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Ján Jakubík

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ON VECTOR LATTICES OF ELEMENTARY CARATHÉODORY FUNCTIONS

Ján Jakubík, Košice

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Abstract. In this paper we deal with the vector lattice $C(B)$ of all elementary Carathéodory functions corresponding to a generalized Boolean algebra B.

Keywords: generalized Boolean algebra, elementary Carathéodory functions, Specker lattice ordered group, (α,β)-distributivity, complete distributivity

MSC 2000: 06F20, 46A40

1. Introduction

The vector lattice of elementary Carathéodory functions corresponding to a Boolean algebra was investigated by Gofman [7]. The author [9] applied elementary Carathéodory functions for studying cardinal properties of lattice ordered groups.

In an analogous way we can deal with elementary Carathéodory functions corresponding to a generalized Boolean algebra. The definition is given in Section 2 below.

For a generalized Boolean algebra B we denote by $C(B)$ the vector lattice of all elementary Carathéodory functions corresponding to B. If the multiplication of elements of $C(B)$ by reals is not taken into account, then we speak about lattice ordered group $C(B)$.

The Specker lattice ordered group $S(B)$ corresponding to B is an ℓ-subgroup of $C(B)$; this notion was investigated by Conrad and Darnel [3], [4], [5], Conrad and Martinez [6] and by the author [11].

Let α and β be cardinals. The (α,β)-distributivity of Boolean algebras and of lattice ordered groups was studied in a rather large series of papers. For the detailed

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bibliography concerning the (α, β)-distributivity in Boolean algebras cf. Sikorski [12]; for the case of lattice ordered groups cf., e.g., Weinberg [13] and the author [10] (and the articles quoted in these papers).

In this paper we deal with the relations between the higher degrees of distributivity concerning the partially ordered structures B, $S(B)$ and $C(B)$.

2. Preliminaries and some results

For lattice ordered groups and vector lattices we apply the terminology and notation as in Birkhoff [1] and Conrad [2].

A generalized Boolean algebra is defined to be a distributive lattice B with the least element 0 such that for each $b \in B$, the interval $[0, b]$ is a Boolean algebra.

Let G be a lattice ordered group and $x, y \in G^+$. The elements x and y are called orthogonal (or disjoint) if $x \wedge y = 0$; in such case we have $x \vee y = x + y$ and $n_1 x \wedge n_2 y = 0$ for any positive integers n_1 and n_2.

We recall the notion of elementary Carathéodory functions corresponding to a generalized Boolean algebra B (cf. [7], [9]; the distinction now is that in the quoted papers B was assumed to be a Boolean algebra).

Let $C(B)$ be the system consisting of all forms

$$f = a_1 b_1 + \ldots + a_n b_n$$

(where a_i are nonzero reals, $b_i \in B$, $b_i > 0$, $b_{i(1)} \wedge b_{i(2)} = 0$ for any $i(1), i(2) \in \{1, 2, \ldots, n\}$, $i(1) \neq i(2)$) and of the "empty form"; if g is another such form,

$$g = a'_1 b'_1 + \ldots + a'_m b'_m,$$

then f and g are considered as equal if $\bigvee\limits_{i=1}^{n} b_i = \bigvee\limits_{j=1}^{m} b'_j$ and if $a_i = a'_j$ whenever $b_i \wedge b'_j \neq 0$.

For $b, b' \in B$ let $b -_1 b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. The operation $+$ in $C(B)$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_1 + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i \Big(b_i -_1 \bigvee_{j=1}^{m} b'_j \Big) + \sum_{j=1}^{m} a'_j \Big(b'_j -_1 \bigvee_{i=1}^{n} b_i \Big),$$

where in the summation only those terms are taken into account in which $a_i + a'_j \neq 0$ and the elements $b_i \wedge b'_j$, $b_i -_1 \bigvee\limits_{j=1}^{m} b'_j$, $b'_j -_1 \bigvee\limits_{i=1}^{n} b_i$ are non-zero. The empty form is considered to be a neutral element in $C(B)$ (with respect to the operation $+$) and it will be identified with the element 0 of B. We put $f > 0$ if $a_i > 0$ for $i = 1, 2, \ldots, n$. Then $C(B)$ turns out to be a lattice ordered group. (We have the same symbol for the zero element of $\mathbb{R}$, the least element of B and for the neutral element of $C(B)$, but the meaning of this symbol will always be clear from the context.) If b is the neutral element of $C(B)$ and $a \in \mathbb{R}$, then we put $ab = b$. If $0 \in \mathbb{R}$ and $b \in B$, we set $0b = 0 \in C(\mathbb{R})$. Further, each element $b \in B$ will be identified with the element $b \in C(B)$; hence $B \subseteq C(B)$. If f is as above and $a \in \mathbb{R}$, then we put $af = (aa_1)b_1 + \ldots + (aa_n)b_n$. Under this definition, $C(B)$ is a vector lattice. The elements of $C(B)$ are called elementary Carathéodory functions corresponding to B.

Let us denote by $S(B)$ the set of all $f \in C(B)$ such that (under the notation as above) either $f = 0$ or all a_i $(i = 1, \ldots, n)$ are integers. Then $S(B)$ is an ℓ-subgroup of $C(B)$; we say that $S(B)$ is a Specker lattice ordered group corresponding to the generalized Boolean algebra B.

A lattice ordered group G will be defined to be a Specker lattice ordered group if there exists a generalized Boolean algebra B such that G is isomorphic to $S(B)$. In Section 3 we verify that this definition is equivalent to that used in the above mentioned paper [5].

Let α and β be nonzero cardinals and let T, S be nonempty sets with $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$. A lattice L is called (α, β)-distributive if the following identities hold in L

$$\bigwedge_{t\in T} \bigvee_{s\in S} x_{t,s} = \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x_{t,\varphi(t)}, \tag{1.1}$$

$$\bigvee_{t\in T} \bigwedge_{s\in S} x_{t,s} = \bigwedge_{\varphi\in S^T} \bigvee_{t\in T} x_{t,\varphi(t)} \tag{1.2}$$

under the assumption that all joins and meets appearing in (1.1) and (1.2) exist in L. Further, L is α-distributive if it is (α, α)-distributive; L is completely distributive if it is α-distributive for every cardinal α.

Let K be a sublattice of a lattice L. The lattice K is a *closed sublattice* of L if the following condition and its dual are satisfied:

(2) whenever $X \subseteq K$ and $\sup_L X$ exists, then $\sup_L X \in K$.

The definition of a regular sublattice is given in Section 4.

Let α be an infinite cardinal. A lattice L is *α-complete* if, whenever X is a nonempty subset of L with $\operatorname{card} X \leqslant \alpha$, then both $\sup X$ and $\inf X$ exist in L. Further, L is conditionally α-complete if each its interval is α-complete.

Let us recall that in accordance with the commonly used terminology, a lattice ordered group is called complete if it is conditionally complete; the analogous terminology will be used for α-completeness.

Conrad and Darnel proved the following result:

(CD)(Cf. [5], Theorem 3.13). *Let B be a generalized Boolean algebra and $G = S(B)$. Then the following conditions are equivalent:*

(i) *G is complete, completely distributive and has a unit;*

(ii) *B is an atomic complete Boolean algebra.*

It is well-known (cf. e.g., Sikorski [12]) that (ii) is equivalent to the condition

(iii) B is a complete and completely distributive Boolean algebra.

Assume that B is a generalized Boolean algebra. Let us mention the following results proven below.

B is a closed and regular sublattice of $S(B)$; further, $S(B)$ is a closed and regular sublattice of $C(B)$.

Let α and β be cardinals. $S(B)$ is (α, β)-distributive if and only if B is (α, β)-distributive. If $C(B)$ is (α, β)-distributive, then $S(B)$ is (α, β)-distributive.

B is conditionally complete and completely distributive if and only if $C(B)$ is complete and completely distributive.

3. Closedness of B and $S(B)$

Assume that B, $S(B)$ and $C(B)$ are as above.

Lemma 3.1. *Let $f, g \in C(B)$.*

a) *There are $b_1, \ldots, b_n \in B$, $0 < b_i$ $(i \in I = \{1, 2, \ldots, n\})$, $b_{i(1)} \wedge b_{i(2)} = 0$ for distinct elements $i(1)$, $i(2)$ of I, and reals $a_1, \ldots, a_n$, $a'_1, \ldots, a'_n$ such that*

$$f = a_1 b_1 + \ldots + a_n b_n, \tag{1}$$

$$g = a'_1 b_1 + \ldots + a'_n b_n. \tag{2}$$

Moreover, if $\circ \in \{+, -, \wedge, \vee\}$, then

$$f \circ g = (a_1 \circ a'_1) b_1 + \ldots + (a_n \circ a'_n) b_n.$$

b) *If $f, g \in S(B)$, then $a_1, \ldots, a_n$, $a'_1, \ldots, a'_n$ are integers.*

P r o o f. If suffices to apply the same method as in [11], Lemma 2.5 (the only distinction is that in [11] the coefficients a_i, a'_i $(i \in I)$ were integers). □

We say that (1) and (2) are canonical representations for the pair (f, g).

Lemma 3.2. *For $b_1, b_2 \in B$, the relation $b_1 < b_2$ as defined in $C(B)$ coincides with the original relation of partial order defined in B.*

Proof. This is an easy consequence of 3.1. □

Let $X \subseteq S(B)$ and $x_0 \in S(B)$. The meaning of the formulas $x_0 = \sup_{S(B)} X$ or $x_0 = \inf_{S(B)} X$ is clear.

Lemma 3.3. *Let $\emptyset \neq X \subseteq B$, $x_0 \in S(B)$. Assume that $x_0 = \sup_{S(B)} X$. Then $x_0 \in B$.*

Proof. Let x be any element of X. In view of 3.1 there exist canonical representations for the pair (x_0, x)

$$x_0 = a_1b_1 + \ldots + a_nb_n, \tag{1'}$$

$$x = a_1'b_1 + \ldots + a_n'b_n. \tag{2'}$$

Since $x = 1x$, in view of the definition of equality in $C(B)$ we must have $a_i' \in \{0,1\}$ for $i = 1, 2, \ldots, n$. Further, $a_i \geqslant a_i'$ for $i = 1, 2, \ldots, n$. Put $I_0 = \{i \in \{1, 2, \ldots, n\} \colon a_i' = 1\}$. Then

$$x = \sum_{i \in I_0} b_i = \bigvee_{i \in I_0} b_i.$$

Further, if $a_i = 0$ for some i, then $a_i' = 0$; thus without loss of generality we can suppose that $a_i \neq 0$ for $i = 1, 2, \ldots, n$. Hence

$$x_0 \geqslant b_1 + b_2 + \ldots + b_n = b_1 \vee b_2 \vee \ldots \vee b_n \geqslant x.$$

Let x_1 be another element of X; consider the canonical representation for the pair (x_0, x_1)

$$x_0 = a_1^1b_1^1 + \ldots + a_m^1b_m^1,$$
$$x_1 = (a_1^1)'b_1^1 + \ldots + (a_m^1)'b_m^1.$$

Similarly as above we can suppose that $b_j^1 \neq 0$ for $j = 1, 2, \ldots, m$. We have

$$a_1^1b_1^1 + \ldots + a_m^1b_m^1 = a_1b_1 + \ldots + a_nb_n;$$

the definition of equality in $C(B)$ yields

$$b_1^1 + \ldots + b_m^1 = b_1 + \ldots + b_n.$$

Hence both x and x_1 are less than or equal to $b_1 + \ldots + b_n$. Therefore $x_0 = \sup_{S(B)} X = b_1 + \ldots + b_n = b_1 \vee \ldots \vee b_n \in B$. □

From the method of the above proof we obtain also the following assertion:

Lemma 3.3.1. *Let $\emptyset \neq X \subseteq B$, $x_0 \in S(B)$. Assume that x_0 is an upper bound of X and that it does not belong to B. Then there exists $b_0 \in B$ such that b_0 is an upper bound of X and $b_0 < x_0$.*

Lemma 3.4. *B is an ideal of the lattice $(S(B))^+$.*

P r o o f. Let $g \in B$, $f \in C(B)$, $0 \leqslant f \leqslant g$. Consider the canonical representation (1) and (2) corresponding to the pair (f, g). In view of $1g = g \in B$ we conclude that $a'_i \in \{0, 1\}$ for $i = \{1, 2, \ldots, n\}$; if $a'_i = 0$, then we can suppose that $a_i = 0$ as well. Hence without loss of generality we can assume that all a'_i are equal to 1; therefore $a_i \in \{0, 1\}$ and $a_i b_i \in B$. Hence $f = a_1 b_1 \vee a_2 b_2 \vee \ldots \vee a_n b_n \in B$. In view of 3.3, the proof is complete. □

From 3.3 and 3.4 we obtain

Proposition 3.5. *Let B be a generalized Boolean algebra. Then B is a closed sublattice of $S(B)$.*

Let G be a lattice ordered group and let Y be the set of all $y \in G^+$ such that the interval $[0, y]$ of G is a Boolean algebra. In [5], G is defined to be a Specker lattice ordered group if it is generated as a group by the set Y; then each element $0 \neq g \in G$ can be expressed in the form

$$g = a_1 y_1 + \ldots + a_n y_n,$$

where $y_1, \ldots, y_n \in Y$, $y_i > 0$, $y_{j(1)} \wedge y_{i(2)} = 0$ for distinct $i(1)$, $i(2)$, and $a_i, \ldots, a_n$ are nonzero reals. It can be shown by a simple calculation that if two elements f and g of G are expressed in this form, then for $f + g$ the formula from Section 2 above is valid. Therefore the definition from [5] is equivalent (up to isomorphisms) to that given in Section 2.

Lemma 3.6. *Let $\emptyset \neq X \subseteq S(B)$, $x_0 \in C(B)$, $x_0 = \sup\limits_{C(B)} X$. Then $x_0 \in S(B)$.*

P r o o f. It is easy to verify (by applying the obvious translation) that it suffices to prove our assertion for the case when $X \subseteq (S(B))^+$. Thus we can assume that $x \geqslant 0$ for each $x \in X$. Hence $x_0 \geqslant 0$.

We apply an analogous idea as in the proof of 3.5. Let $x \in X$. Assume that (1′) and (2′) are canonical representations corresponding to the pair (x_0, x). Similarly as in the above proofs we can suppose that $a_i > 0$ for $i = 1, 2, \ldots, n$. The relation $x \in S(B)$ implies that all a'_i are integers. We denote by a^0_i the greatest integer with

$a_i^0 \leqslant a_i$. Hence $a_i' \leqslant a_i^0$. Put

$$x_0(x) = a_1^0 b_1 + \ldots + a_n^0 b_n.$$

Then $x \leqslant x_0$.

Let y be another element of X. Consider the canonical representations

$$(1'') \qquad x_0 = a_1^* b_1' + \ldots + a_m^* b_m',$$

$$(2'') \qquad y = y_1'' b_1' + \ldots + a_m'' b_m'$$

for the pair (x_0, y). Similarly as above we can suppose that all $a_1^*, \ldots, a_m^*$ are nonzero. Further, by an analogous construction as above we define

$$x_0(y) = a_1^{*0} b_1' + \ldots + a_m^{*0} b_m'.$$

In view of the definition of equality in $C(B)$ we have

$$b_1 \vee \ldots \vee b_n = b_1' \vee \ldots \vee b_m'.$$

Put $I = \{1, 2, \ldots, n\}$, $J = \{1, 2, \ldots, m\}$. Let $j \in J$. Then

$$b_j' = b_j' \wedge (b_1 \vee b_2 \vee \ldots \vee b_n) = (b_j' \wedge b_1) \vee \ldots \vee (b_j' \wedge b_n).$$

Denote $I(j) = \{i \in I\colon\ b_j' \wedge b_i > 0\}$. Then $I(j) \neq \emptyset$. By using again the definition of equality in $C(B)$ we obtain that for each $i \in I(j)$ we have $a_j^* = a_i$, whence $a_j^{*0} = a_i^0$. Then

$$\begin{aligned} a_j^{*0} b_j' = a_j^{*0} \Big(\bigvee_{i \in I(j)} (b_j' \wedge b_i) \Big) &= a_j^{*0} \sum_{i \in I(j)} (b_j' \wedge b_i) \\ \leqslant \sum_{i \in I(j)} a_j^* b_i &= \sum_{i \in I(j)} a_i^0 b_i \leqslant x_0(x). \end{aligned}$$

Since this holds for each $j \in J$, we get

$$x_0(y) = \sum_{j \in J} a_j^{*0} b_j' = \bigvee_{j \in J} a_j^{*0} b_j' \leqslant x_0(x).$$

Clearly $y \leqslant x_0(y)$, hence $y \leqslant y_0(x)$. Thus $x_0(x)$ is an upper bound of X. Because $x_0(x) \leqslant x_0 = \sup\limits_{C(B)} X$, we get $x_0(x) = x_0$. In view of the definition of $x_0(x)$ we have $x_0(x) \in S(B)$. $\square$

Similarly as in 3.3.1, the above proof yields also the following assertion:

Lemma 3.6.1. *Let $\emptyset \neq X \subseteq S(B)$, $x_0 \in C(B)$. Assume that x_0 is an upper bound of X and that it does not belong to $S(B)$. Then there exists $y_0 \in S(B)$ such that y_0 is an upper bound of X and $y_0 < x_0$.*

Lemma 3.7. *Let $\emptyset \neq X \subseteq S(B)$, $x_0 \in C(B)$, $x_0 = \inf_{C(B)} X$. Then $x_0 \in S(B)$.*

P r o o f. Similarly as in the proof of 3.6, it suffices to consider the case $X \subseteq (S(B))^+$. Thus $x_0 \geqslant 0$. Let $x \in X$. Again, let $(1')$ and $(2')$ be canonical representations of the pair (x_0, x). Let I be as above and $i \in I$. Then $a'_i \geqslant a_i$. We denote by a_i^0 the least integer with $a_i^0 \geqslant a_i$. Hence $a'_i \geqslant a_i^0$. Put

$$x_0(x) = a_1^0 b_1 + \ldots + a_n^0 b_n.$$

Then $x_0 \geqslant x_0(x) \geqslant x$.

Let $y \in X$ and let $(1'')$ and $(2'')$ be canonical representations of the pair (x_0, y). By means of these representations we define

$$x_0(y) = a_1^{*0} b'_1 + \ldots + a_m^{*0} b'_m$$

analogously as in the case of $x_0(x)$.

For $j \in J$ let $I(j)$ be as in the proof of 3.6. Further, for $i \in I$ let $J(i) = \{j \in J\colon\ b_i \wedge b'_j > 0\}$. In the proof of 3.6 we verified that, for each $j \in J$,

$$b'_j = \bigvee_{i \in I(j)} (b'_j \wedge b_i);$$

similarly, for each $i \in I$ we have

$$b_i = \bigvee_{j \in J(i)} (b_i \wedge b'_j).$$

Next, in view of the definition of the equality in $C(B)$ we infer that whenever $b_i \wedge b'_j > 0$, then $a_i = a_j^*$, whence

$$a_i^0 = a_j^{*0}. \tag{3}$$

This yields

$$x_0(x) = \bigvee_{i \in I} a_i^0 b_i = \bigvee_{i \in I} a_i^0 \bigvee_{j \in J(i)} (b_i \wedge b'_j) = \bigvee_{i \in I} \bigvee_{j \in J(i)} a_i^0 (b_i \wedge b'_j), \tag{4}$$

$$x_0(y) = \bigvee_{j \in J} a_j^* b'_j = \bigvee_{j \in J} a_j^* \bigvee_{i \in I(j)} (b'_j \wedge b_i) = \bigvee_{j \in J} \bigvee_{i \in I(j)} a_j^* (b'_j \wedge b_i). \tag{5}$$

We remark that in (4) we take all $b_i \wedge b'_j$ which are nonzero, and the same situation is in (5). Hence according to (3) we have $x_0(y) = x_0(x)$. Thus $y \geqslant x_0(x)$ for each $y \in X$. Therefore we must have $x_0(x) = x_0$. Since $x_0(x) \in S(B)$, we get $x_0 \in S(B)$. $\square$

In view of the proof of 3.7 we obtain (analogously as in 3.6.1)

Lemma 3.7.1. *Let $\emptyset \neq X \subseteq S(B)$, $x_0 \in C(B)$. Assume that x_0 is a lower bound of X and that it does not belong to $S(B)$. Then there exists $y_0 \in S(B)$ such that y_0 is a lower bound of X and $y_0 > x_0$.*

Proposition 3.8. *Let B be a generalized Boolean algebra. Then $S(B)$ is a closed ℓ-subgroup of $C(B)$.*

Proof. Since $S(B)$ is a subgroup of the group $C(B)$, the assertion follows from 3.6 and 3.7. □

In view of 3.5 we have

Corollary 3.9. *Let B be a generalized Boolean algebra. Then B' is a closed sublattice of $C(B)$.*

4. Regularity

Assume that L_1 is a sublattice of a lattice L_2. Consider the following condition:
(r$_1$) Whenever $x_1 \in L_1$, $\emptyset \neq X \subseteq L_1$ such that $x_1 = \sup_{L_1} X$, then $x_1 = \sup_{L_2} X$.

Further, let (r$_2$) be the condition dual to (r$_1$). If (r$_1$) and (r$_2$) are valid, then L_1 is said to be a regular sublattice of L_2.

Lemma 4.1. *Let L_1 be a sublattice of a lattice L_2. The condition* (r$_1$) *is implied by the condition*
(r$'_1$) *Whenever $\emptyset \neq X \subseteq L_1$, $x_0 \in L_2$, $x_0 \notin L_1$ such that x_0 is an upper bound of X, then there exists $y \in L_1$ such that y is an upper bound of X and $y < x_0$.*

Proof. Suppose that (r$'_1$) is satisfied. If (r$_1$) does not hold, then there are $x_1 \in L_1$, $\emptyset \neq X \subseteq L_1$ such that $x_1 = \sup_{L_1} X$ and x_1 fails to be the supremum of X in L_2. Hence there exists $y_1 \in L_2$ such that y_1 is an upper bound of X and $y_1 \ngeq x_1$. Put $y = y_1 \wedge x$. Thus $y < x_1$ and y is an upper bound of X. Then we must have $y \notin L_1$. In view of (r$'_1$), there is $x_2 \in L_1$ such that x_2 is an upper bound of X and $x_2 < y$. But then $x_2 < x_1$ and hence the relation $x_1 = \sup_{L_1} X$ cannot hold; we arrived at a contradiction. □

Let (r$'_2$) be the condition dual to (r$'_1$). Similarly as in 4.1 we have

Lemma 4.1.1. *Let L_1 be a sublattice of L_2. Then* (r_2) *is implied by* (r_2').

Proposition 4.2. *Let B be a generalized Boolean algebra. Then B is a regular sublattice of $S(B)$.*

P r o o f. Put $L_1 = B$, $L_2 = S(B)$. In view of 3.3.1 and 4.1, the condition (r_1) is satisfied. Further, according to 3.4 and 4.1.1, the condition (r_2) holds. □

Proposition 4.3. *Let B be a generalized Boolean algebra. Then $S(B)$ is a regular sublattice of $C(B)$.*

P r o o f. Put $L_1 = S(B)$, $L_2 = C(B)$. In view of 3.6.1 and 4.1, we obtain that the condition (r_1) holds. In view of 3.7.1 and 4.1.1, the condition (r_2) is valid. □

Corollary 4.4. *Let B be a generalized Boolean algebra. Then B is a regular sublattice of $C(B)$.*

5. Higher degrees of distributivity

Let α and β be cardinals; consider the relations (1.1) and (1.2) defining the (α, β)-distributivity of a lattice.

Proposition 5.1. *Let B be a generalized Boolean algebra. Then the following conditions are equivalent:*
(i) *B is (α, β)-distributive.*
(ii) *$S(B)$ is (α, β)-distributive.*

P r o o f. The case $B = \{0\}$ is trivial; suppose that $B \neq \{0\}$. Assume that (i) is valid. It is easy to verify that $S(B)$ is (α, β)-distributive if and only if all intervals of $S(B)$ are (α, β)-distributive. If $[u, v]$ is an interval in $S(B)$ and $a \in S(B)$, then $[u, v]$ is (α, β)-distributive if and only if the interval $[u+a, v+a]$ is (α, β)-distributive. Therefore, without loss of generality, it suffices to deal with intervals of the form $[0, v]$ with $0 < v \in S(B)$.

Let $\{x_{t,s}\}_{t \in T,\, s \in S} \subseteq [0, v]$; assume that $T \neq \emptyset \neq S$ and $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$. Further, suppose that all joins and meets appearing in (1.1) and (1.2) exist in $S(B)$; then these elements belong to the interval $[0, v]$. By way of contradiction, suppose that $S(B)$ fails to be (α, β)-distributive; e.g., suppose that (1.1) does not hold. Thus

$$(1) \qquad v_1 = \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} > \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t, \varphi(t)} = u_1.$$

There exists $0 < b \in B$ with $b \leqslant v_1 - u_1$. Denote

$$(s_{t,s} - u_1) \wedge b = x'_{t,s}.$$

From (1) we obtain

$$(2) \qquad 0 < b = (v_1 - u_1) \wedge b = \bigwedge_{t\in T} \bigvee_{s\in S} x'_{t,s} \geqslant \bigvee_{\varphi\in S^T} \bigwedge_{t\in T} x'_{t,\varphi(t)} = (u_1 - u_1) \wedge b = 0.$$

The joins and meets in (2) are taken with respect to $S(B)$; since B is closed in $S(B)$ (cf. Proposition 3.5), these operations give the same results in B. But then, in view of (2), B is not (α,β)-distributive, which is a contradiction.

b) Assume that (ii) holds and let $\{x_{t,s}\}_{t\in T,\, s\in S} \subseteq B$, $\operatorname{card} T \leqslant \alpha$, $\operatorname{card} S \leqslant \beta$. Suppose that all the joins and meets appearing in (1.1) and (1.2) exist in B. Then, since B is a regular sublattice of $S(B)$ (cf. 4.2), these operations give the same results in $S(B)$. Because $S(B)$ is (α,β)-distributive, (1.1) and (1.2) hold. Hence B is (α,β)-distributive. □

Proposition 5.2. *Let B be a generalized Boolean algebra. Assume that $C(B)$ is (α,β)-distributive. Then $S(B)$ is (α,β)-distributive as well.*

Proof. We can apply analogous argument as in the part b) of the proof of 5.1 with the distinction that instead of 4.2 we use 4.3. □

Proposition 5.3. *Let B be a generalized Boolean algebra and let α be an infinite cardinal.*

a) *B is α-complete if and only if $S(B)$ is α-complete.*

b) *If $C(B)$ is α-complete, then $S(B)$ is α-complete.*

Proof. Each interval of B is projective to an interval of type $[0, b_1]$ in B. Also, each interval of $S(B)$ is isomorphic to an interval of the form $[0, x]$, $x \in S(B)$, and an analogous assertion is valid for $C(B)$. Hence, when investigating the conditional completeness of B, $S(B)$ or $C(B)$, it suffices to consider only the intervals of the above mentioned types.

a1) Assume that $S(B)$ is conditionally α-complete. Since B is a closed sublattice of $S(B)$, in view of 3.5 we conclude that B is conditionally complete as well.

a2) Suppose that B is α-complete. Let $0 < x \in S(B)$. Then there are mutually orthogonal elements $b_1, \ldots, b_n$ in B and positive integers $a_1, \ldots, a_n$ such that

$$x = a_1 b_1 + \ldots + a_n b_n.$$

Put $a = \max\{a_1, \ldots, a_n\}$, $b = b_1 \vee \ldots \vee b_n$. Hence $[0, x] \subseteq [0, ab]$. The interval $[0, b]$ of B is α-complete. Since B is a regular subset of $S(B)$, the interval $[0, b]$ is

α-complete also as a subset of $S(B)$ (i.e., if we consider the operations $\wedge$ and $\vee$ as defined in $S(B)$). Now by applying the results of [9] we get that the interval $[0, ab]$ of $S(B)$ is α-complete as well.

b) Assume that $C(B)$ is conditionally α-complete. By the same method as in a1) (applying Proposition 3.8) we obtain that $S(B)$ is conditionally α-complete. □

Proposition 5.4. *Let B be a generalized Boolean algebra. The following conditions are equivalent:*

(i) *B is a Boolean algebra.*

(ii) *$S(B)$ has a strong unit.*

(iii) *$C(B)$ has a strong unit.*

Proof. The equivalence of (i) and (ii) is a consequence of Proposition 3.1 in [5]. For each element $x > 0$ of $C(B)$ there exists $y \in S(B)$ with $y \geqslant x$; from this we immediately obtain that (ii) and (iii) are equivalent. □

An element $0 < u$ of a lattice ordered group G is a *weak unit* if, whenever $0 < g \in G$, then $u \wedge g > 0$.

Proposition 5.4.1. *Let $B \neq \{0\}$ be a generalized Boolean algebra and $u \in C(B)$. The following conditions are equivalent:*

(i) *u is a strong unit of $C(B)$.*

(ii) *u is a weak unit of $C(B)$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) is valid. The element u can be represented in the form

$$u = a_1 b_1 + \ldots + a_n b_n$$

with $0 < b_i \in B$, $0 < a_i \in \mathbb{R}$ such that the system $\{b_1, \ldots, b_n\}$ is orthogonal. Let $0 < b \in B$. In view of (ii) we have

$$0 < u \wedge b = (a_1 b_1 + \ldots + a_n b_n) \wedge b = (a_1 b_1 \vee \ldots \vee a_n b_n) \wedge b = \bigvee_{i=1}^{n} (a_i b_i \wedge b).$$

Hence there is $i \in \{1, 2, \ldots, n\}$ such that $a_i b_i \wedge b > 0$. This yields that $b_i \wedge b > 0$. Therefore $\{b_1, \ldots, b_n\}$ is a maximal disjoint system in B.

It is easy to verify that whenever $\{b'_i\}_{i \in I}$ is a maximal disjoint system of a generalized Boolean algebra B' such that $\sup\{b'_i\}_{i \in I}$ exists in B', then the element $\sup\{b'_i\}_{i \in I}$ is the greatest element of B'.

Hence, in our case, the element $b = b_1 \vee \ldots \vee b_n = b_1 + \ldots + b_n$ is the greatest element of B. There exists $n \in N$ such that $na_i > 1$ for each $i = 1, 2, \ldots, n$. Thus $nu > b$; from this we conclude that u is a strong unit of $C(B)$. □

The analogous result for $S(B)$ was proved in [5, Theorem 3.1] by using a different idea of the proof.

We remark that Theorem 3.13 of [5] (cf. (CD) in Section 2 above) is a consequence of 5.1, 5.3, 5.4 and 5.4.1.

Lemma 5.5. *Let B be a generalized Boolean algebra and let b_0 be an atom of B. Then the interval $[0, b_0]$ in $C(B)$ is a complete chain.*

Proof. Let $0 < x$ be an element of the interval $[0, b_0]$ in $C(B)$. Then x can be represented in the form

$$x = a_1 b_1 + \ldots + a_n b_n,$$

where $b_1, \ldots, b_n$ are mutually orthogonal strictly positive elements of B and $a_1, \ldots, a_n$ are positive reals. Since $x \leqslant b_0$, we get $b_i \leqslant b_0$ $(i = 1, 2, \ldots, n)$. But b_0 is an atom in B, hence $b_0 = b_1 = \ldots = b_n$. We get $n = 1$, $x = a_1 b_0$. Then $0 < a_1 \leqslant 1$. If y is another element belonging to the interval $[0, b_0]$ in $C(B)$, then there is a'_1 with $0 < a'_1 \leqslant 1$, $y = a'_1 b_0$. Thus the elements x and y are comparable. Moreover, for $a_2 \in \mathbb{R}$, $a_2 b_0$ belongs to the interval $[0, b_0]$ in $C(B)$ iff $0 \leqslant a_2 \leqslant 1$, hence the interval under consideration is isomorphic to the interval $[0, 1]$ of reals; thus it is a complete lattice. □

Proposition 5.6. *Let B be a generalized Boolean algebra. The following conditions are equivalent:*

(i) *B is conditionally complete and completely distributive;*
(ii) *$S(B)$ is complete and completely distributive;*
(iii) *$C(B)$ is complete and completely distributive.*

Proof. (iii) $\Rightarrow$ (ii): This is a consequence of 5.2 and 5.3.

(ii) $\Rightarrow$ (i): This follows from 5.1 and 5.3.

(i) $\Rightarrow$ (iii): Assume that (i) is valid. If suffices to verify that if $0 < x \in c(B)$, then the interval $[0, x]$ of $C(B)$ is complete and completely distributive.

There exists $b \in B$ and a positive integer a such that $x \leqslant ab$, hence $[0, x] \subseteq [0, ab]$. In view of the assumption, the interval $[0, b]$ of B is complete and completely distributive. Therefore, since this interval is a Boolean algebra, it is atomic and hence there is a set $\{b_i\}_{i \in I}$ of its atoms such that

$$b = \bigvee_{i \in I} b_i \tag{3}$$

is valid in B. In view of Proposition 3.9, the relation (3) is valid also in $C(B)$. Thus in $C(B)$ we have

$$ab = \bigvee_{i \in I} ab_i. \tag{4}$$

For each $i \in I$ let X_i be the interval $[0, b_i]$ of $C(B)$. In view of 5.5, X_i is a complete chain. Thus according to [8], there exists a linearly ordered direct factor $\overline{X}_i$ of $C(B)$ such that $X_i \subseteq \overline{X}_i$. From $b_i \in \overline{X}_i$ we obtain $ab_i \in \overline{X}_i$ and so $[0, ab_i]$ (the interval in $C(B)$) is a chain; therefore it is completely distributive.

The system $\{ab_i\}_{i \in I}$ is orthogonal. From this and from the infinite distributivity of $C(B)$ we conclude that the relation (4) implies the existence of an isomorphisms of $[0, ab]$ onto the direct product $\prod_{i \in I} [0, ab_i]$. From the complete distributivity of the direct factors $[0, ab_i]$ we infer that $[0, ab]$ is completely distributive. Further, since $[0, x] \subseteq [0, ab]$, we obtain that $[0, x]$ is completely distributive.

In the part a2) of the proof of 5.3 we have already used the fact that from the completeness of $[0, b]$ it follows that $[0, ab]$ is complete as well. Thus $[0, x]$ is complete. □

References

[1] *G. Birkhoff*: Lattice Theory, Third Edition. Providence, 1967.
[2] *P. Conrad*: Lattice Ordered Groups. Tulane University, Tulane, 1970.
[3] *P. Conrad and M. R. Darnel*: Lattice-ordered groups whose lattices determine their additions. Trans. Amer. Math. Soc. *330* (1992), 575–598.
[4] *P. F. Conrad and M. R. Darnel*: Generalized Boolean algebras in lattice-ordered groups. Order *14* (1998), 295–319.
[5] *P. F. Conrad and M. R. Darnel*: Subgroups and hulls of Specker lattice-ordered groups. Czechoslovak Math. J. To appear.
[6] *P. F. Conrad and J. Martinez*: Signatures and S-discrete lattice ordered groups. Alg. Universalis *29* (1992), 521–545.
[7] *C. Gofman*: Remarks on lattice ordered groups and vector lattices. I. Carathéodory functions. Trans. Amer. Math. Soc. *88* (1958), 107–120.
[8] *J. Jakubík*: Konvexe Ketten in ℓ-Gruppen. Čas. pěst. mat. *83* (1959), 53–63.
[9] *J. Jakubík*: Cardinal properties of lattice ordered groups. Fundamenta Math. *74* (1972), 85–98.
[10] *J. Jakubík*: Distributivity in lattice ordered groups. Czechoslovak Math. J. *22* (1972), 108–125.
[11] *J. Jakubík*: Torsion classes of Specker lattice ordered groups. Czechoslovak Math. J. *52* (2002), 469–482.
[12] *R. Sikorski*: Boolean Algebras. Second Edition, Springer-Verlag, Berlin, 1964.
[13] *E. C. Weinberg*: Higher degrees of distributivity in lattices of continuous functions. Trans. Amer. Math. Soc. *114* (1962), 334–346.

Author's address: Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovak Republic, e-mail: kstefan@saske.sk.