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A MORITA TYPE THEOREM FOR A SORT OF QUOTIENT CATEGORIES

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Abstract. We consider the quotient categories of two categories of modules relative to the Serre classes of modules which are bounded as abelian groups and we prove a Morita type theorem for some equivalences between these quotient categories.

Keywords: Morita theorem, quotient category, equivalent categories, adjoint functors

MSC 2000: 16D90, 16A50

1. INTRODUCTION

The notions “quasi-isomorphism”, “quasi-direct decomposition” and other similar notions became important in the torsion free Abelian Groups Theory because they allowed B. Jönson to enunciate a Krull-Schmidt type theorem for torsion free groups of finite rank (see [5, Corollary 7.9]). In [13], E. Walker extended these notions to the category of abelian groups, observing that they originate in the quotient category $\mathcal{A}b/\mathcal{B}$, where $\mathcal{A}b$ is the category of abelian groups and \mathcal{B} is the class of all bounded abelian groups. In [3], the authors introduced the notion of almost-flat modules in order to answer the question which of the properties of torsion free abelian groups as modules over their endomorphism rings are preserved by quasi-isomorphisms. The notion of almost (quasi-)projective module was used in [11] and [12] in order to characterize the torsion free abelian groups projective as modules over their endomorphism rings. In the same context Albrecht extended in [1] the classes of A -static groups and A -adstatic modules to the classes of almost A -static groups and almost A -adstatic modules requesting the arrows of adjunction, induced by the pair of adjoints functors $\text{Hom}(A, -): \mathcal{A}b \rightleftarrows \text{Mod-}E: - \otimes_E A$, to be quasi-isomorphisms. The class of almost A -static modules was used in [6] in order to characterize the modules which are almost-flat over their endomorphism rings.

In [7], using the Elbert Walker's techniques which were developed in [13], the authors give natural interpretations for the notions "almost projective" and "almost flat". We recall here some of the constructions and notions used in this paper: Let Σ be a multiplicatively closed set of non-zero integers. We will say that an abelian group A is Σ -bounded if there exists $n \in \Sigma$ such that $nA = 0$. If R is a unital ring and $\text{Mod-}R$ is the category of right R -modules, then the class \mathcal{S} of all right R -modules which are Σ -bounded as abelian groups is a Serre class. Hence the quotient category $\text{Mod-}R/\mathcal{S}$ exists and it is proved (analogously to [13, Theorem 3.1]) that this category is equivalent to the category $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ which has as objects all the right R -modules and if $M, N \in \text{Mod-}R$, then

$$\text{Hom}_{\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R}(M, N) = \mathbb{Z}[\Sigma^{-1}] \otimes_{\mathbb{Z}} \text{Hom}_R(M, N).$$

We will denote by $\mathbf{q}: \text{Mod-}R \rightarrow \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ the canonical functor. Note that $\mathbf{q}(M) = M$ for any $M \in \text{Mod-}R$ and $\mathbf{q}(f) = 1 \otimes f$ for all R -homomorphisms f . Observe that every homomorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ can be written as $\frac{1}{n}f \stackrel{\text{not}}{=} \frac{1}{n} \otimes f$ where f is an R -homomorphism and, as the multiplication by $n \in \Sigma$ represents an automorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ for all $n \in \Sigma$, $\frac{1}{n}f$ is respectively a monomorphism, an epimorphism, an isomorphism if and only if $\mathbf{q}(f)$ has the same property. We will say: a R -homomorphism f is a **q-monomorphism** if $\mathbf{q}(f)$ is a monomorphism (this means $\text{Ker}(f)$ is a Σ -bounded group), it is a **q-epimorphism** if $\mathbf{q}(f)$ is an epimorphism (equivalently, $\text{Coker}(f)$ is a Σ -bounded group), and it is a **q-isomorphism** if $\mathbf{q}(f)$ is an isomorphism. Note that $f: M \rightarrow N$ is a **q-isomorphism** if and only if there exists an integer $n \in \Sigma$ and an R -homomorphism $g: N \rightarrow M$ such that $gf = n1_M$ and $fg = n1_N$. We say that g is a **q-inverse** for f . A **q-epimorphism** (**q-monomorphism**) $f: M \rightarrow N$ **q-splits** if $\mathbf{q}(f)$ splits in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$. This means that there exists an R -homomorphism $g: N \rightarrow M$ and an integer $n \in \Sigma$ such that $fg = n1_N$ ($gf = n1_M$). Observe that in the case $\Sigma = \mathbb{Z}^*$ is the set of all non-zero integers we find again Albrecht's "quasi-notions" presented in [2].

The definition of the quotient category modulo a Serre subcategory as a category of additive fractions is given in [9, Section 4.7] (see also [8, Corollaire 3.2]). Let us observe that if $F: \text{Mod-}R \rightarrow \text{Mod-}S$ is an additive functor, then it induces a canonical functor $qF: \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightarrow \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$ such that $\mathbf{q}F = qF\mathbf{q}$, where \mathbf{q} denotes both the canonical functors $\text{Mod-}R \rightarrow \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ and $\text{Mod-}S \rightarrow \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$. In [7] it is proved that a right R -module P is projective in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ (called Σ -almost projective) if and only if the functor $q\text{Hom}_R(P, -): \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightarrow \mathbb{Z}[\Sigma^{-1}]\text{Ab}$ is exact. In the case $\Sigma = \mathbb{Z}^*$ this notion coincides with the notion presented and used in [12] and [11]. It is proved that an R -module P is Σ -almost projective if and only if there exist a projective (free) module F and a **q-epimorphism** $\alpha: F \rightarrow P$

such that α \mathbf{q} -splits. Note that the almost flat left R -modules, introduced in [3], are just the left R -modules A such that the functor $q(- \otimes_R A): \mathbb{Q}\text{Mod-}R \rightarrow \mathbb{Q}Ab$ is exact.

The main result of this paper is Theorem 3.10, which proves a statement analogous to the Morita Theorem, [4, Theorem 22.2] and [14, 46.4], for the special case of equivalences between two quotient categories $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ and $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}S$. As the Morita Theorem justifies the introduction of static and adstatic modules, our theorem justifies, in the case $\Sigma = \mathbb{Z}^*$, the definitions of almost static and almost adstatic modules.

2. \mathbf{q} -GENERATORS

Recall that if G is a right R -module with the endomorphism ring E , then G becomes a left E -module and we have a pair of adjoint functors

$$H_G = \text{Hom}_R(G, -): \text{Mod-}R \rightleftarrows \text{Mod-}E: - \otimes_E G = T_G$$

with the canonical arrows $\varphi: T_G H_G \rightarrow 1$, $\varphi_M(\alpha \otimes g) = \alpha(g)$ and $\psi: 1 \rightarrow H_G T_G$, $\psi_X(x)(g) = x \otimes g$, for all $M \in \text{Mod-}R$ and $X \in \text{Mod-}E$. They induce the pair of adjoint functors

$$qH_G: \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]\text{Mod-}E: qT_G.$$

We mention that $qH_G(M) = H_G(M)$ for all $M \in \text{Mod-}R$ and that for every R -homomorphism $\gamma: M \rightarrow N$ we have $qH_G(\frac{1}{n}\gamma) = \frac{1}{n}H_G(\gamma)$.

We will say that $G \in \text{Mod-}R$ is a \mathbf{q} -generator if $qH_G: q\text{Mod-}R \rightarrow q\text{Mod-}E$ is a faithful functor.

Proposition 2.1. *If $G \in \text{Mod-}R$, then the following assertions are equivalent:*

- a) G is a \mathbf{q} -generator;
- b) for each $M \in \text{Mod-}R$, there exists a set Λ and a \mathbf{q} -epimorphism $\varphi: G^{(\Lambda)} \rightarrow M$;
- c) for every right R -module M the canonical homomorphism $\varphi_M: T_G H_G(M) \rightarrow M$ is a \mathbf{q} -epimorphism.

Proof. a) \Rightarrow b) If $\Lambda = \text{Hom}_R(G, M)$, we will prove that the canonical homomorphism $\varphi: G^{(\Lambda)} \rightarrow M$, induced by the direct sum and the family of homomorphisms $\{\alpha: G \rightarrow M \mid \alpha \in \Lambda\}$, is an epimorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$. Let $(1/n)\gamma: M \rightarrow N$ be a homomorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ such that $((1/n)\gamma)\varphi = 0$ (in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$). Then $\gamma\varphi = 0$ in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$, hence $\text{Im}(\gamma\varphi)$ is Σ -bounded. Therefore, there exists an integer $n \in \Sigma$ such that $n\gamma\varphi_i \alpha = 0$ for all $\alpha \in \Lambda$

($i_\alpha: G \rightarrow G^{(\Lambda)}$ is the canonical injection). It follows that $n\gamma\alpha = 0$ for all $\alpha \in \Lambda$ and we obtain $qH_G(\gamma) = 0$. Because qH_G is faithful we obtain $\gamma = 0$ in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$, hence $(1/n)\gamma = 0$. Then φ is an epimorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$.

b) \Rightarrow c) Let Λ be a set such that there exists an R -homomorphism $\varphi: G^{(\Lambda)} \rightarrow M$ which is a \mathbf{q} -epimorphism. Then we can find an integer $n \in \Sigma$ with $n \text{Coker}(\varphi) = 0$. We obtain that for every $x \in M$ there exist $\lambda_1, \dots, \lambda_k \in \Lambda$ and $g_1 \in G_{\lambda_1} = G, \dots, g_k \in G_{\lambda_k} = G$ such that $nx = \varphi(g_1 \oplus \dots \oplus g_k)$. It follows that $\varphi_M(\varphi_{i_{\lambda_1}} \otimes g_1 + \dots + \varphi_{i_{\lambda_k}} \otimes g_k) = nx$ and this shows that φ_M is a \mathbf{q} -epimorphism.

c) \Rightarrow a) If $\alpha: M \rightarrow N$ is an R -homomorphism such that $qH_G(\alpha) = 0$, we use the commutative diagram

$$\begin{array}{ccc} \text{T}_G H_G(M) & \xrightarrow{\varphi_M} & M \\ \downarrow & & \downarrow \alpha \\ \text{T}_G H_G(N) & \xrightarrow{\varphi_N} & N \end{array}$$

to obtain $\alpha\varphi_M = 0$ in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$. Then $\alpha = 0$ in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ because φ_M represents an epimorphism in this category. \square

Example 2.2. If p is a prime, the module \mathbb{Q} is a \mathbf{q} -generator in the category $\text{Mod-}\mathbb{Q} \times \mathbb{Z}(p)$ which is not a generator.

Lemma 2.3. *If G is a \mathbf{q} -generator in $\text{Mod-}R$ and $M \in \text{Mod-}R$ is \mathbf{q} -isomorphic with a finitely generated right R -module, then there exists an integer $k > 0$ and an R -homomorphism $G^k \rightarrow M$ which is a \mathbf{q} -epimorphism.*

Proof. Observe that we can suppose without loss of generality that M is a finitely generated right R -module. If $\langle x_1, \dots, x_m \rangle = M$ and $\varphi: G^{(I)} \rightarrow M$ is an R -homomorphism such that $n \text{Coker}(\varphi) = 0$ for an integer $n \in \Sigma$, we fix $g_1, \dots, g_m \in G$ such that $\varphi(g_i) = nx_i$ for all $i = 1, \dots, m$. Then there exists a submodule $G^{(J)}$ in $G^{(I)}$ which is a finite direct sum of copies of G such that $g_i \in G^{(J)}$ for all $i = 1, \dots, m$. The restriction $\varphi|_{G^{(J)}}: G^{(J)} \rightarrow M$ is a \mathbf{q} -epimorphism. \square

We will say that an R -module M is an Σ -almost finitely generated module if it is \mathbf{q} -isomorphic to a finitely generated R -module.

Proposition 2.4. *If G is a \mathbf{q} -generator in $\text{Mod-}R$ where E is the endomorphism ring of G and B the biendomorphism ring of G , then:*

- a) G is Σ -almost projective and Σ -almost finitely generated as a left E -module;
- b) if $\vartheta: R \rightarrow B$, $\vartheta(r)(g) = gr$ is the canonical ring homomorphism, then $\text{Ker}(\vartheta)$ and $B/\text{Im}(\vartheta)$ are Σ -bounded as abelian groups.

P r o o f. Using the previous lemma we find an exact sequence $G^k \xrightarrow{\varphi} R \rightarrow H \rightarrow 0$ in $\text{Mod-}R$ such that H is Σ -bounded as an abelian group. Because R is Σ -almost projective, there exist $n \in \Sigma$ and $\psi: R \rightarrow G^k$ such that $\varphi\psi = n1_R$.

a) We apply the contravariant functor $\text{Hom}_R(-, G): \text{Mod-}R \rightarrow E\text{-Mod}$ to the previous exact sequence concluding that the sequence

$$0 \rightarrow \text{Hom}_R(H, G) \rightarrow \text{Hom}_R(R, G) \xrightarrow{\text{Hom}_R(\varphi, G)} \text{Hom}_R(G^k, G)$$

is exact. Therefore $\text{Hom}_R(\varphi, G)$ is a \mathbf{q} -monomorphism which splits in the category $\mathbb{Z}[\Sigma^{-1}](E\text{-Mod})$ because we have $\text{Hom}_R(\psi, G) \text{Hom}_R(\varphi, G) = n1_{\text{Hom}_R(R, G)}$, hence $G \cong \text{Hom}_R(R, G)$ is Σ -almost projective as an E -module.

b) If $\mu: B(G^k) \rightarrow B$ is the canonical isomorphism which is presented in [4, 14.2] and $\lambda: R \rightarrow B(G^k)$ is the canonical ring homomorphism, then $\vartheta = \mu\lambda$. Consider $r \in R$ such that $\lambda(r) = 0$ and $x \in G^k$ with $\varphi(x) = n1$. Then $nr = \varphi(xr) = \varphi(\lambda(r)(x)) = 0$, hence the group $\text{Ker}(\vartheta) = \text{Ker}(\varphi)$ is bounded by $n \in \Sigma$.

To prove that $\text{Coker}(\vartheta)$ is Σ -bounded, we apply the density theorem, [14, 15.7], to the following hypothesis: G is Σ -almost finitely generated as a left E -module, hence there exist an integer $n \in \Sigma$ and an E -homomorphism $\alpha: E^k \rightarrow G$ with $n \text{Coker}(\alpha) = 0$. We consider $x_1, \dots, x_k \in G$ such that they generate an E -submodule H with $nG/H = 0$. It follows that for every $\beta \in B$, the homomorphism $n\beta$ is determined by $\beta(x_1), \dots, \beta(x_k)$. Then the density theorem shows that $n\beta$ is a multiplication by an element $r \in R$. Therefore, $n(B/\text{Im}(\vartheta)) = 0$. \square

3. THE MAIN THEOREM

Let R and S be unital rings. Consider a pair of functors $F: \text{Mod-}R \rightleftarrows \text{Mod-}S: G$ such that

- i) there exists a natural transformation $\varphi: GF \rightarrow 1_{\text{Mod-}R}$ such that φ_A is a \mathbf{q} -isomorphism for all $A \in \text{Mod-}R$,
- ii) there exists a natural transformation $\psi: 1_{\text{Mod-}S} \rightarrow FG$ such that ψ_X is a \mathbf{q} -isomorphism for all $X \in \text{Mod-}S$.

We will say that F is a *good \mathbf{q} -equivalence* and G is a *good \mathbf{q} -inverse for F* . Observe that under these conditions the functors $qF: \mathbb{Z}[\Sigma^{-1}]\text{Mod-}R \rightleftarrows \mathbb{Z}[\Sigma^{-1}]\text{Mod-}S: qG$ are equivalences.

Remark 3.1. In this case it follows that

- i) for every $A \in \text{Mod-}R$ there exist an integer $n_A \in \Sigma$ and $\varphi'_A: A \rightarrow GF(A)$ such that $\varphi_A \varphi'_A = n_A 1_A$ and $\varphi'_A \varphi_A = n_A 1_{GF(A)}$,

ii) for every $X \in \text{Mod-}S$ there exist an integer $m_X \in \Sigma$ and $\psi'_A: FG(X) \rightarrow X$ such that $\psi_X \psi'_X = m_X 1_{FG(X)}$ and $\psi'_A \psi_A = m_X 1_X$.

Moreover, if $A \cong A'$ and $X \cong X'$, we can suppose that $n_A = n_{A'}$ and $m_X = m_{X'}$.

If $\varphi: R \rightarrow S$ is a unital ring homomorphism, then we obtain a pair of adjoint functors (φ_*, φ^*) , where

$$\varphi_*: S\text{-Mod} \rightarrow R\text{-Mod}, \quad \varphi_*(A) = A, \quad \varphi_*(f) = f$$

is the restriction of scalars and

$$\varphi^*: R\text{-Mod} \rightarrow S\text{-Mod}, \quad \varphi^*(C) = S \otimes_R C, \quad \varphi^*(f) = 1_S \otimes f$$

is the extension of scalars [10, IV.9]. They induce the natural transformations

$$\xi: \varphi^* \varphi_* \rightarrow 1_{S\text{-Mod}} \quad \text{with } \xi_A: \varphi^* \varphi_*(A) \rightarrow A, \quad s \otimes a \mapsto sa$$

and

$$\zeta: 1_{R\text{-Mod}} \rightarrow \varphi_* \varphi^* \quad \text{with } \zeta_C: C \rightarrow \varphi_* \varphi^*(C), \quad c \mapsto 1 \otimes c$$

for all $A \in S\text{-Mod}$ and $C \in R\text{-Mod}$.

In [6, Lemma 3.1] the following result is proved which will be useful in the enunciation of our main theorem.

Proposition 3.2 [6, Lemma 3.1]. *If R and S are rings such that there exists a unital ring homomorphism $\varphi: R \rightarrow S$ such that the groups $\text{Ker}(\varphi)$ and $S/\text{Im}(\varphi)$ are Σ -bounded, then φ_* is a good \mathbf{q} -equivalence and φ^* is a good \mathbf{q} -inverse for φ_* .*

We revert to the general case.

Proposition 3.3. *Let $F: \text{Mod-}R \rightleftarrows \text{Mod-}S$ be a good \mathbf{q} -equivalence and let G be a good \mathbf{q} -inverse for F . Then*

a) *the group homomorphism*

$$F: \text{Hom}_R(A, B) \rightarrow \text{Hom}_S(F(A), F(B))$$

is a \mathbf{q} -isomorphism for all $A, B \in \text{Mod-}R$;

a') *the group homomorphism*

$$G: \text{Hom}_S(X, Y) \rightarrow \text{Hom}_R(G(X), G(Y))$$

is a \mathbf{q} -isomorphism for all $X, Y \in \text{Mod-}S$;

b) *If $X \in \text{Mod-}S$, then there exists $A \in \text{Mod-}R$ such that $F(A)$ and X are \mathbf{q} -isomorphic.*

Proof. a), a') Let $f: A \rightarrow B$ be an R -homomorphism such that $F(f) = 0$. We obtain that $GF(f) = 0$ and the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \varphi_A \uparrow & & \uparrow \varphi'_B \\ GF(A) & \xrightarrow{GF(f)} & GF(B), \end{array}$$

shows that $\text{Im}(\varphi_A) \subseteq \text{Ker}(f)$. It follows that $\text{Im}(f) \cong A/\text{Ker}(f)$ is a homomorphic image of $A/\text{Im}(\varphi_A)$, hence $n_A \text{Ker}(F) = 0$.

Analogously, it follows that $\text{Ker}(G)$ is bounded by m_Y .

If $g \in \text{Hom}_S(F(A), F(B))$ and $h = \varphi_B G(g) \varphi'_A$, we obtain the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \varphi_A \uparrow & & \uparrow \varphi_B \\ GF(A) & \xrightarrow{G(n_A g)} & GF(B) \end{array}$$

and it follows that $\varphi_B G(n_A g) = \varphi_B GF(h)$, hence we have $n_B G(n_A g) = n_B G(F(h))$. Because the kernel of G is bounded by $m_{F(B)}$ we find

$$m_{F(B)} n_A n_B g = F(m_{F(B)} n_B h)$$

and this shows that F is \mathbf{q} -epic.

The proof of the fact that G is a \mathbf{q} -epimorphism is analogous. We obtain that for every $f: G(X) \rightarrow G(Y)$ there exists $h = \psi'_Y F(f) \varphi_X$ such that

$$n_{G(X)} m_X m_Y f = G(n_{G(X)} m_X h).$$

The definition of good \mathbf{q} -equivalences ensures that b) is valid. □

Proposition 3.4. Let $F: \text{Mod-}R \rightleftarrows \text{Mod-}S: G$ be a pair of functors such that F is a good \mathbf{q} -equivalence and G is a good \mathbf{q} -inverse for F .

a) For every family $(X_i)_{i \in I}$ of right S -modules, the canonical monomorphism

$$\gamma: \bigoplus_{i \in I} G(X_i) \rightarrow G\left(\bigoplus_{i \in I} X_i\right)$$

is a \mathbf{q} -isomorphism.

b) If M is a right R -module and I is a set, then the canonical monomorphism $\gamma: F(M)^{(I)} \rightarrow F(M^{(I)})$ is a \mathbf{q} -isomorphism.

Proof. a) It is enough to prove that every R -homomorphism $f: G\left(\bigoplus_{i \in I} X_i\right) \rightarrow M$ such that $f\gamma = 0$ represents the zero homomorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$.

Let $f: G\left(\bigoplus_{i \in I} X_i\right) \rightarrow M$ be an R -homomorphism with $f\gamma = 0$. Then $fG(\alpha_i) = 0$ for all $i \in I$ (if $i \in I$, $\alpha_i: X_i \rightarrow \bigoplus_{i \in I} X_i$ denotes the canonical injection). It follows that $\varphi'_M f G(\alpha_i): G(X_i) \rightarrow G(F(M))$ is the zero homomorphism. From the proof of Proposition 3.3 it follows that there exists $\bar{f} \in \text{Hom}_S\left(\bigoplus_{i \in I} X_i, F(M)\right)$ such that

$$n_{G\left(\bigoplus_{i \in I} X_i\right)} m_{\bigoplus_{i \in I} X_i} m_{F(M)} \varphi'_M f = G\left(n_{G\left(\bigoplus_{i \in I} X_i\right)} m_{\bigoplus_{i \in I} X_i} \bar{f}\right)$$

and this implies that

$$G\left(n_{G\left(\bigoplus_{i \in I} X_i\right)} m_{\bigoplus_{i \in I} X_i} \bar{f} \alpha_i\right) = 0$$

for all $i \in I$. Using again Proposition 3.3 we obtain

$$m_{F(M)} n_{G\left(\bigoplus_{i \in I} X_i\right)} m_{\bigoplus_{i \in I} X_i} \bar{f} \alpha_i = 0$$

for all $i \in I$. As the family $(\alpha_i)_{i \in I}$ is an epimorphic family, we have

$$m_{F(M)} n_{G\left(\bigoplus_{i \in I} X_i\right)} m_{\bigoplus_{i \in I} X_i} \bar{f} = 0$$

and this shows that $\varphi'_M f$ represents the zero homomorphism in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$. As φ'_M is a \mathbf{q} -isomorphism, it follows that f is zero in $\mathbb{Z}[\Sigma^{-1}]\text{Mod-}R$ and the proof is complete.

In the same way b) follows. Remark that here we use the fact that for every right R -module N the kernel of the homomorphism $F: \text{Hom}_R(M, N) \rightarrow \text{Hom}_S(F(M), F(N))$ is bounded by n_M and we have the direct sum of I copies of M . \square

Corollary 3.5. *If $F: \text{Mod-}R \rightleftarrows \text{Mod-}S: G$ is a pair of functors such that F is a good \mathbf{q} -equivalence and G is a good \mathbf{q} -inverse for F , then F and G preserve the \mathbf{q} -generators.*

Proof. Let X be a \mathbf{q} -generator in $\text{Mod-}S$. If M is a right R -module, then there exists a set I and a \mathbf{q} -epimorphism $\alpha: X^{(I)} \rightarrow F(M)$. If $\gamma: G(X)^{(I)} \rightarrow G(X^{(I)})$ is the canonical homomorphism, then $\varphi_M G(\alpha)\gamma: G(X)^{(I)} \rightarrow M$ is a \mathbf{q} -epimorphism, hence G preserves the \mathbf{q} -generators. Similarly, we obtain that F preserves the \mathbf{q} -generators. \square

Proposition 3.6. *Let $F: \text{Mod-}R \rightleftarrows \text{Mod-}S: G$ be a pair of functors such that F is a good \mathbf{q} -equivalence and G is a good \mathbf{q} -inverse for F . Then*

a) *the natural transformation*

$$\overline{\varphi}: \text{Hom}_S(-, F(-)) \rightarrow \text{Hom}_R(G(-), -), \quad \overline{\varphi}_{X,A}(\alpha) = \varphi_A G(\alpha),$$

has the property that $\overline{\varphi}_{X,A}$ is a \mathbf{q} -isomorphism for all $A \in \text{Mod-}R$ and $X \in \text{Mod-}S$;

b) *the natural transformation*

$$\overline{\psi}: \text{Hom}_S(F(-), -) \rightarrow \text{Hom}_R(-, G(-)), \quad \overline{\psi}_{A,X}(\beta) = F(\beta)\psi_X$$

has the property that $\overline{\psi}_{A,X}$ is a \mathbf{q} -isomorphism for all $A \in \text{Mod-}R$ and $X \in \text{Mod-}S$.

P r o o f. If $A \in \text{Mod-}R$ and $X \in \text{Mod-}S$, then the composition

$$\text{Hom}_S(X, F(A)) \xrightarrow{G} \text{Hom}_R(G(X), G(F(A))) \xrightarrow{\text{Hom}_R(G(X), \varphi_A)} \text{Hom}_R(G(X), A)$$

gives the homomorphism

$$\overline{\varphi}_{X,A}: \text{Hom}_S(X, F(A)) \rightarrow \text{Hom}_R(G(X), A),$$

hence $\overline{\varphi}_{X,A}$ is a \mathbf{q} -isomorphism. □

Lemma 3.7. *Let $F_1, F_2: \text{Mod-}R \rightarrow \text{Mod-}S$ be functors and $\mu: F_1 \rightarrow F_2$ a natural transformation. If A and B are right R -modules and $f: A \rightarrow B: g$ are R -homomorphisms such that μ_A is an isomorphism and $fg = n1_B$ or $gf = n1_A$, then μ_B is a \mathbf{q} -isomorphism.*

P r o o f. Using the fact that μ is a natural transformation, we obtain

$$F_2(f)\mu_A = \mu_B F_1(f) \quad \text{and} \quad \mu_A F_1(g) = F_2(g)\mu_B.$$

It follows that

$$F_1(f)\mu_A^{-1}F_2(g)\mu_B = n1_{F_1(B)} \quad \text{and} \quad \mu_B F_1(f)\mu_A^{-1}F_2(g) = n1_{F_2(B)}.$$

In the other case, the proof is similar. □

The next lemma is analogous to [4, Proposition 20.11].

Lemma 3.8. *Let P be a left S -module, U an S - R -bimodule and N a right R -module. If P is Σ -almost finitely generated and Σ -almost projective then the canonical homomorphism*

$$\mu: \text{Hom}_R(U, N) \otimes_S P \rightarrow \text{Hom}_R(\text{Hom}_S(P, U), N), \quad \mu(\alpha \otimes p)(\beta) = \alpha\beta(p)$$

is a \mathbf{q} -isomorphism.

Proof. Because P is Σ -almost finitely generated and Σ -almost projective, there exist an integer $n \in \Sigma$ and S -homomorphisms $f: S^k \rightrightarrows P: g$ such that $fg = n1_P$. We obtain the commutative diagram

$$\begin{array}{ccc} \text{Hom}_R(U, N) \otimes_S S^k & \xrightarrow{\mu_{S^k}} & \text{Hom}_R(\text{Hom}_S(S^k, U), N) \\ \Downarrow & & \Downarrow \\ \text{Hom}_S(U, N) \otimes_S P & \xrightarrow{\mu_P} & \text{Hom}_R(\text{Hom}_S(P, U), N) \end{array}$$

and, using [4, Proposition 20.11] and the previous lemma, we conclude μ_P is a \mathbf{q} -isomorphism. \square

In the same way we find an analogue of Proposition 20.10 in [4]:

Lemma 3.9. *Let P be a right R -module, U an R - S -bimodule and N a right S -module. If P is Σ -almost projective and Σ -almost finitely generated, then the canonical homomorphism*

$$\nu: N \otimes_S \text{Hom}_R(P, U) \rightarrow \text{Hom}_R(P, N \otimes_S U), \quad \nu(n \otimes \alpha)(p) = n \otimes \alpha(p)$$

is a \mathbf{q} -isomorphism.

Theorem 3.10. *If R and S are rings, then the following are equivalent:*

- a) *There exists a functor $F: \text{Mod-}R \rightarrow \text{Mod-}S$ which is a good \mathbf{q} -equivalence.*
- b) *There exists a right R -module P which is Σ -almost finitely generated, Σ -almost projective, \mathbf{q} -generator and there exists a unital ring homomorphism $\varphi: S \rightarrow \text{End}_R(P)$ such that the groups $\text{Ker}(\varphi)$ and $S/\text{Im}(\varphi)$ are Σ -bounded.*
- c) *There exist a right R -module P and a unital ring homomorphism*

$$\varphi: S \rightarrow \text{End}_R(P)$$

with $\text{Ker}(\varphi)$ and $S/\text{Im}(\varphi)$ Σ -bounded groups such that

$$\varphi_* \text{Hom}_R(P, -): \text{Mod-}R \rightarrow \text{Mod-}S$$

is a good \mathbf{q} -equivalence with

$$(- \otimes_{\text{End}_R(P)} P)\varphi^*: \text{Mod-}S \rightarrow \text{Mod-}R$$

a good \mathbf{q} -inverse.

Under these conditions there exists a natural transformation

$$\mu: F \rightarrow \varphi_* \text{Hom}_R(P, -)$$

such that μ_M is a \mathbf{q} -isomorphism for all $M \in \text{Mod-}R$.

Proof. a) \Rightarrow b) If G is a good \mathbf{q} -inverse for F , we denote $P = G(S)$. By virtue of [4, Lemma 20.3], $\varphi = G: \text{Hom}_S(S, S) \rightarrow \text{Hom}_R(P, P)$ is a unital ring homomorphism. We apply Proposition 3.3 and it follows that $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are Σ -bounded as abelian groups. Because an equivalence preserves the projective objects, P is Σ -almost projective. Corollary 3.5 shows that P is a \mathbf{q} -generator. Moreover, we observe that $F(R)$ is a \mathbf{q} -generator, and it follows that there exists an S -homomorphism $\alpha: F(R)^n \rightarrow S$ such that $\text{Coker}(\alpha)$ is Σ -bounded. If δ is a \mathbf{q} -inverse for the canonical \mathbf{q} -isomorphism $\gamma: F(R)^n \rightarrow F(R^n)$, then $\alpha\delta: F(R^n) \rightarrow S$ is a \mathbf{q} -epimorphism. It follows that $G(\alpha\delta): GF(R^n) \rightarrow G(S)$ is a \mathbf{q} -epimorphism (qG is a equivalence, hence it preserves the epimorphisms). $G(\alpha\delta)\varphi'_{R^n}: R^n \rightarrow G(S)$ is a \mathbf{q} -epimorphism, hence $P = G(S)$ is Σ -almost finitely generated.

b) \Rightarrow c) Let $X \in \text{Mod-End}_R(P)$. Then Lemma 3.9 shows that there exist \mathbf{q} -isomorphisms

$$X \rightarrow X \otimes_{\text{End}_R(P)} \text{Hom}_R(P, P) \rightarrow \text{Hom}_R(P, X \otimes_{\text{End}_R(P)} P)$$

which are natural in X .

Recall that if $B = \text{Hom}_{\text{End}_R(P)}(P, P)$ is the biendomorphism ring of P and $\vartheta: R \rightarrow B$ is the canonical ring homomorphism then $\text{Ker}(\vartheta)$ and $\text{Coker}(\vartheta)$ are Σ -bounded groups (Proposition 2.4). Using Lemma 3.8, we obtain the \mathbf{q} -isomorphisms

$$\text{Hom}_R(P, A) \otimes_{\text{End}_R(P)} P \rightarrow \text{Hom}_R(\text{Hom}_{\text{End}_R(P)}(P, P), A) \rightarrow A$$

which are natural in A and it follows that

$$\text{Hom}_R(P, -): \text{Mod-}R \rightarrow \text{Mod-End}_R(P)$$

is a good \mathbf{q} -equivalence with $- \otimes_{\text{End}_R} P$ a good \mathbf{q} -inverse. Under these conditions

$$\varphi_* \text{Hom}_R(P, -): \text{Mod-}R \rightarrow \text{Mod-}S$$

is a good \mathbf{q} -equivalence and $(- \otimes_{\text{End}_R} P)\varphi^*$ is a good \mathbf{q} -inverse.

c) \Rightarrow a) is obvious.

The last statement follows from the fact that the arrows

$$F(M) \rightarrow \text{Hom}_S(S, F(M)) \rightarrow \text{Hom}_R(G(S), M) = \varphi_* \text{Hom}_R(G(S), M)$$

are \mathbf{q} -isomorphisms natural in M . □

Remark 3.11. In the case $\Sigma = \{1\}$, the theorem is the classical Morita Theorem.

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