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Toshiaki Adachi; Sadahiro Maeda

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CHARACTERIZATION OF TOTALLY UMBILIC HYPERSURFACES IN A SPACE FORM BY CIRCLES

Toshiaki Adachi, Nagoya, and Sadahiro Maeda, Shimane

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Abstract. In this paper we characterize totally umbilic hypersurfaces in a space form by a property of the extrinsic shape of circles on hypersurfaces. This characterization corresponds to characterizations of isoparametric hypersurfaces in a space form by properties of the extrinsic shape of geodesics due to Kimura-Maeda.

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1. Introduction

A smooth curve $\gamma = \gamma(s)$ on a Riemannian manifold M parameterized by its arc length s is called a *circle* if it satisfies $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -k^2\dot\gamma$ with some nonnegative constant k, where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ on M. This condition is equivalent to the condition that there exist a nonnegative constant k and a field of unit vectors $Y = Y(s)$ along this curve which satisfy the following differential equations: $\nabla_{\dot\gamma}\dot\gamma = kY$ and $\nabla_{\dot\gamma}Y = -k\dot\gamma$. We call the constant k the *curvature* of γ. As $k = \|\nabla_{\dot\gamma}\dot\gamma\|$, we treat geodesics as circles of null curvature. For given a point $x \in M$, an orthonormal pair of tangent vectors $u, v \in T_xM$ and a positive constant k, by the existence and uniqueness theorem for solutions of ordinary differential equations we have locally a unique circle $\gamma = \gamma(s)$ with the initial condition that $\gamma(0) = x$, $\dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = kv$. It is well-known

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that in Euclidean space a circle of positive curvature k is nothing but a circle of radius $1/k$ in the sense of Euclidean geometry.

We are interested in getting some properties of a submanifold by observing the extrinsic shape of circles on this submanifold. In this paper we restrict ourselves to hypersurfaces in a space form. Here, a space form $\tilde{M}^{n+1}(c)$ of constant curvature c is the Euclidean space $\mathbb{R}^{n+1}$, the standard sphere $S^{n+1}(c)$ or the hyperbolic space $H^{n+1}(c)$ according as c is zero, positive or negative. The purpose of this paper is to prove the following:

Theorem 1. *A connected hypersurface M^n in a space form $\tilde{M}^{n+1}(c)$ of constant curvature c is totally umbilic in $\tilde{M}^{n+1}(c)$ if and only if there exists $k > 0$ with the following property: At each point $x \in M$, there is an orthonormal basis $\{v_1, \dots, v_n\}$ of T_xM such that for each distinct i, j the circles $\gamma_{i,j}$, $\gamma_{i,-j}$ of curvature k on M with the initial conditions that*

$$\gamma_{i,j}(0) = \gamma_{i,-j}(0) = x, \quad \dot{\gamma}_{i,j}(0) = \dot{\gamma}_{i,-j}(0) = v_i,$$
$$\nabla_{\dot{\gamma}_{i,j}}\dot{\gamma}_{i,j}(0) = kv_j, \quad \nabla_{\dot{\gamma}_{i,-j}}\dot{\gamma}_{i,-j}(0) = -kv_j$$

are circles in the ambient space $\tilde{M}^{n+1}(c)$.

The readers should compare our result with characterizations of isoparametric hypersurfaces in a space form by the extrinsic shape of geodesics (see Theorems 2 and 5 in [1]). Our study on circles on a hypersurface gives much information on the hypersurface.

2. Proof of our result

The "only if" part of Theorem 1 follows from the following well-known result.

Proposition. *Let M^n be a hypersurface isometrically immersed into a space form $\tilde{M}^{n+1}(c)$. Then the following three conditions are equivalent:*
(1) *M^n is totally umbilic in $\tilde{M}^{n+1}(c)$.*
(2) *Every geodesic on M^n is a circle in $\tilde{M}^{n+1}(c)$.*
(3) *Every circle on M^n is a circle in $\tilde{M}^{n+1}(c)$.*

The "if" part of Theorem 1 follows from the following result on a hypersurface in a *general* Riemannian manifold.

Theorem 2. *A connected hypersurface M^n in a general Riemannian manifold $\tilde{M}^{n+1}$ is totally umbilic in $\tilde{M}^{n+1}$ if there exists $k > 0$ satisfying the following condition. At each point $x \in M$, there is an orthonormal basis $\{v_1, \ldots, v_n\}$ of T_xM such that for each distinct i, j the circles $\gamma_{i,j}$, $\gamma_{i,-j}$ of curvature k on M with the initial conditions that*

$$\gamma_{i,j}(0) = \gamma_{i,-j}(0) = x, \quad \dot{\gamma}_{i,j}(0) = \dot{\gamma}_{i,-j}(0) = v_i,$$
$$\nabla_{\dot{\gamma}_{i,j}}\dot{\gamma}_{i,j}(0) = kv_j, \quad \nabla_{\dot{\gamma}_{i,-j}}\dot{\gamma}_{i,-j}(0) = -kv_j$$

are circles in the ambient space $\tilde{M}^{n+1}$.

Proof. We denote by $\tilde{\nabla}$ the Riemannian connection of $\tilde{M}$. Let $\gamma_{a,b} = \gamma_{a,b}(s)$ be a circle of curvature k satisfying the hypothesis at an arbitrary point $x = \gamma_{a,b}(0)$ on the hypersurface M. By use of the formulae of Gauss and Weingarten which assure

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N \quad \text{and} \quad \tilde{\nabla}_X N = -AX$$

for vector fields X, Y on M, we find by regarding $\gamma_{a,b}$ as a curve on $\tilde{M}$ that

$$\tilde{\nabla}_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b} = \nabla_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b} + \langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle N, \tag{1}$$

$$\begin{aligned}\tilde{\nabla}_{\dot{\gamma}_{a,b}}\tilde{\nabla}_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b} = & -k^2\dot{\gamma}_{a,b} - \langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle A\dot{\gamma}_{a,b} \\ & + \{3\langle A\dot{\gamma}_{a,b}, \nabla_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b}\rangle + \langle(\nabla_{\dot{\gamma}_{a,b}}A)\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle\}N.\end{aligned} \tag{2}$$

Thus we have $\|\tilde{\nabla}_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b}\|^2 = k^2 + \langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle^2$. Since $\gamma_{a,b}$ is also a circle as a curve in $\tilde{M}$, we find $\langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle$ is constant along this curve and obtain

$$\begin{aligned}&\langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle\{\langle A\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle\dot{\gamma}_{a,b} - A\dot{\gamma}_{a,b}\} \\ &\quad + \{3\langle A\dot{\gamma}_{a,b}, \nabla_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b}\rangle + \langle(\nabla_{\dot{\gamma}_{a,b}}A)\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle\}N = 0\end{aligned} \tag{3}$$

by comparing the equality (2) with

$$\tilde{\nabla}_{\dot{\gamma}_{a,b}}\tilde{\nabla}_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b} + \|\tilde{\nabla}_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b}\|^2\dot{\gamma}_{a,b} = 0.$$

Taking the normal component of the equality (3) for the hypersurface we get

$$3\langle A\dot{\gamma}_{a,b}, \nabla_{\dot{\gamma}_{a,b}}\dot{\gamma}_{a,b}\rangle + \langle(\nabla_{\dot{\gamma}_{a,b}}A)\dot{\gamma}_{a,b}, \dot{\gamma}_{a,b}\rangle = 0.$$

Evaluating this equation at $s = 0$, we have

$$\pm 3k\langle Av_i, v_j\rangle + \langle(\nabla_{v_i}A)v_i, v_i\rangle = 0,$$

where the double sign takes plus if $(a,b)=(i,j)$ and takes minus if $(a,b)=(i,-j)$. From these equations for (i,j) and $(i,-j)$ we obtain

$$\langle Av_i, v_j\rangle = 0 \quad \text{and} \quad \langle (\nabla_{v_i} A)v_i, v_i\rangle = 0 \quad \text{for every distinct } i,\ j. \tag{4}$$

On the other hand, taking the tangential component of the equality (3), we have

$$\langle A\dot\gamma_{a,b}, \dot\gamma_{a,b}\rangle \{\langle A\dot\gamma_{a,b}, \dot\gamma_{a,b}\rangle \dot\gamma_{a,b} - A\dot\gamma_{a,b}\} = 0.$$

When $\langle Av_i, v_i\rangle \neq 0$, the constant $\langle A\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle$ is not 0 for every b. Therefore $A\dot\gamma_{i,b} = \langle A\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle \dot\gamma_{i,b}$ holds for every b. Differentiating both sides of this equality along $\gamma_{i,b}$, we get

$$\begin{aligned}&(\nabla_{\dot\gamma_{i,b}} A)\dot\gamma_{i,b} + A\nabla_{\dot\gamma_{i,b}}\dot\gamma_{i,b}\\ &\qquad = \{\langle (\nabla_{\dot\gamma_{i,b}} A)\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle + 2\langle A\dot\gamma_{i,b}, \nabla_{\dot\gamma_{i,b}}\dot\gamma_{i,b}\rangle\}\dot\gamma_{i,b} + \langle A\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle \nabla_{\dot\gamma_{i,b}}\dot\gamma_{i,b}.\end{aligned}$$

Evaluating this equation at $s=0$, from the equality (4) we have

$$(\nabla_{v_i} A)v_i \pm kAv_j = \pm k\langle Av_i, v_i\rangle v_j$$

for every $j\ (\neq i)$, where the double signs take plus if $b=j$ and take minus if $b=-j$. Thus we obtain $(\nabla_{v_i} A)v_i = 0$ and $Av_j = \langle Av_i, v_i\rangle v_j$ for every $j\ (\neq i)$ in this case.

When $\langle Av_i, v_i\rangle = 0$, we have $\langle A\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle = 0$ for every b. Differentiating this equation we get

$$\langle (\nabla_{\dot\gamma_{i,b}} A)\dot\gamma_{i,b}, \dot\gamma_{i,b}\rangle + 2\langle A\dot\gamma_{i,b}, \nabla_{\dot\gamma_{i,b}}\dot\gamma_{i,b}\rangle = 0.$$

Evaluating this equation at $s=0$, we have $\langle (\nabla_{v_i} A)v_i, v_i\rangle \pm 2k\langle Av_i, v_j\rangle = 0$ for every $j\ (\neq i)$, where the rule of double sign is the same as above. Thus we obtain $\langle Av_i, v_j\rangle = 0$ for every j. As $\{v_1, \ldots, v_n\}$ is a basis of T_xM, we get $Av_i = 0$.

We now show that M is umbilic at x. Since we have already seen that $\langle Av_i, v_j\rangle = 0$ for every distinct $i,\ j$, it is enough to verify $\langle Av_i, v_i\rangle = \langle Av_j, v_j\rangle$. When $Av_j = \langle Av_i, v_i\rangle v_j$ holds, this is trivial. When $Av_i = 0$, we have $\langle Av_j, v_j\rangle = 0$, because either $Av_j = 0$ or $Av_i = \langle Av_j, v_j\rangle v_i$ holds. Thus in this case we have $\langle Av_i, v_i\rangle = \langle Av_j, v_j\rangle = 0$. As $x \in M$ is an arbitrary point we get our conclusion. □

3. Concluding remark

If we substitute $k = 0$ for $k > 0$ in Theorem 1, the same statement does not hold. We call a hypersurface in a space form isoparametric if all of its principal curvatures are constant. Such hypersurfaces in a space form were characterized by Kimura and the second author in terms of the extrinsic shape of geodesics. Their results correspond to the case $k = 0$. Since they are closely related to our result we reproduce them here for the readers' convenience.

Remark ([1]).

(1) A connected hypersurface M^n in a space form $\tilde{M}^{n+1}(c)$ is isoparametric with nonzero principal curvatures if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of T_xM such that all geodesics on M through x in the direction v_i $(1 \leqslant i \leqslant n)$ are circles of *positive* curvature in $\tilde{M}^{n+1}(c)$.

(2) A connected hypersurface M^n in a space form $\tilde{M}^{n+1}(c)$ is isoparametric if and only if at each point $x \in M$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of T_xM as *principal curvature vectors* such that all geodesics on M through x in the direction v_i $(1 \leqslant i \leqslant n)$ are circles in $\tilde{M}^{n+1}(c)$.

It should be noted that the classification of isoparametric hypersurfaces in a sphere is not completed (cf. [2], [3]).

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Authors' address: T. Adachi, Dept. of Mathematics, Nagoya Institute of Technology, Gokiso, Nagoya, 466-8555, Japan, e-mail: adachi.toshiaki@nitech.ac.jp; S. Maeda, Dept. of Mathematics, Shimane University, Matsue, Shimane, 690-8504, Japan, e-mail: smaeda@math.shimane-u.ac.jp.