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ORBITS CONNECTING SINGULAR POINTS IN THE PLANE

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Abstract. This paper concerns the global structure of planar systems. It is shown that if a positively bounded system with two singular points has no closed orbits, the set of all bounded solutions is compact and simply connected. Also it is shown that for such a system the existence of connecting orbits is tightly related to the behavior of homoclinic orbits. A necessary and sufficient condition for the existence of connecting orbits is given. The number of connecting orbits is also discussed.

Keywords: connecting orbit, homoclinic orbit, positively bounded system

MSC 2000: 34C35

1. Introduction

In this paper, we consider the differential equations:

$$\text{(E)} \qquad \frac{\mathrm{d}x_1}{\mathrm{d}t} = X(x_1, x_2), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = Y(x_1, x_2)$$

in the plane $\mathbb{R}^2$, where X and Y are continuous, and assume that solutions of arbitrary initial value problems are unique. Suppose that the vector field $V = (X, Y)$ defines a flow $f(p,t)$. Further assume that the system (E) has exactly two singular points $p_1, p_2 \in \mathbb{R}^2$, i.e., $V(p_1) = V(p_2) = 0$. For $A \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, we denote $A \cdot I = \{f(x,t) \mid x \in A,\ t \in I\}$, in particular $x \cdot t = f(x,t)$. For a subset $C \subset \mathbb{R}^2$, $\overline{C}$, ∂C and $\operatorname{Int} C$ denote respectively the closure, the boundary and the interior of C. A set S is invariant if $S \cdot \mathbb{R} = S$ holds. Let $B_r = \{x = (x_1, x_2) \mid d(x, O) = \sqrt{{x_1}^2 + {x_2}^2} \leqslant r\}$ be the closed disc with center O and radius $r (> 0)$, where O is the origin of $\mathbb{R}^2$.

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Definition 1. If there exists a point $p \in \mathbb{R}^2$ such that $\lim\limits_{t\to-\infty} p\cdot t = p_1$ and $\lim\limits_{t\to+\infty} p\cdot t = p_2$, then the set $p\cdot\mathbb{R} = f(p,\mathbb{R})$ is called an orbit connecting p_1 and p_2 (sometimes called a heteroclinic orbit).

The existence of connecting orbits was first proposed by I. M. Gelfand [6] as an important problem, it relates to the studies of shock-wave solutions [2], [3]. On the other hand, as special invariant sets, such orbits are crucial objects of the global structure of dynamical systems. A considerable number of papers have been written in connection with this subject, recently the research is still going on (see [5], [8], [10], [11], [12], [13] and their references). It is obvious that for the system (E) connecting orbits and closed orbits circling a singular point do not coexist. Among those papers (e.g., [8], [10], [11], [12]), another crucial condition for the existence of connecting orbits is the absence of singular closed orbits.

Definition 2. The system (E) is called a positively bounded system, if for any $x \in \mathbb{R}^2$ there exists an $r = r(x) > 0$ such that the positive semi-orbit $O^+(x) = x\cdot\mathbb{R}^+$ lies in the closed disc B_r.

In [8] it was proved that for a positively bounded system (E), if there exist no closed orbits and singular closed orbits (for definition see the next section), then the system has a connecting orbit. In this paper, our purpose is to give a necessary and sufficient condition for the existence of connecting orbits without the assumption of absence of singular closed orbits, also the sufficiency strengthens the result of [8]. It is shown that the existence of connecting orbits is tightly related to the behavior of homoclinic orbits. Further, the number of connecting orbits is dicussed in the third section. In the last section our result is applied to a concrete system with a singular closed orbit.

2. Preliminaries

In this section we state some definitions and lemmas that will be used in the sequel.

Definition 2.1. A simple closed curve is called a singular closed orbit if it is the union of alternating nonclosed whole orbits and singular points, and is contained in the ω- (or α-)limit set of an orbit.

Definition 2.2. For a singular point p in $\mathbb{R}^2$, an orbit $O(x) = x\cdot\mathbb{R}$ $(x \neq p)$ is called a homoclinic orbit with respect to p provided that $\lim\limits_{t\to-\infty} x\cdot t = \lim\limits_{t\to+\infty} x\cdot t = p$.

Definition 2.3. If Y is a subset of $\mathbb{R}^2$, the ω-limit set of Y is defined to be the set $\omega(Y) = \bigcap\limits_{t\geqslant 0} \overline{Y\cdot[t,\infty)}$ (see [1]).

Remark. For the definition of $\omega(Y)$ in a more general setting and its basic properties we refer to [1], [9]. In this paper we only need the following result:

Lemma 2.4 [1]**.** *If $Y \cdot [0,+\infty) = f(Y,[0,+\infty))$ is compact and Y is connected, then $\omega(Y)$ is a compact and connected set, furthemore it is the maximal invariant set in $Y \cdot [0,+\infty)$.*

Lemma 2.5. *For $r > 0$, let $B_r = \{(x_1,x_2) \mid x_1^2 + x_2^2 \leqslant r^2\}$. If the system* (E) *is positively bounded, then $\omega(B_r)$ is compact and connected.*

P r o o f. By Lemma 2.4 we only need to prove that there is a $\lambda > 0$ such that for any point $p \in B_r$ the semi-orbit $O^+(p) = p \cdot \mathbb{R}^+$ is contained in B_λ, i.e., $B_r \cdot [0,+\infty) \subset B_\lambda$. Otherwise, there exists a sequence $\{z_n\}$ in B_r satisfying $O^+(z_n) \not\subset B_n$ for any positive integer n. Since B_r is compact, by passing to a subsequence we may suppose that $\lim\limits_{n\to\infty} z_n = z \in B_r$. Thus by the continuous dependence on initial conditions it is easy to verify that the positive semi-orbit $O^+(z)$ is unbounded, which is a contradiction. $\square$

Proposition 2.6. *Let the planar system* (E) *have a homoclinic orbit L with respect to a singular point p and assume that there are no singular points in the interior of the region D_L surrounded by $L \cup \{p\}$. Then any orbit passing through a point in D_L is homoclinic with respect to p.*

P r o o f. Let $x \in D_L$ be a regular point. Since D_L is an invariant and bounded set, by the Poincaré-Bendixson theorem we get that the limit set $\omega(x)$ is a singular point, or a closed orbit, or a connected set composed of some singular points and some orbits whose positive semi-orbit and negative semi-orbit tend to a singular point respectively. From the condition of the proposition, we know that the second case never takes place, since any closed orbit circles at least a singular point. If the last situation happens, let $q \in \omega(x)$ be a regular point and denote by J a transversal at q of the flow defined by the system. Then the positive semi-orbit $O^+(x)$ crosses J in the same direction successively at $0 < t_1 < t_2 < \ldots$ and $x \cdot t_n$ tends monotonously to q along J. For the argument above we refer to [7, Chap. 7]. Thus a simple closed curve consisting of the solution arc $x \cdot [t_n, t_{n+1}]$ and a segment of J between $x \cdot t_n$ and $x \cdot t_{n+1}$ surrounds a bounded region $Z \subset D_L$. Hence a (negative or positive) semi-orbit of the point $x \cdot t_n$ or $x \cdot t_{n+1}$ lies in Z, also by the Poincaré-Bendixson theorem it follows that there exists at least a singular point in Z, which is a contradiction. Thus we conclude that $\omega(x) = \{p\}$. Similarly $\alpha(x) = \{p\}$ holds, and now it follows that the orbit $O(x)$ is homoclinic. $\square$

Remark. The result of Proposition 2.6 is true for any planar systems.

Notation. In this paper, S always denotes the set of all bounded orbits. If L is a homoclinic orbit with respect to p, let D_L denote the region surrounded by $L \cup \{p\}$.

Definition 2.7. Let L_1 and L_2 be two homoclinic orbits with respect to the same singular point p. We call L_1 and L_2 in the same class, if $L_1 \subset \operatorname{Int} D_{L_2}$ holds or there exists another homoclinic orbit L with respect to p satisfying $L_1 \cup L_2 \subset \operatorname{Int} D_L$. By a maximal elliptic sector (with respect to p) we mean the union of $\{p\}$ and the set consisting of all homoclinic orbits in the same class with respect to p.

3. Main results

Theorem 3.1. *Suppose that the positively bounded system* (E) *has exactly two singular points and there exist no closed orbits. Then the set of all bounded orbits is simply connected and compact.*

Proof. Let $B_r = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leqslant r^2, r > 0\}$. We take an $r > 0$ such that both singular points lie in the interior of B_r. Since the system has no closed orbits, the ω-limit set of any point in $\mathbb{R}^2$ contains at least a singular point. Hence, for any $p \in \mathbb{R}^2$ the positive semi-orbit $O^+(p)$ meets B_r and $\omega(p) \cap B_r \neq \emptyset$. Now if the orbit $O(p)$ is a bounded orbit, we assert that $O(p) \subset B_r \cdot [0, +\infty)$. In fact, as in the argument above we also have $\alpha(p) \cap B_r \neq \emptyset$ for the bounded orbit $O(p)$. Thus for any point $p \cdot t \in O(p)$ there exists a $\tau > 0$ such that $(p \cdot t) \cdot (-\tau) \in B_r$; it follows that $p \cdot t \in B_r \cdot \tau \subset B_r \cdot [0, +\infty)$, so $O(p) \subset B_r \cdot [0, +\infty)$ holds. Since $\omega(B_r)$ is the maximal invariant set in $B_r \cdot [0, +\infty)$, we get $O(p) \subset \omega(B_r)$. On the other hand, by Lemma 2.5 all the orbits in $\omega(B_r)$ are bounded, thus $\omega(B_r)$ is just the set of all bounded orbits, and it is compact and connected. The simple connectedness is directly derived from the fact that $\omega(B_r)$ is the maximal invariant set in the simply connected set $B_r \cdot [0, +\infty)$. In fact, any loop C in $\omega(B_r)$ surrounds a bounded region that is also contained in $\omega(B_r)$, so C is contractible in $\omega(B_r)$. □

Remark 3.2. The conclusion of Theorem 3.1 is still true if the positively bounded system (E) has a finite number of singular points, or the set of singular points is bounded.

Theorem 3.3. *Suppose that the positively bounded system* (E) *has exactly two singular points and also assume that there exist no closed orbits and homoclinic orbits. Then the set of all bounded orbits is composed of singular points and connecting orbits.*

Proof. Let S be the set of all bounded orbits. Then by Theorem 3.1 S is compact. For any $x \in S$, suppose that x is not a singular point, and consider the limit set $\omega(x)$. First we prove that $\omega(x)$ has at most one singular point. Otherwise, since two different singular points separate, it follows from the connectedness of $\omega(x)$ that there exists a regular point p in $\omega(x)$. Let J be a transversal of the flow at p and let the positive semi-orbit $O^+(x)$ cross J successively at $0 < t_1 < t_2 < \ldots$, and $x \cdot t_n$ tend to q. Thus the simple closed curve consisting of the solution arc $x \cdot [t_n, t_{n+1}]$ and a segment of J separates $\omega(x)$ from the negative semi-orbit $O^-(x)$. Now it follows from the Poincaré-Bendixson theorem that there exists another singular point in $\alpha(x)$, which is a contradiction since we find three singular points. Further, since now there exists only one singular point in $\omega(x)$, we assert that there exist no regular points in $\omega(x)$, otherwise $\omega(x)$ has a homoclinic orbit, which is contradictory to the assumption of the theorem. Hence $\omega(x)$ is just a singular point, and so is $\alpha(x)$. Since the system has no homoclinic orbits, $O(x)$ is a connecting orbit and the theorem follows.

Lemma 3.4. *If the positively bounded system* (E) *has exactly two singular points, then $\omega(x)$ has at most one singular point for any point $x \in S$.*

P r o o f. Suppose that $\omega(x)$ has two singular points for $x \in S$. By a similar argument as in the proof of Theorem 3.3, we can find the third singular point in $\alpha(x)$. This is a contradiction. □

Lemma 3.5. *Let the positively bounded system* (E) *have exactly two singular points and no closed orbits. If for any regular point $x \in S$ the omega limit set $\omega(x)$ contains no homoclinic orbits, then $\omega(x)$ is a singular point.*

P r o o f. Since S is compact and $\omega(x) \subset S$, by the Poincaré-Bendixson theorem we know that the limit set $\omega(x)$ is a singular point, or a closed orbit, or a connected set composed of some singular points and some orbits whose positive semi-orbit and negative semi-orbit tend to a singular point respectively. Now the second case does not take place. If the third case happens, there is a regular point $q \in \omega(x)$. By Lemma 3.4, $\omega(x)$ contains a unique singular point. It follows that there exist homoclinic orbits in $\omega(x)$, which is contradictory to the assumption of the lemma. So only the first case holds, i.e., $\omega(x)$ is a singular point. □

Remark 3.6. Obviously, a similar result of Lemma 3.5 for $\alpha(x)$ holds.

Theorem 3.7. *Assume that the positively bounded system* (E) *has exactly two singular points and no closed orbits. Then the system has a connecting orbit if and only if for any homoclinic orbit L neither $L \subset \omega(x)$ nor $L \subset \alpha(x)$ $(x \in S)$ holds, i.e., L is not contained in a limit set of a point in S.*

P r o o f. Suppose that the system has a connecting orbit H between singular points p_1 and p_2. Without loss of generality, let L be a homoclinic orbit with respect to p_1. Let $L \subset \omega(x)$ for some regular point $x \in S$ (of course $x \notin H$) and let J be a transversal at a regular point p on L. Then the positive semi-orbit $O^+(x)$ crosses J successively at $0 < t_1 < t_2 < \ldots$ and $x \cdot t_n$ tends monotonously to q. Since $L \cup H \cup \{p_1, p_2\}$ is an invariant and connected set, thus the simple closed curve consisting of the solution arc $x \cdot [t_n, t_{n+1}]$ and a segment of L separates $\alpha(x)$ and $L \cup H \cup \{p_1, p_2\}$. Now $\alpha(x)$ contains at least a singular point, hence we find three singular points, which is a contradiction. A similar argument works for the case $L \subset \alpha(x)$.

Sufficiency: Let each homoclinic orbit L be not contained in a limit set of a point in S. Denote by S_i the union of $\{p_i\}$ and all homoclinic orbits with respect to p_i, respectively, for $i = 1, 2$. Then we have $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 \subset S$. If $S \setminus (S_1 \cup S_2) \neq \emptyset$, by Lemma 3.5 and Remark 3.6 for any $p \in S \setminus (S_1 \cup S_2)$ the orbit $O(p)$ is a connecting orbit. If $S = S_1 \cup S_2$, since S is compact and simply connected, it follows that $(S_1 \cap \overline{S_2}) \cup (\overline{S_1} \cap S_2) \neq \emptyset$. Without loss of generality, we assume $p \in S_1 \cap \overline{S_2}$. Of course, this implies $p \in S_1 \cap \partial S_2$ from $S_1 \cap S_2 = \emptyset$. Thus, since ∂S_2 is an invariant set, it follows from $p \in S_1$ that $p_1 \in \omega(p) \in \partial S_2$. Next, because S_2 is composed of maximal elliptic sectors and there exist at most a finite number of elliptic sectors with respect to a Jordan curve circling the singular point p_2 (see [7, p. 164]), we obtain $\{p_1, p_2\} \subset \partial S_2'$, where S_2' is a maximal elliptic sector with respect to p_2. Now by the definition of S_2', there exists a point $p \in \partial S_2'$ such that $O(p)$ is a connecting orbit. This is contradictory to $S = S_1 \cup S_2$. Hence $S = S_1 \cup S_2$ does not hold. So the proof is complete. □

Corollary 3.8 [8]**.** *If there exist no closed orbits and singular closed orbits for the positively bounded system* (E) *with just two singular points, then the system has a connecting orbit.*

P r o o f. Since a homoclinic orbit lying in a limit set of a point in S is a singular closed orbit, the result follows directly from Theorem 3.7. □

Theorem 3.9. *Let the planar system* (E) *have exactly two singular points. If there exist no homoclinic orbits, then the possible numbers of connecting orbits are one and uncountable.*

P r o o f. Suppose that there exist two connecting orbits $O(p)$ and $O(q)$ between p_1 and p_2. Then $O(p) \cup O(q) \cup \{p_1, p_2\}$ constitutes a simple closed curve surrounding a bounded region W. Since p_1 and p_2 lie in the boundary of W, there exist no singular points and closed orbits in $\operatorname{Int} W$. Choose a point $z \in \operatorname{Int} W$ arbitrarily. By Lemma 3.4 $\omega(z)$ contains a singular point. Since the system has no homoclinic orbits, $\omega(z)$ is a singular point, and so is $\alpha(z)$. Thus the orbit $O(z)$ must be a connecting orbit. Hence all the orbits in $\operatorname{Int} W$ are connecting orbits. $\square$

Remark 3.10. Theorem 3.9 remains true for any planar systems, because we may restrict our discussion to the invariant and bounded set W. Moreover, by Theorem 3.9 if the system (E) has two or more (finite) connecting orbits, then there exists at least a homoclinic orbit. On the other hand, if the system admits homoclinic orbits, we think that the number of connecting orbits can be any positive integer. In the following we give a system with two connecting orbits. A system with an uncountable number of connecting orbits will be given in the next section. Examples of systems with a unique connecting orbit are trivial.

Example 3.11. To give an example with two connecting orbits, we consider the well-known Liénard system [4, p. 33]:

$$\dot{x} = y - \Big(\frac{1}{3}x^3 - \frac{3}{2}x^2\Big), \quad \dot{y} = -x^3. \tag{3.1}$$

In the phase-portrait of (3.1) (see [4, p. 34]) there is a maximal elliptic sector S consisting of all homoclinic orbits with respect to $O = (0,0)$. Define a smooth function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying $\varphi(x,y) \geqslant 0$ and $\varphi(x,y) = 0$ only at a point p in the interior of S. Then the following system is a positively bounded system with two singular points p and O:

$$\dot{x} = \Big[y - \Big(\frac{1}{3}x^3 - \frac{3}{2}x^2\Big)\Big] \cdot \varphi(x,y), \quad \dot{y} = -x^3 \cdot \varphi(x,y). \tag{3.2}$$

Now the homoclinic orbit passing through p of (3.1) becomes two connecting orbits and a singular point p of (3.2), and the other orbits of (3.1) remain unchanged.

4. An example

Consider the following planar system in polar coordinates:

$$\dot{r} = r(1-r), \quad \dot{\theta} = \sin^2\frac{\theta}{2} + (r - 1 + |r-1|). \tag{4.1}$$

This system has exactly two singular points $O = (0,0)$ and $p = (1,0)$. The circle $C = \{(r,\theta) \mid r = 1\}$ is an invariant set, which is composed of a unique homoclinic orbit L and a singular point p. Obviously the system (4.1) is positively bounded, and for any point x outside the disc $B_1 = \{(r,\theta) \mid r \leqslant 1\}$ we have $\omega(x) = C$. Thus C is a singular closed orbit, the results of [8], [12] do not work. However, the segment

$\overline{Op} = \{(r, \theta) \mid 0 \leqslant r \leqslant 1,\, \theta = 0\}$ is also invariant, and it is easy to see that for any $x \in \operatorname{Int} B_1$ the relation $L \subset \omega(x)$ or $L \subset \alpha(x)$ does not hold. Otherwise, we may suppose that $L \subset \omega(x)$ for some x in the interior of $B_1 \setminus \overline{Op}$. Let J be a transversal at a regular point $q \in L$. Then the positive semi-orbit $O^+(x)$ crosses J successively at $0 < t_1 < t_2 < \ldots$ and $x \cdot t_n$ tends to q. Thus a simple closed curve consisting of the solution arc $x \cdot [t_n, t_{n+1}]$ and a segment of J surrounds a bounded region D. By the Poincaré-Bendixson theorem it follows that there exists at least a singular point in D, which is a contradiction since the system has only two singular points O and p. On the other hand, it is straightforward that L is not contained in the limit set of a point lying in $\overline{Op}$ or the boundary of B_1. Thus L is not contained in a limit set of a point in B_1. From Theorem 3.7 we conclude that the system has a connecting orbit. In fact, each orbit passing through a point in $\operatorname{Int} B_1 \setminus \{O\}$ is a connecting orbit.

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