Jan Kühr
Ideals of noncommutative $DRL$-monoids


Terms of use:
© Institute of Mathematics AS CR, 2005

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
IDEALS OF NONCOMMUTATIVE $DR\ell$-MONOIDS

JAN KUHR, Olomouc

(Received March 21, 2002)

Abstract. In this paper, we introduce the concept of an ideal of a noncommutative dually residuated lattice ordered monoid and we show that congruence relations and certain ideals are in a one-to-one correspondence.

Keywords: dually residuated lattice ordered monoid, ideal, normal ideal

MSC 2000: 06F05, 06D35

1. Introduction

Commutative $DR\ell$-monoids (called $DR\ell$-semigroups) were introduced by K. L. N. Swamy in [11] as a common generalization of commutative $\ell$-groups and Brouwerian algebras. A noncommutative extension of $DR\ell$-semigroups is mentioned in [12], but the present definition, due to [8], is more general. In fact, Swamy’s noncommutative $DR\ell$-semigroup was considered as an algebra $(A, +, 0, \lor, \land, \neg)$, where “$\neg$” agrees with “$\rightarrow$”.

Definition. An algebra $\mathfrak{A} = (A, +, 0, \lor, \land, \neg, \rightarrow)$ is a dually residuated lattice ordered monoid, or simply a $DR\ell$-monoid, iff

(1) $(A, +, 0, \lor, \land)$ is an $\ell$-monoid, that is, $(A, +, 0)$ is a monoid, $(A, \lor, \land)$ is a lattice and, for any $x, y, s, t \in A$, the following distributive laws are satisfied:

\[
s + (x \lor y) + t = (s + x + t) \lor (s + y + t),
\]

\[
s + (x \land y) + t = (s + x + t) \land (s + y + t);
\]

(2) for any $x, y \in A$, $x \rightarrow y$ is the least $s \in A$ such that $s + y \geq x$, and $x \leftarrow y$ is the least $t \in A$ such that $y + t \geq x$;
(3) $\mathfrak{A}$ fulfills the identities
\[
((x \to y) \lor 0) + y \leq x \lor y, \quad y + ((x \leftarrow y) \lor 0) \leq x \lor y,
\]
\[x \to x \geq 0, \quad x \leftarrow x \geq 0.
\]

Note that the condition (2) is equivalent to the following system of identities (see [10]):
\[
(x \to y) + y \geq x, \quad y + (x \leftarrow y) \geq x,
\]
\[x \to y \leq (x \lor z) \to y, \quad x \leftarrow y \leq (x \lor z) \leftarrow y,
\]
\[(x + y) \to y \leq x, \quad (y + x) \leftarrow y \leq x.
\]

Also, Swamy introduced the notion of an ideal of a commutative $DR\ell$-monoid as a nonempty subset closed under “+” containing with any $x$ also all $y$ such that $y \ast 0 \leq x \ast 0$ (where $a \ast b = (a - b) \lor (b - a)$ is the symmetric difference of $a$ and $b$). In addition, ideals and congruence relations are in a one-to-one correspondence; for any ideal $I$ of a commutative $DR\ell$-monoid $\mathfrak{A}$, the corresponding congruence relation $\Theta(I)$ is defined by $(x, y) \in \Theta(I)$ iff $x \ast y \in I$.

We generalize the notion of an ideal and, in an attempt to describe congruence kernels of noncommutative $DR\ell$-monoids, we introduce normal ideals which in the case that a $DR\ell$-monoid is an $\ell$-group coincide with $\ell$-ideals.

The concepts of distance functions and normal ideals are motivated by $GMV$-algebras (pseudo $MV$-algebras) which are included among $DR\ell$-monoids (see [10]).

Recall that $GMV$-algebras were introduced by J. Rachůnek in [10] (and independently by G. Georgescu and A. Iorgulescu in [4] under the name pseudo $MV$-algebras) to be a noncommutative generalization of $MV$-algebras. As shown in [10], if $(A, \oplus, \neg, \sim, 0, 1)$ is a $GMV$-algebra with the additional binary operation “$\odot$” defined by $x \odot y = \sim (\neg x \oplus y)$ and if we put $x \lor y = (\neg x \odot y) \oplus x$, $x \land y = (\neg x \oplus y) \odot x$, $x \rightarrow y = \neg y \odot x$, and $x \leftarrow y = x \odot \sim y$, then $(A, \oplus, 0, \lor, \land, \rightarrow, \leftarrow)$ is a bounded $DR\ell$-monoid whose greatest element is 1.

2. Distance functions, absolute value

**Definition.** Let $\mathfrak{A}$ be a $DR\ell$-monoid. We define the distance functions by
\[
d_1(x, y) := (x \rightarrow y) \lor (y \rightarrow x),
\]
\[
d_2(x, y) := (x \leftarrow y) \lor (y \leftarrow x),
\]
for any $x, y \in A$. 

98
Further, for each \( x \in A \), \(|x| := d_1(x, 0)\) is the absolute value of \( x \), and \( x^+ := x \lor 0 \) is the positive part of \( x \).

Before stating some results concerning the above notions, it is useful to mention basic properties of \( DR \ell \)-monoids.

**Lemma 1** [8, Lemmas 1.1.7, 1.1.5, 1.1.8, 1.1.12]. In any \( DR \ell \)-monoid we have

1. \( x \lor y = (x \to y)^+ + y = y + (x \leftarrow y)^+ \);
2. \( x \to x = x \leftarrow x = 0 \);
3. \( x \geq y \implies x \to z \geq y \to z, x \leftarrow z \geq y \leftarrow z, z \to x \leq z \to y, \) and \( z \leftarrow x \leq z \leftarrow y \);
4. \( x \to (y + z) = (x \to z) \to y, x \leftarrow (y + z) = (x \leftarrow y) \leftarrow z \).

**Lemma 2.** Suppose that all joins and meets on the left-hand side exist. Then the following is valid:

1. \( x + \bigwedge_{\lambda \in \Lambda} y_\lambda = \bigwedge_{\lambda \in \Lambda} (x + y_\lambda), \bigwedge_{\lambda \in \Lambda} y_\lambda + x = \bigwedge_{\lambda \in \Lambda} (y_\lambda + x) \);
2. \( x \to \bigwedge_{\lambda \in \Lambda} y_\lambda = \bigvee_{\lambda \in \Lambda} (x \to y_\lambda), x \leftarrow \bigwedge_{\lambda \in \Lambda} y_\lambda = \bigvee_{\lambda \in \Lambda} (x \leftarrow y_\lambda) \);
3. \( \bigvee_{\lambda \in \Lambda} x_\lambda \to y = \bigvee_{\lambda \in \Lambda} (x_\lambda \to y), \bigvee_{\lambda \in \Lambda} x_\lambda \leftarrow y = \bigvee_{\lambda \in \Lambda} (x_\lambda \leftarrow y) \);
4. \( x \lor \bigwedge_{\lambda \in \Lambda} y_\lambda = \bigwedge_{\lambda \in \Lambda} (x \lor y_\lambda) \).

**Remark.** (2) and (3) extend [8, Lemma 1.1.9] for the arbitrary existing joins and meets, respectively.

**Proof.** (1) From \( y_\lambda \geq \bigwedge_{\lambda \in \Lambda} y_\lambda \) it follows that \( x + y_\lambda \geq x + \bigwedge_{\lambda \in \Lambda} y_\lambda \), for any \( \lambda \in \Lambda \). Conversely, if there is \( z \in A \) with \( x + y_\lambda \geq z \), for all \( \lambda \in \Lambda \), then \( y_\lambda \geq z \leftarrow x \), for all \( \lambda \in \Lambda \), and so \( \bigwedge_{\lambda \in \Lambda} y_\lambda \geq z \leftarrow x \), whence \( x + \bigwedge_{\lambda \in \Lambda} y_\lambda \geq z \), proving the first identity in (1). The rest of (1), and (2) and (3) have a similar proof.

(4) Obviously, \( x \lor \bigwedge_{\lambda \in \Lambda} y_\lambda \leq x \lor y_\lambda \) for all \( \lambda \in \Lambda \). Choose \( z \in A \) such that \( z \leq x \lor y_\lambda = (x \to y_\lambda)^+ + y_\lambda \) for each \( \lambda \in \Lambda \). Then \( y_\lambda \geq z \leftarrow (x \to y_\lambda)^+ \), for all \( \lambda \in \Lambda \), and therefore \( \bigwedge_{\lambda \in \Lambda} y_\lambda \geq z \leftarrow (x \to \bigwedge_{\lambda \in \Lambda} y_\lambda)^+ \) which gives \( z \leq (x \to \bigwedge_{\lambda \in \Lambda} y_\lambda)^+ + \bigwedge_{\lambda \in \Lambda} y_\lambda = x \lor \bigwedge_{\lambda \in \Lambda} y_\lambda \).

**Corollary 3** [8, Theorem 1.1.23]. For any \( DR \ell \)-monoid \( \mathfrak{A} \), the lattice \( \mathcal{L}(\mathfrak{A}) = (A, \lor, \land) \) is distributive.
Lemma 4 [8, Lemma 1.1.11]. For all $x, y$ of any $DR\ell$-monoid, it holds

$$
(x \rightarrow y) \lor (y \rightarrow x) = (x \lor y) \rightarrow (x \land y),
$$
$$
(x \leftarrow y) \lor (y \leftarrow x) = (x \lor y) \leftarrow (x \land y).
$$

Proof. Using Lemma 2, (2) and (3), we obtain $(x \lor y) \rightarrow (x \land y) = (x \rightarrow y) \lor (y \rightarrow x) \lor 0$. However, $(x \rightarrow y) \lor (y \rightarrow x) \geq (x \rightarrow (x \lor y)) \lor (y \rightarrow (x \lor y)) = (x \lor y) \rightarrow (x \lor y) = 0$, again by Lemma 2.

Lemma 5 [8, Lemma 1.1.15]. If $x \geq y \geq z$ then

$$
(x \rightarrow y) + (y \rightarrow z) = x \rightarrow z \quad \text{and} \quad (y \leftarrow z) + (x \leftarrow y) = x \leftarrow z.
$$

Proof. If $y \geq z$ then $y \rightarrow z \geq 0$ and $y = y \lor z = (y \rightarrow z)^+ + z = (y \rightarrow z) + z$. Hence $x \rightarrow y = x \rightarrow ((y \rightarrow z) + z) = (x \rightarrow z) \rightarrow (y \rightarrow z)$. Similarly, $x \geq y$ entails $x \rightarrow z \geq y \rightarrow z$ which yields $x \rightarrow z = ((x \rightarrow z) \rightarrow (y \rightarrow z)) + (y \rightarrow z)$. Summarizing, $x \rightarrow z = (x \rightarrow y) + (y \rightarrow z)$.

Lemma 6 [8, Lemmas 1.1.5, 1.1.13]. The following holds in any $DR\ell$-monoid:

1. $0 \rightarrow x = 0 \leftarrow x$,
2. $(x \rightarrow y) + (y \rightarrow z) \geq x \rightarrow z$,
3. $(y \leftarrow z) + (x \leftarrow y) \geq x \leftarrow z$.

Proof. (1) From $(x + (0 \rightarrow x)) + x = x + ((0 \rightarrow x) + x) \geq x + 0 = x$ it follows that $x + (0 \rightarrow x) \geq x \rightarrow x = 0$, whence $0 \rightarrow x \geq 0 \leftarrow x$. Similarly, $0 \leftarrow x \geq 0 \rightarrow x$.

(2) and similarly (3) $(x \rightarrow y) + (y \rightarrow z) + z \geq (x \rightarrow y) + y \geq x$ implies $(x \rightarrow y) + (y \rightarrow z) \geq x \rightarrow z$.

Applying (2) and (3), we immediately get

Lemma 7. In every $DR\ell$-monoid we have

1. $y \rightarrow x \geq (z \rightarrow x) \leftarrow (z \rightarrow y)$,
2. $y \leftarrow x \geq (z \leftarrow x) \rightarrow (z \leftarrow y)$,
3. $y \rightarrow x \geq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
4. $y \leftarrow x \geq (y \leftarrow z) \leftarrow (x \leftarrow z)$.
Proposition 8. The distance functions have the following properties:

1. \( d_1(x, y) = d_1(y, x), \)
2. \( d_2(x, y) = d_2(y, x), \)
3. \( d_1(x, 0) = d_2(x, 0), \)
4. \( d_1(x, y) = (x \rightarrow y)^+ + (y \rightarrow x)^+, \)
5. \( d_2(x, y) = (y \leftarrow x)^+ + (x \leftarrow y)^+, \)
6. \( d_1(x, y) \geq 0 \) with \( d_1(x, y) = 0 \) iff \( x = y, \)
7. \( d_2(x, y) \geq 0 \) with \( d_2(x, y) = 0 \) iff \( x = y, \)
8. \( d_1(x, y) \leq d_1(x, z) + d_1(z, y) + d_1(x, z), \)
9. \( d_2(x, y) \leq d_2(x, z) + d_2(z, y) + d_2(x, z), \)
10. \( d_1(x, y) \leq d_2(x, z) + d_2(z, y) + d_2(x, z), \)
11. \( d_2(x, y) \leq d_2(x, z) + d_2(z, y) + d_2(x, z), \)
12. \( d_1(x, y) \vee d_1(s, t) \geq d_1(x \vee s, y \vee t), d_1(x \wedge s, y \wedge t), \)
13. \( d_2(x, y) \vee d_2(s, t) \geq d_2(x \vee s, y \vee t), d_2(x \wedge s, y \wedge t), \)
14. \( d_2(z \rightarrow x, z \rightarrow y) \leq d_1(x, y), \)
15. \( d_1(z \leftarrow x, z \leftarrow y) \leq d_2(x, y), \)
16. \( d_2(x \leftarrow z, y \leftarrow z) \leq d_1(x, y), \)
17. \( d_2(x \leftarrow z, y \leftarrow z) \leq d_2(x, y). \)

Proof. Obviously, (1) and (2) hold; (3) follows by Lemma 6 (1). To check (4), and similarly (5), we compute

\[
\begin{align*}
   d_1(x, y) &= (x \rightarrow y) \vee (y \rightarrow x) = (x \vee y) \rightarrow (x \wedge y) & \text{by Lemma 4} \\
   &= [(x \vee y) \rightarrow y] + [y \rightarrow (x \wedge y)] & \text{by Lemma 5} \\
   &= [(x \rightarrow y) \vee (y \rightarrow y)] + [(y \rightarrow x) \vee (y \rightarrow y)] & \text{by Lemma 2} \\
   &= [(x \rightarrow y) \vee 0] + [(y \rightarrow x) \vee 0] \\
   &= (x \rightarrow y)^+ + (y \rightarrow x)^+.
\end{align*}
\]

Further, (6) follows from (4) and (7) from (5), respectively, since

\[
d_1(x, y) = (x \rightarrow y)^+ + (y \rightarrow x)^+ \geq 0.
\]

It is clear that \( x = y \) entails \( d_1(x, y) = 0 \). Conversely, if

\[
d_1(x, y) = (x \rightarrow y)^+ + (y \rightarrow x)^+ = 0
\]

then \( (x \rightarrow y)^+ = (y \rightarrow x)^+ = 0 \). Hence \( x \rightarrow y \leq 0 \) and \( y \rightarrow x \leq 0 \), and so \( x \leq y \) and \( y \leq x \), thus \( x = y \).
Now, we will prove (8) (similarly (9), (10) and (11)):

\[ d_1(x, z) + d_1(z, y) + d_1(x, z) \]
\[ = [(x \rightarrow z) \vee (z \rightarrow x)] + [(z \rightarrow y) \vee (y \rightarrow z)] + [(x \rightarrow z) \vee (z \rightarrow x)] \]
\[ = [(x \rightarrow z) + (z \rightarrow y) + (x \rightarrow z)] \vee [(x \rightarrow z) + (z \rightarrow y) + (z \rightarrow x)] \]
\[ \vee [(x \rightarrow z) + (z \rightarrow y) + (x \rightarrow z)] \vee [(x \rightarrow z) + (z \rightarrow y) + (z \rightarrow x)] \]
\[ \vee [(x \rightarrow z) + (z \rightarrow y) + (x \rightarrow z)] \vee [(z \rightarrow x) + (y \rightarrow z) + (z \rightarrow x)] \]
\[ \geq [(x \rightarrow z) + (z \rightarrow y) + (x \rightarrow z)] \vee [(x \rightarrow z) + (z \rightarrow y) + (z \rightarrow x)] \]
\[ \vee [(x \rightarrow z) + (z \rightarrow y) + (x \rightarrow z)] \vee [(z \rightarrow x) + (y \rightarrow z) + (z \rightarrow x)] \]
\[ = [[(x \rightarrow z) + (z \rightarrow y)] + ((x \rightarrow z) \vee (z \rightarrow x))] \]
\[ \vee [[(x \rightarrow z) \vee (z \rightarrow x)] + ((y \rightarrow z) + (z \rightarrow y))] \]
\[ (\text{using } (x \rightarrow z) \vee (z \rightarrow x) \geq 0, \text{ by (4)}) \]
\[ \geq [(x \rightarrow z) + (z \rightarrow y)] \vee [(y \rightarrow z) + (z \rightarrow x)] \]
\[ \geq (x \rightarrow y) \vee (y \rightarrow x) = d_1(x, y). \]

Let us verify (12):

\[ d_1(x, y) \vee d_1(s, t) = (x \rightarrow y) \vee (y \rightarrow x) \vee (s \rightarrow t) \vee (t \rightarrow s) \]
\[ = (x \rightarrow y) \vee (s \rightarrow t) \vee (y \rightarrow x) \vee (t \rightarrow s) \]
\[ \geq [x \rightarrow (y \vee t)] \vee [s \rightarrow (y \vee t)] \vee [y \rightarrow (x \vee s)] \vee [t \rightarrow (x \vee s)] \]
\[ (\text{by Lemma 2}) \]
\[ = [(x \vee s) \rightarrow (y \vee t)] \vee [(y \vee t) \rightarrow (x \vee s)] = d_1(x \vee s, y \vee t). \]

The rest of (12) and (13) is analogous. Finally, (14)–(17) are consequences of Lemma 7. \[\square\]

**Proposition 9.** The following holds in any DRℓ-monoid:

1. \(|x| \geq 0 \text{ with } |x| = 0 \text{ iff } x = 0\),
2. \(|x| = x \text{ iff } x \geq 0\),
3. \(|x + y| \leq |x| + |y| + |x|, |x + y| \leq |y| + |x| + |y|\),
4. \(|x \vee y| \leq |x| \vee |y|\).

**Proof.** (1) follows immediately by Proposition 8 (6), (7); (4) is a consequence of Proposition 8 (12).

(2) If \(x \geq 0\) then \(x \geq 0 \geq 0 \rightarrow x\), whence \(|x| = x \vee (0 \rightarrow x) = x\). Obviously, \(x = |x| \text{ entails } x \geq 0\).
(3) Since

\[ d_1(x + y, y) = [(x + y) \rightarrow y] \lor [y \rightarrow (x + y)] \]
\[ = [(x + y) \rightarrow y] \lor [(y \rightarrow y) \rightarrow x] \]
\[ = [(x + y) \rightarrow y] \lor (0 \rightarrow x) \]
\[ \leq x \lor (0 \rightarrow x) = |x|, \]

it follows that

\[ |x + y| = d_1(x + y, 0) \leq d_1(x + y, y) + d_1(y, 0) + d_1(x + y, y) \leq |x| + |y| + |x|. \]

\[ \square \]

3. Ideals

**Definition.** Let \( \mathfrak{A} \) be a DR\( \ell \)-monoid. A subset \( I \subseteq A \) is said to be an *ideal of \( \mathfrak{A} \)* if the following conditions are fulfilled:

(I1) \( 0 \in I \);

(I2) if \( x, y \in I \) then \( x + y \in I \);

(I3) if \( x \in I, y \in A \) and \( |y| \leq |x| \) then \( y \in I \).

It can be easily seen that the intersection of any family of ideals of \( \mathfrak{A} \) is still an ideal. For any \( M \subseteq A \), the smallest ideal containing \( M \), i.e., the intersection of all ideals \( I \) such that \( M \subseteq I \), is called the *ideal generated by \( M \)*. It will be denoted by \( I(M) \).

**Proposition 10.** Let \( \mathfrak{A} \) be a DR\( \ell \)-monoid. Then for any \( \emptyset \neq M \subseteq A \), for any \( a \in A \), and for any ideal \( J \) we have

(1) \( I(M) = \{ x \in A ; |x| \leq |a_1| + \ldots + |a_n| \text{ for some } a_1, \ldots, a_n \in M, n \geq 1 \}; \)

(2) \( I(a) = \{ x \in A ; |x| \leq n|a| \text{ for some } n \geq 1 \}; \)

(3) \( I(J \cup \{a\}) = \{ x \in A ; |x| \leq \sum_{i=1}^{k} (a_i + n_i |a|), \text{ for some } a_1, \ldots, a_k \in J, n_1, \ldots, n_k \geq 0, k \geq 1 \}. \)

**Proof.** (1) Suppose that \( x, y \in I(M) \), i.e., \( |x| \leq |a_1| + \ldots + |a_n|, |y| \leq |b_1| + \ldots + |b_m| \text{ for some } a_1, \ldots, a_n, b_1, \ldots, b_m \in M \) and \( n, m \geq 1 \). Then

\[ |x + y| \leq |x| + |y| + |x| \]
\[ \leq |a_1| + \ldots + |a_n| + |b_1| + \ldots + |b_m| + |a_1| + \ldots + |a_n|. \]
Hence \( x + y \in I(M) \). It is easy to see that \(|y| \leq |x|, x \in I(M)\), implies \( y \in I(M) \). Thus \( I(M) \) is an ideal. Finally, if \( I \) is an ideal such that \( M \subseteq I \) then \( I(M) \subseteq I \).

(2) and (3) follow by (1); note only that \( a_i \in J \) iff \( |a_i| \in J \) since \( J \) is an ideal. \( \square \)

**Lemma 11.** For each \( 0 \leq x, y, z \in A \), it holds \( x \wedge (y + z) \leq (x \wedge y) + (x \wedge z) \).

**Proof.** We compute \((x \wedge y) + (x \wedge z) = (x + x) \wedge (x + z) \wedge (y + x) \wedge (y + z) \geq x \wedge x \wedge (y + z) = x \wedge (y + z) \). \( \square \)

**Proposition 12.** If \( \mathfrak{A} \) is a \( DR\ell \)-monoid then for all \( x, y \in A \) we have
\[
I(x) \cap I(y) = I(|x| \wedge |y|) \quad \text{and} \quad I(x) \lor I(y) = I(|x| \lor |y|) = I(|x| + |y|).
\]

**Proof.** Since \(|x| = |x|\) it is obvious that \( I(x) = I(|x|) \). Further, \(|x| \wedge |y| \leq |x|, |y|\) implies \(|x| \wedge |y| \in I(x) \cap I(y)\). Thus \( I(|x| \wedge |y|) \subseteq I(x) \cap I(y) \). Conversely, \( z \in I(x) \cap I(y) \) iff \(|z| \leq n|x|\) and \(|z| \leq m|y|\) for some \( n, m \in \mathbb{N} \). Hence \(|z| \leq n|x| \wedge m|y| \leq nm(|x| \wedge |y|)\), by Lemma 11. Therefore, \( z \in I(|x| \wedge |y|) \), and so \( I(x) \cap I(y) \subseteq I(|x| \wedge |y|) \).

It is easy to see that \( I(x) \lor I(y) \subseteq I(|x| \lor |y|) \subseteq I(|x| + |y|) \). Suppose that \( J \) is an ideal such that \( I(x), I(y) \subseteq J \) and \( z \in I(|x| + |y|) \). Then \(|z| \leq n(|x| + |y|)\) for some \( n \in \mathbb{N} \). But \(|x|, |y| \in J\), thus \(|x| + |y| \in J \) and \( z \in J \). This yields \( I(x) \lor I(y) = I(|x| \lor |y|) = I(|x| + |y|) \). \( \square \)

**Theorem 13.** If \( \mathfrak{A} \) is a \( DR\ell \)-monoid then any ideal \( I \) is a convex subalgebra in \( \mathfrak{A} \). Conversely, if \( C \) is a convex subalgebra of \( \mathfrak{A} \) such that, for each \( x \in A \), \(|x| \in C \) iff \( x \in C \), then \( C \) is an ideal of \( \mathfrak{A} \).

**Proof.** If \( x, y \in I \) then, by Proposition 8,
\[
|d_1(x, y)| = d_1(x, y) \leq d_1(0, y) + d_1(x, 0) + d_1(0, y) = |y| + |x| + |y| \in I.
\]
Hence \( d_1(x, y) \in I \). Further,
\[
|x \rightarrow y| = (x \rightarrow y) \lor (0 \rightarrow (x \rightarrow y)) \leq (x \rightarrow y) \lor (y \rightarrow x) = d_1(x, y) \in I
\]
since \( y \rightarrow x \geq 0 \rightarrow (x \rightarrow y) \). Thus \( x \rightarrow y \in I \). Similarly, \( d_2(x, y) \in I \), \(|x \leftarrow y| \leq d_2(x, y) \in I \) and hence \( x \leftarrow y \in I \).

To prove that \( I \) is a convex subset, suppose \( a, b \in I \) and \( a \wedge b \leq x \leq a \lor b \) for some \( x \in A \). Then
\[
|x| = x \lor (0 \rightarrow x) \leq (a \lor b) \lor (0 \rightarrow (a \land b)) = a \lor b \lor (0 \rightarrow a) \lor (0 \rightarrow b) = a \lor (0 \rightarrow a) \lor b \lor (0 \rightarrow b) = |a| \lor |b| \leq |a| + |b| \in I
\]
Hence \( x \in I \).

The proof of the second statement is straightforward. \( \square \)
As argued at the beginning of this section, it is obvious that the set of all ideals of an arbitrary $DR\ell$-monoid, ordered by set inclusion, is a complete lattice.

**Theorem 14.** For any $DR\ell$-monoid $\mathfrak{A}$, the lattice $Id(\mathfrak{A})$ of all its ideals is algebraic and Brouwerian.

**Proof.** It suffices to show that $Id(\mathfrak{A})$ is distributive and algebraic. (It is well-known that every algebraic distributive lattice satisfies the join infinite distributive identity and any such a lattice is Brouwerian.)

Let $I, J, K \in Id(\mathfrak{A})$ and suppose that $x \in I \cap (J \vee K)$. Then $|x| \leq a_1 + \ldots + a_n$, for some $0 \leq a_1, \ldots, a_n \in J \cup K$. Hence $|x| = |x| \wedge (a_1 + \ldots + a_n) \leq (|x| \wedge a_1) + \ldots + (|x| \wedge a_n)$. But $|x| \wedge a_i \in (I \cap J) \cup (I \cap K) \subseteq (I \cap J) \vee (I \cap K)$, for all $i = 1, \ldots, n$, and so $I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K)$, proving the distributivity of $Id(\mathfrak{A})$.

Let $\emptyset \neq M \subseteq A$. For any $x \in A$, $x \in I(M)$ iff there are $a_1, \ldots, a_n \in M$ such that $|x| \leq |a_1| + \ldots + |a_n|$. Hence $x \in I(\{a_1, \ldots, a_n\})$ and therefore

$$I(M) = \bigcup \{I(X); X \subseteq M, |X| < \aleph_0\}.$$ 

Thus $M \mapsto I(M)$ is an algebraic closure operator and, consequently, $Id(\mathfrak{A})$ is an algebraic lattice. 

The following result describes relative pseudocomplements in the lattice $Id(\mathfrak{A})$.

**Proposition 15.** For any ideals $J, K$ of $\mathfrak{A}$, the relative pseudocomplement of $J$ with respect to $K$ is given by

$$J \ast K = \{x \in A; |x| \wedge |a| \in K \text{ for any } a \in J\}.$$ 

**Proof.** Let us denote by $H$ the set on the right-hand side. First, we will prove that $H$ is an ideal. (I1) $0 \in H$, because $|0| \wedge |a| = 0 \in K$ for all $a \in J$. (I2) If $x, y \in H$ then, for each $a \in J$,

$$|x + y| \wedge |a| \leq (|x| + |y| + |x|) \wedge |a| \leq (|x| \wedge |a|) + (|y| \wedge |a|) + (|x| \wedge |a|) \in K;$$

so that $x + y \in H$. (I3) If $x \in H$ and $|y| \leq |x|$ then $|y| \wedge |a| \leq |x| \wedge |a| \in K$, for any $a \in J$, whence $y \in H$.

Now, we have to prove that $H = J \ast K$. If $x \in J \cap H$ then $|x| \wedge |x| \in K$, thus $x \in K$ and therefore $J \cap H \subseteq K$. In addition, from

$$J \ast K = \bigvee \{I \in Id(\mathfrak{A}); I \cap J \subseteq K\}$$

it follows that $H \subseteq J \ast K$. Conversely, if $x \in J \ast K$ then, for each $a \in J$, $|x| \wedge |a| \in J \cap (J \ast K) \subseteq K$ since $|x| \wedge |a| \leq |a| \in J$ and $|x| \wedge |a| \leq |x| \in J \ast K$. Hence $x \in H$. So $H = J \ast K$. 

\[105\]
The pseudocomplement of an ideal $I$ is $I^* := I \setminus \{0\}$.

**Corollary 16.** $I^* = \{x \in A; \ |x| \land |a| = 0 \text{ for each } a \in I\}$.

Let $\mathfrak{A}$ be a $DRL$-monoid and $I \in \operatorname{Id}(\mathfrak{A})$. Let us define two binary relations on $A$ by

\[
\langle x, y \rangle \in \Theta_1(I) \iff d_1(x, y) \in I,
\]

\[
\langle x, y \rangle \in \Theta_2(I) \iff d_2(x, y) \in I,
\]

for each $x, y \in A$.

**Lemma 17.** For any ideal $I$, $\Theta_1(I)$ and $\Theta_2(I)$ are equivalence relations.

**Proof.** It is obvious that $\Theta_1(I)$ is reflexive and symmetric. The transitivity follows from Proposition 8. Indeed, if $\langle x, y \rangle, \langle y, z \rangle \in \Theta_1(I)$ then $d_1(x, z) \leq d_1(x, y) + d_1(y, z) + d_1(x, y) \in I$, hence $d_1(x, z) \in I$. Similarly for $\Theta_2(I)$.

**Theorem 18.** For any ideal $I$ of $\mathfrak{A}$, the relations $\Theta_1(I)$ and $\Theta_2(I)$ are congruence relations on the lattice $\mathcal{L}(\mathfrak{A})$. Moreover, $I = \langle 0 \rangle_{\Theta_1(I)} = \langle 0 \rangle_{\Theta_2(I)}$.

**Proof.** Let $\langle x, y \rangle, \langle s, t \rangle \in \Theta_1(I)$, i.e., $d_1(x, y), d_1(s, t) \in I$. Then, by Proposition 8,

\[
d_1(x \lor s, y \lor t) \leq d_1(x, y) \lor d_1(s, t) \leq d_1(x, y) + d_1(s, t) \in I,
\]

\[
d_1(x \land s, y \land t) \leq d_1(x, y) \lor d_1(s, t) \leq d_1(x, y) + d_1(s, t) \in I.
\]

Hence $\langle x \lor s, y \lor t \rangle, \langle x \land s, y \land t \rangle \in \Theta_1(I)$. Similarly for $\Theta_2(I)$.

For each $x \in A$, $x \in \langle 0 \rangle_{\Theta_1(I)}$ iff $\langle x, 0 \rangle \in \Theta_1(I)$ iff $d_1(x, 0) = |x| \in I$ iff $x \in I$.

**Theorem 19.** Let $I$ be an ideal of a $DRL$-monoid $\mathfrak{A}$. Then $\mathcal{L}(\mathfrak{A})/\Theta_1(I)$ is a distributive lattice whose partial order relation is defined by

\[
[x]_{\Theta_1(I)} \leq [y]_{\Theta_1(I)} \iff (x \leftarrow y)^+ \in I.
\]

Similarly, $\mathcal{L}(\mathfrak{A})/\Theta_2(I)$ is a distributive lattice in which

\[
[x]_{\Theta_2(I)} \leq [y]_{\Theta_2(I)} \iff (x \leftarrow y)^+ \in I.
\]

**Proof.** Since $\mathcal{L}(\mathfrak{A})$ is a distributive lattice, by Corollary 3, so is $\mathcal{L}(\mathfrak{A})/\Theta_1(I)$. Further, for each $x, y \in A$, $[x]_{\Theta_1(I)} \leq [y]_{\Theta_1(I)}$ iff $[x]_{\Theta_1(I)} \lor [y]_{\Theta_1(I)} = [x \lor y]_{\Theta_1(I)} = [x \lor y]_{\Theta_1(I)} = [x \lor y]_{\Theta_1(I)}$.
\[ \Theta_1(I) \text{ iff } \langle x \lor y, y \rangle \in \Theta_1(I) \text{ iff } d_1(x \lor y, y) \in I \text{ iff } (x \rightarrow y)^+ \in I. \]
Indeed, since
\[ d_1(x \lor y, y) = \left[ ((x \lor y) \rightarrow y) \lor 0 \right] + \left[ (y \rightarrow (x \lor y)) \lor 0 \right] = (x \rightarrow y) \lor 0 = (x \rightarrow y)^+. \]
The proof of the other statement is analogous. \hfill \Box

4. Normal ideals

**Definition.** An ideal \( I \) is said to be normal if it satisfies the following condition, for each \( x, y \in A \):
\[ (x \rightarrow y)^+ \in I \iff (x \Leftarrow y)^+ \in I. \]

The set of all normal ideals of a \( DR \ell \)-monoid \( \mathfrak{A} \) will be denoted by \( N(\mathfrak{A}) \). For an ideal \( I \), we denote \( I^+ = \{ x \in I; \ x \geq 0 \} \).

**Proposition 20.** For any \( I \in \operatorname{Id}(\mathfrak{A}) \), the following conditions are equivalent:

(i) \( I \in N(\mathfrak{A}) \);
(ii) \( x + I^+ = I^+ + x, \) for any \( x \in A \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( x \in A, \ h \in I^+ \) and set \( y = h + x \in I^+ + x \). It is clear that \( y \geq x \) and, consequently, \( y = x \lor y = (y - x)^+ + x = x + (y \leftarrow x)^+ \). From \( h + x \geq y \) it follows that \( h \geq y \sim x \geq 0 \), since \( y \geq x \). Hence \( (y \rightarrow x)^+ = y - x \in I^+ \). But \( I \in N(\mathfrak{A}) \), so that \( (y \leftarrow x)^+ \in I^+ \). Thus \( y \in x + I^+ \). Similarly, \( x + I^+ \subseteq I^+ + x \).

(ii) \( \Rightarrow \) (i) If \( (y \rightarrow x)^+ \in I \) then \( x \lor y = (y \rightarrow x)^+ + x = x + h \) for some \( h \in I^+ \). Therefore \( y \leq x + h \), which yields \( (y \leftarrow x)^+ \leq ((x + h) \leftarrow x)^+ \leq h \lor 0 = h \in I^+ \). Thus \( (y \rightarrow x)^+ \in I \). The converse is analogous. \hfill \Box

**Lemma 21.** If \( J \) and \( K \) are normal ideals of a \( DR \ell \)-monoid \( \mathfrak{A} \) then
\[ J \lor K = \{ x \in A; \ |x| \leq a + b \text{ for some } a \in J^+, \ b \in K^+ \}. \]

In addition, \( J \lor K \) is a normal ideal of \( \mathfrak{A} \).

**Proof.** Let us denote the set on the right-hand side by \( M \). (I1) and (I3) are obviously satisfied. To prove (I2), let \( x, y \in M \), i.e., \( |x| \leq a + b \) and \( |y| \leq c + d \) for some \( a, c \in J^+ \) and \( b, d \in K^+ \). Then \( |x + y| \leq |x| + |y| + |x| \leq a + b + c + d + a + b = a' + b' \) for some \( a' \in J^+ \), \( b' \in K^+ \), by Proposition 20. Consequently, \( M \in \operatorname{Id}(\mathfrak{A}) \). Finally, it is easy to see that any ideal \( H \) such that \( J, K \subseteq H \) contains \( M \).
If $(x \rightarrow y)^+ \in J \lor K$ then $(x \rightarrow y)^+ \leq a + b$ for some $a \in J^+$, $b \in K^+$. Hence $a + b \geq x \rightarrow y$ iff $a + b + y \geq x$. Since $J, K \in N(\mathfrak{A})$, there exist $a' \in J^+$ and $b' \in K^+$ such that $a + b + y = y + a' + b'$. Therefore $y + a' + b' \geq x$ iff $a' + b' \geq x \leftarrow y$, whence $a' + b' \geq (x \rightarrow y)^+$ and $(x \rightarrow y)^+ \in J \lor K$, proving that $J \lor K$ is a normal ideal. \(\square\)

**Proposition 22.** Let $\mathfrak{A}$ be a $\text{DR} \ell$-monoid. Then $N(\mathfrak{A})$ is a complete sublattice of $\text{Id}(\mathfrak{A})$.

**Proof.** Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of normal ideals. Obviously, $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a normal ideal. Let us assume $(x \rightarrow y)^+ \in \bigvee_{\lambda \in \Lambda} I_\lambda$, for some $x, y \in A$; then $(x \rightarrow y)^+ \in \bigvee_{\lambda \in \Lambda_0} I_\lambda$, for some finite subset $\Lambda_0$ of $\Lambda$. Hence $(x \leftarrow y)^+ \in \bigvee_{\lambda \in \Lambda_0} I_\lambda$ since it is a normal ideal. Thus $(x \leftarrow y)^+ \in \bigvee_{\lambda \in \Lambda} I_\lambda$. The converse is analogous. \(\square\)

**Proposition 23.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\text{DR} \ell$-monoids and $\varphi: A \to B$ a homomorphism. Then $\varphi^{-1}(0) = \{x \in A; \varphi(x) = 0\}$ is a normal ideal of $\mathfrak{A}$.

**Proof.** Clearly, the conditions (I1) and (I2) hold. Suppose $\varphi(x) = 0$ and $|y| \leq |x|$. Then $\varphi(|x|) = \varphi(x \lor (0 \rightarrow x)) = \varphi(x) \lor (0 \rightarrow \varphi(x)) = 0$ and, consequently, $\varphi(|y|) = 0$. Hence $\varphi(y \lor (0 \rightarrow y)) = \varphi(y) \lor (0 \rightarrow \varphi(y)) = 0$, which gives $\varphi(y) = 0$. Thus, $\varphi^{-1}(0)$ is an ideal in $\mathfrak{A}$.

Finally, $(x \rightarrow y)^+ \in \varphi^{-1}(0)$ iff $\varphi((x \rightarrow y) \lor 0) = (\varphi(x) \rightarrow \varphi(y)) \lor 0 = 0$. Hence $0 \geq \varphi(x) \rightarrow \varphi(y)$ iff $\varphi(y) \geq \varphi(x)$ iff $0 \geq \varphi(x) \leftarrow \varphi(y)$. Therefore $0 = (\varphi(x) \leftarrow \varphi(y)) \lor 0 = \varphi((x \leftarrow y) \lor 0)$, thus $(x \leftarrow y)^+ \in \varphi^{-1}(0)$. \(\square\)

**Proposition 24.** If $I \in N(\mathfrak{A})$ then, for all $x, y \in A$, $d_1(x, y) \in I$ iff $d_2(x, y) \in I$.

**Proof.** If $d_1(x, y) = (x \rightarrow y)^+ + (y \rightarrow x)^+ \in I$ then $(x \rightarrow y)^+, (y \rightarrow x)^+ \in I$. Since $I$ is a normal ideal, this implies $(x \leftarrow y)^+, (y \leftarrow x)^+ \in I$. Hence $d_2(x, y) = (x \leftarrow y)^+ + (y \leftarrow x)^+ \in I$. \(\square\)

**Corollary 25.** If $I$ is a normal ideal then $\Theta_1(I) = \Theta_2(I)$; it will be denoted by $\Theta(I)$.

**Lemma 26.** Let $I \in N(\mathfrak{A})$. If $(x, y) \in \Theta(I)$ then, for each $z \in A$,

$$
\langle x \leftarrow z, y \rightarrow z \rangle \in \Theta(I), \quad \langle x \leftarrow z, y \leftarrow z \rangle \in \Theta(I),
$$

$$
\langle z \rightarrow x, z \leftarrow y \rangle \in \Theta(I), \quad \langle z \leftarrow x, z \leftarrow y \rangle \in \Theta(I).
$$

**Proof.** This follows from Proposition 8 (14)–(17). \(\square\)
**Theorem 27.** If $I$ is a normal ideal of a $DR\ell$-monoid $A$ then $\Theta(I)$ is a congruence relation on $A$. In addition, $[0]_\Theta = I$.

**Proof.** Let $\langle x, y \rangle \in \Theta(I)$ and $\langle s, t \rangle \in \Theta(I)$. Then $(x \rightarrow y)^+, (s \rightarrow t)^+ \in I$. Obviously, $x \leq x \lor y = (x \rightarrow y)^+ + y$ and $s \leq s \lor t = (s \rightarrow t)^+ + t$. Hence, it follows that

$$
x + s \leq (x \rightarrow y)^+ + y + (s \rightarrow t)^+ + t
= (x \rightarrow y)^+ + (y + (s \rightarrow t)^+) + t
= (x \rightarrow y)^+ + (h + y) + t
= ((x \rightarrow y)^+ + h) + (y + t)
$$

for some $h \in I^+$ such that $y + (s \rightarrow t)^+ = h + y$. However, $((x \rightarrow y)^+ + h) + (y + t) \geq x + s$ iff $(x \rightarrow y)^+ + h \geq (x + s) \rightarrow (y + t)$. Therefore, $((x + s) \rightarrow (y + t))^+ \leq ((x \rightarrow y)^+ + h)^+ = (x \rightarrow y)^+ + h \in I$. So $((x + s) \rightarrow (y + t))^+ \in I$. We can similarly show that $((y + t) \rightarrow (x + s))^+ \in I$. Hence, we conclude that $d_1(x + s, y + t) = ((x + s) \rightarrow (y + t))^+ + ((y + t) \rightarrow (x + s))^+ + 1 \in I$ and $\langle x + s, y + t \rangle \in \Theta(I)$.

By Lemma 26, $\langle x \rightarrow s, y \rightarrow s \rangle \in \Theta(I)$ and $\langle y \rightarrow s, y \rightarrow t \rangle \in \Theta(I)$. This yields $\langle x \rightarrow s, y \rightarrow t \rangle \in \Theta(I)$. Similarly, $\langle x \leftarrow s, y \rightarrow t \rangle \in \Theta(I)$.

The rest follows by Theorem 18. \hfill \Box

**Theorem 28.** If $\Theta$ is a congruence on $A$ then $[0]_\Theta = \{x \in A; \langle x, 0 \rangle \in \Theta\}$ is a normal ideal in $A$. Moreover, $\Theta = \Theta([0]_\Theta)$.

**Proof.** The first part follows by Proposition 23. Further, we claim that

(C) \hspace{1cm} \langle x, y \rangle \in \Theta \iff \langle d_1(x, y), 0 \rangle \in \Theta,

or equivalently,

$$
\langle x, y \rangle \in \Theta \iff \langle d_2(x, y), 0 \rangle \in \Theta.
$$

Indeed, if $\langle x, y \rangle \in \Theta$ then $\langle x \rightarrow y, 0 \rangle \in \Theta$ and $\langle y \rightarrow x, 0 \rangle \in \Theta$, whence $\langle d_1(x, y), 0 \rangle = \langle (x \rightarrow y) \lor (y \rightarrow x), 0 \rangle \in \Theta$. Conversely, $\langle d_1(x, y), 0 \rangle \in \Theta$ iff $d_1(x, y) \in [0]_\Theta$ which implies $(x \rightarrow y)^+, (y \rightarrow x)^+ \in [0]_\Theta$. This gives

$$
x \lor y = (x \rightarrow y)^+ + y \equiv 0 + y = y \quad (\Theta),
$$

$$
x \lor y = (y \rightarrow x)^+ + x \equiv 0 + x = x \quad (\Theta).
$$

Thus, by the transitivity, $\langle x, y \rangle \in \Theta$.

Now, $\Theta = \Theta([0]_\Theta)$ is an immediate consequence of (C). \hfill \Box
Corollary 29. In any $DR\ell$-monoid, there is a one-to-one correspondence between congruences and normal ideals.

5. Deductive systems

It was proved in [8] that the variety of $DR\ell$-monoids is weakly regular, that is, $[0]_{\Phi} = [0]_{\Psi}$ entails $\Phi = \Psi$, for any congruences $\Phi, \Psi$ on an arbitrary $DR\ell$-monoid. Hence it follows that congruence kernels of $DR\ell$-monoids can also be described by means of so-called deductive systems (see [6]).

Definition. Let $A$ be a $DR\ell$-monoid and $D \subseteq A$. Then $D$ is said to be a deductive system if the following conditions are fulfilled:

(D1) $0 \in D$;
(D2) if $x \in D$ and $d_1(x, y) \in D$ then $y \in D$;
(D3) if $x \in D$ then $d_1(x, 0) \in D$.

A deductive system $D$ is called compatible iff the following holds:

If $d_1(x, y) \in D$ and $d_1(s, t) \in D$, for $x, y, s, t \in A$, then $d_1(f(x, s), f(y, t)) \in D$, for each $f \in \{+, \lor, \land, \rightarrow, \leftarrow\}$.

The following result is only a special case of [6, Theorems 1, 2] and it generalizes the analogous property of $GMV$-algebras ([7, Theorems 2.8, 2.9]).

Theorem 30. Let $A$ be a $DR\ell$-monoid, $D \subseteq A$. Let us define a binary relation $\Theta_D$ via

$$\langle x, y \rangle \in \Theta_D \iff d_1(x, y) \in D,$$

for every $x, y \in A$. If $D$ is a compatible deductive system then $\Theta_D$ is a congruence on $A$ such that $[0]_{\Theta_D} = D$. Conversely, if $\Theta$ is a congruence relation on $A$ then $[0]_{\Theta}$ is a compatible deductive system and $\Theta_{[0]_{\Theta}} = \Theta$.

Therefore by Theorems 27 and 28 we immediately obtain

Corollary 31. If $A$ is a $DR\ell$-monoid and $D \subseteq A$ then the following conditions are equivalent:

(i) $D$ is a normal ideal;
(ii) $D$ is a compatible deductive system;
(iii) $D = [0]_{\Theta}$ for some congruence relation $\Theta$ on $A$. 
References


Author’s address: Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: kuhr@inf.upol.cz.