\langle x \vee s, y \vee t\rangle, \langle x \wedge s, y \wedge t\rangle \in \Theta_1(I)$. Similarly for $\Theta_2(I)$.

For each $x \in A$, $x \in [0]_{\Theta_1(I)}$ iff $\langle x, 0\rangle \in \Theta_1(I)$ iff $d_1(x,0) = |x| \in I$ iff $x \in I$. $\square$

Theorem 19. *Let I be an ideal of a $DR\ell$-monoid $\mathfrak{A}$. Then $\mathfrak{L}(\mathfrak{A})/\Theta_1(I)$ is a distributive lattice whose partial order relation is defined by*

$$[x]_{\Theta_1(I)} \leqslant [y]_{\Theta_1(I)} \Longleftrightarrow (x \rightharpoonup y)^+ \in I.$$

Similarly, $\mathfrak{L}(\mathfrak{A})/\Theta_2(I)$ is a distributive lattice in which

$$[x]_{\Theta_2(I)} \leqslant [y]_{\Theta_2(I)} \Longleftrightarrow (x \leftharpoondown y)^+ \in I.$$

Proof. Since $\mathfrak{L}(\mathfrak{A})$ is a distributive lattice, by Corollary 3, so is $\mathfrak{L}(\mathfrak{A})/\Theta_1(I)$. Further, for each $x, y \in A$, $[x]_{\Theta_1(I)} \leqslant [y]_{\Theta_1(I)}$ iff $[x]_{\Theta_1(I)} \vee [y]_{\Theta_1(I)} = [x \vee y]_{\Theta_1(I)} =$

$[y]_{\Theta_1(I)}$ iff $\langle x \vee y, y\rangle \in \Theta_1(I)$ iff $d_1(x \vee y, y) \in I$ iff $(x \rightharpoonup y)^+ \in I$. Indeed, since

$$\begin{aligned} d_1(x \vee y, y) &= [((x \vee y) \rightharpoonup y) \vee 0] + [(y \rightharpoonup (x \vee y)) \vee 0] \\ &= [(x \rightharpoonup y) \vee (y \rightharpoonup y) \vee 0] + 0 \\ &= (x \rightharpoonup y) \vee 0 = (x \rightharpoonup y)^+. \end{aligned}$$

The proof of the other statement is analogous. □

4. Normal ideals

Definition. An ideal I is said to be *normal* if it satisfies the following condition, for each $x, y \in A$:

$$(x \rightharpoonup y)^+ \in I \Longleftrightarrow (x \leftharpoondown y)^+ \in I.$$

The set of all normal ideals of a $DR\ell$-monoid $\mathfrak{A}$ will be denoted by $\mathrm{N}(\mathfrak{A})$. For an ideal I, we denote $I^+ = \{x \in I;\ x \geqslant 0\}$.

Proposition 20. *For any $I \in \mathrm{Id}(\mathfrak{A})$, the following conditions are equivalent:*
(i) *$I \in \mathrm{N}(\mathfrak{A})$;*
(ii) *$x + I^+ = I^+ + x$, for any $x \in A$.*

Proof. (i) ⇒ (ii) Let $x \in A$, $h \in I^+$ and set $y = h + x \in I^+ + x$. It is clear that $y \geqslant x$ and, consequently, $y = x \vee y = (y \rightharpoonup x)^+ + x = x + (y \leftharpoondown x)^+$. From $h + x \geqslant y$ it follows that $h \geqslant y \rightharpoonup x \geqslant 0$, since $y \geqslant x$. Hence $(y \rightharpoonup x)^+ = y \rightharpoonup x \in I^+$. But $I \in \mathrm{N}(\mathfrak{A})$, so that $(y \leftharpoondown x)^+ \in I^+$. Thus $y \in x + I^+$. Similarly, $x + I^+ \subseteq I^+ + x$.

(ii) ⇒ (i) If $(y \rightharpoonup x)^+ \in I$ then $x \vee y = (y \rightharpoonup x)^+ + x = x + h$ for some $h \in I^+$. Therefore $y \leqslant x + h$, which yields $(y \leftharpoondown x)^+ \leqslant ((x + h) \leftharpoondown x)^+ \leqslant h \vee 0 = h \in I^+$. Thus $(y \leftharpoondown x)^+ \in I$. The converse is analogous. □

Lemma 21. *If J and K are normal ideals of a $DR\ell$-monoid $\mathfrak{A}$ then*

$$J \vee K = \{x \in A;\ |x| \leqslant a + b \text{ for some } a \in J^+,\ b \in K^+\}.$$

In addition, $J \vee K$ is a normal ideal of $\mathfrak{A}$.

Proof. Let us denote the set on the right-hand side by M. (I1) and (I3) are obviously satisfied. To prove (I2), let $x, y \in M$, i.e., $|x| \leqslant a+b$ and $|y| \leqslant c+d$ for some $a, c \in J^+$ and $b, d \in K^+$. Then $|x+y| \leqslant |x| + |y| + |x| \leqslant a+b+c+d+a+b = a' + b'$ for some $a' \in J^+$, $b' \in K^+$, by Proposition 20. Consequently, $M \in \mathrm{Id}(\mathfrak{A})$. Finally, it is easy to see that any ideal H such that $J, K \subseteq H$ contains M.

If $(x \rightharpoonup y)^+ \in J \vee K$ then $(x \rightharpoonup y)^+ \leqslant a + b$ for some $a \in J^+$, $b \in K^+$. Hence $a+b \geqslant x \rightharpoonup y$ iff $a+b+y \geqslant x$. Since $J, K \in \mathrm{N}(\mathfrak{A})$, there exist $a' \in J^+$ and $b' \in K^+$ such that $a+b+y = y+a'+b'$. Therefore $y+a'+b' \geqslant x$ iff $a'+b' \geqslant x \leftharpoondown y$, whence $a'+b' \geqslant (x \leftharpoondown y)^+$ and $(x \leftharpoondown y)^+ \in J \vee K$, proving that $J \vee K$ is a normal ideal. $\square$

Proposition 22. *Let $\mathfrak{A}$ be a DRℓ-monoid. Then $\mathrm{N}(\mathfrak{A})$ is a complete sublattice of $\mathrm{Id}(\mathfrak{A})$.*

Proof. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of normal ideals. Obviously, $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a normal ideal. Let us assume $(x \rightharpoonup y)^+ \in \bigvee_{\lambda \in \Lambda} I_\lambda$, for some $x, y \in A$; then $(x \rightharpoonup y)^+ \in \bigvee_{\lambda \in \Lambda_0} I_\lambda$, for some finite subset Λ_0 of Λ. Hence $(x \leftharpoondown y)^+ \in \bigvee_{\lambda \in \Lambda_0} I_\lambda$ since it is a normal ideal. Thus $(x \leftharpoondown y)^+ \in \bigvee_{\lambda \in \Lambda} I_\lambda$. The converse is analogous. $\square$

Proposition 23. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be DRℓ-monoids and $\varphi\colon A \to B$ a homomorphism. Then $\varphi^{-1}(0) = \{x \in A;\ \varphi(x) = 0\}$ is a normal ideal of $\mathfrak{A}$.*

Proof. Clearly, the conditions (I1) and (I2) hold. Suppose $\varphi(x) = 0$ and $|y| \leqslant |x|$. Then $\varphi(|x|) = \varphi(x \vee (0 \rightharpoonup x)) = \varphi(x) \vee (0 \rightharpoonup \varphi(x)) = 0$ and, consequently, $\varphi(|y|) = 0$. Hence $\varphi(y \vee (0 \rightharpoonup y)) = \varphi(y) \vee (0 \rightharpoonup \varphi(y)) = 0$, which gives $\varphi(y) = 0$. Thus, $\varphi^{-1}(0)$ is an ideal in $\mathfrak{A}$.

Finally, $(x \rightharpoonup y)^+ \in \varphi^{-1}(0)$ iff $\varphi((x \rightharpoonup y) \vee 0) = (\varphi(x) \rightharpoonup \varphi(y)) \vee 0 = 0$. Hence $0 \geqslant \varphi(x) \rightharpoonup \varphi(y)$ iff $\varphi(y) \geqslant \varphi(x)$ iff $0 \geqslant \varphi(x) \leftharpoondown \varphi(y)$. Therefore $0 = (\varphi(x) \leftharpoondown \varphi(y)) \vee 0 = \varphi((x \leftharpoondown y) \vee 0)$, thus $(x \leftharpoondown y)^+ \in \varphi^{-1}(0)$. $\square$

Proposition 24. *If $I \in \mathrm{N}(\mathfrak{A})$ then, for all $x, y \in A$, $d_1(x,y) \in I$ iff $d_2(x,y) \in I$.*

Proof. If $d_1(x,y) = (x \rightharpoonup y)^+ + (y \rightharpoonup x)^+ \in I$ then $(x \rightharpoonup y)^+, (y \rightharpoonup x)^+ \in I$. Since I is a normal ideal, this implies $(x \leftharpoondown y)^+, (y \leftharpoondown x)^+ \in I$. Hence $d_2(x,y) = (x \leftharpoondown y)^+ + (y \leftharpoondown x)^+ \in I$. $\square$

Corollary 25. *If I is a normal ideal then $\Theta_1(I) = \Theta_2(I)$; it will be denoted by $\Theta(I)$.*

Lemma 26. *Let $I \in \mathrm{N}(\mathfrak{A})$. If $\langle x, y\rangle \in \Theta(I)$ then, for each $z \in A$,*

$$\langle x \rightharpoonup z, y \rightharpoonup z\rangle \in \Theta(I), \quad \langle x \leftharpoondown z, y \leftharpoondown z\rangle \in \Theta(I),$$
$$\langle z \rightharpoonup x, z \rightharpoonup y\rangle \in \Theta(I), \quad \langle z \leftharpoondown x, z \leftharpoondown y\rangle \in \Theta(I).$$

Proof. This follows from Proposition 8 (14)–(17). $\square$

Theorem 27. *If I is a normal ideal of a $DR\ell$-monoid $\mathfrak{A}$ then $\Theta(I)$ is a congruence relation on $\mathfrak{A}$. In addition, $[0]_\Theta = I$.*

P r o o f. Let $\langle x, y\rangle \in \Theta(I)$ and $\langle s, t\rangle \in \Theta(I)$. Then $(x \rightharpoonup y)^+, (s \rightharpoonup t)^+ \in I$. Obviously, $x \leqslant x \vee y = (x \rightharpoonup y)^+ + y$ and $s \leqslant s \vee t = (s \rightharpoonup t)^+ + t$. Hence, it follows that

$$\begin{aligned} x + s &\leqslant (x \rightharpoonup y)^+ + y + (s \rightharpoonup t)^+ + t \\ &= (x \rightharpoonup y)^+ + (y + (s \rightharpoonup t)^+) + t \\ &= (x \rightharpoonup y)^+ + (h + y) + t \\ &= ((x \rightharpoonup y)^+ + h) + (y + t) \end{aligned}$$

for some $h \in I^+$ such that $y + (s \rightharpoonup t)^+ = h + y$. However, $((x \rightharpoonup y)^+ + h) + (y + t) \geqslant x + s$ iff $(x \rightharpoonup y)^+ + h \geqslant (x + s) \rightharpoonup (y + t)$. Therefore, $((x + s) \rightharpoonup (y + t))^+ \leqslant ((x \rightharpoonup y)^+ + h)^+ = (x \rightharpoonup y)^+ + h \in I$. So $((x + s) \rightharpoonup (y + t))^+ \in I$. We can similarly show that $((y + t) \rightharpoonup (x + s))^+ \in I$. Hence, we conclude that $d_1(x + s, y + t) = ((x + s) \rightharpoonup (y + t))^+ + ((y + t) \rightharpoonup (x + s))^+ \in I$ and $\langle x + s, y + t\rangle \in \Theta(I)$.

By Lemma 26, $\langle x \rightharpoonup s, y \rightharpoonup s\rangle \in \Theta(I)$ and $\langle y \rightharpoonup s, y \rightharpoonup t\rangle \in \Theta(I)$. This yields $\langle x \rightharpoonup s, y \rightharpoonup t\rangle \in \Theta(I)$. Similarly, $\langle x \leftharpoondown s, y \leftharpoondown t\rangle \in \Theta(I)$.

The rest follows by Theorem 18. $\square$

Theorem 28. *If Θ is a congruence on $\mathfrak{A}$ then $[0]_\Theta = \{x \in A;\ \langle x, 0\rangle \in \Theta\}$ is a normal ideal in $\mathfrak{A}$. Moreover, $\Theta = \Theta([0]_\Theta)$.*

P r o o f. The first part follows by Proposition 23. Further, we claim that

$$\text{(C)} \qquad \langle x, y\rangle \in \Theta \Longleftrightarrow \langle d_1(x, y), 0\rangle \in \Theta,$$

or equivalently,

$$\langle x, y\rangle \in \Theta \Longleftrightarrow \langle d_2(x, y), 0\rangle \in \Theta.$$

Indeed, if $\langle x, y\rangle \in \Theta$ then $\langle x \rightharpoonup y, 0\rangle \in \Theta$ and $\langle y \rightharpoonup x, 0\rangle \in \Theta$, whence $\langle d_1(x, y), 0\rangle = \langle (x \rightharpoonup y) \vee (y \rightharpoonup x), 0\rangle \in \Theta$. Conversely, $\langle d_1(x, y), 0\rangle \in \Theta$ iff $d_1(x, y) \in [0]_\Theta$ which implies $(x \rightharpoonup y)^+, (y \rightharpoonup x)^+ \in [0]_\Theta$. This gives

$$\begin{aligned} x \vee y = (x \rightharpoonup y)^+ + y \equiv 0 + y = y \quad (\Theta), \\ x \vee y = (y \rightharpoonup x)^+ + x \equiv 0 + x = x \quad (\Theta). \end{aligned}$$

Thus, by the transitivity, $\langle x, y\rangle \in \Theta$.

Now, $\Theta = \Theta([0]_\Theta)$ is an immediate consequence of (C). $\square$

Corollary 29. *In any $DR\ell$-monoid, there is a one-to-one correspondence between congruences and normal ideals.*

5. Deductive systems

It was proved in [8] that the variety of $DR\ell$-monoids is weakly regular, that is, $[0]_\Phi = [0]_\Psi$ entails $\Phi = \Psi$, for any congruences Φ, Ψ on an arbitrary $DR\ell$-monoid. Hence it follows that congruence kernels of $DR\ell$-monoids can also be described by means of so-called deductive systems (see [6]).

Definition. Let $\mathfrak{A}$ be a $DR\ell$-monoid and $D \subseteq A$. Then D is said to be a *deductive system* if the following conditions are fulfilled:

(D1) $0 \in D$;

(D2) if $x \in D$ and $d_1(x, y) \in D$ then $y \in D$;

(D3) if $x \in D$ then $d_1(x, 0) \in D$.

A deductive system D is called *compatible* iff the following holds:

If $d_1(x, y) \in D$ and $d_1(s, t) \in D$, for $x, y, s, t \in A$, then $d_1(f(x, s), f(y, t)) \in D$, for each $f \in \{+, \vee, \wedge, \rightharpoonup, \leftharpoondown\}$.

The following result is only a special case of [6, Theorems 1, 2] and it generalizes the analogous property of GMV-algebras ([7, Theorems 2.8, 2.9]).

Theorem 30. *Let $\mathfrak{A}$ be a $DR\ell$-monoid, $D \subseteq A$. Let us define a binary relation Θ_D via*

$$\langle x, y\rangle \in \Theta_D \Longleftrightarrow d_1(x, y) \in D,$$

for every $x, y \in A$. If D is a compatible deductive system then Θ_D is a congruence on $\mathfrak{A}$ such that $[0]_{\Theta_D} = D$. Conversely, if Θ is a congruence relation on $\mathfrak{A}$ then $[0]_\Theta$ is a compatible deductive system and $\Theta_{[0]_\Theta} = \Theta$.

Therefore by Theorems 27 and 28 we immediately obtain

Corollary 31. *If $\mathfrak{A}$ is a $DR\ell$-monoid and $D \subseteq A$ then the following conditions are equivalent:*

(i) *D is a normal ideal;*

(ii) *D is a compatible deductive system;*

(iii) *$D = [0]_\Theta$ for some congruence relation Θ on $\mathfrak{A}$.*

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