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OSCILLATION OF NONLINEAR DIFFERENTIAL SYSTEMS
WITH RETARDED ARGUMENTS

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Abstract. In this work we investigate some oscillatory properties of solutions of non-linear differential systems with retarded arguments. We consider the system of the form

$$\begin{aligned} y'_i(t) - p_i(t)y_{i+1}(t) &= 0, \quad i = 1, 2, \dots, n-2, \\ y'_{n-1}(t) - p_{n-1}(t)|y_n(h_n(t))|^\alpha \operatorname{sgn}[y_n(h_n(t))] &= 0, \\ y'_n(t) \operatorname{sgn}[y_1(h_1(t))] + p_n(t)|y_1(h_1(t))|^\beta &\leq 0, \end{aligned}$$

where $n \geq 3$ is odd, $\alpha > 0$, $\beta > 0$.

Keywords: nonlinear differential system, oscillatory (nonoscillatory) solution

MSC 2000: 34K15, 34K40

1. INTRODUCTION

We consider systems of nonlinear differential inequalities with retarded arguments of the form

$$(S) \quad \begin{aligned} y'_i(t) - p_i(t)y_{i+1}(t) &= 0, \quad i = 1, 2, \dots, n-2, \\ y'_{n-1}(t) - p_{n-1}(t)|y_n(h_n(t))|^\alpha \operatorname{sgn}[y_n(h_n(t))] &= 0, \\ y'_n(t) \operatorname{sgn}[y_1(h_1(t))] + p_n(t)|y_1(h_1(t))|^\beta &\leq 0, \end{aligned}$$

where the following conditions are always assumed: $n \geq 3$ is odd, $\alpha > 0$, $\beta > 0$, $p_i: [a, \infty) \rightarrow [0, \infty)$, $a \in \mathbb{R}$, $i = 1, 2, \dots, n$, are continuous functions not identically

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equal to zero on any subinterval of $[a, \infty)$,

$$\int_a^\infty p_i(t) dt = \infty, \quad i = 1, 2, \dots, n - 1,$$

$h_1: [a, \infty) \rightarrow \mathbb{R}$, $h_n: [a, \infty) \rightarrow \mathbb{R}$ are continuous nondecreasing functions and $h_1(t) < t$, $h_n(t) < t$ on $[a, \infty)$, $\lim_{t \rightarrow \infty} h_1(t) = \lim_{t \rightarrow \infty} h_n(t) = \infty$. Denote by W the set of all solutions $y(t) = (y_1(t), \dots, y_n(t))$ of the system (S) which exist on some ray $[T_y, \infty) \subset [a, \infty)$ and satisfy $\sup \left\{ \sum_{i=1}^n |y_i(t)| : t \geq T \right\} > 0$ for any $T \geq T_y$.

As far as the authors know there is no oscillatory result for the system (S) in the case when $n \geq 3$ is odd. It is to be pointed out that Theorems 1 and 2 extend the result of Theorem 3 in [4]. Moreover, Theorems 3 and 4 consider the case when $\alpha\beta = 1$, which is not treated in [4].

Definition 1. A solution $y \in W$ is called oscillatory (weakly oscillatory) if each component (at least one component) has arbitrarily large zeros. A solution $y \in W$ is called nonoscillatory (weakly oscillatory) if each component (at least one component) is eventually of a constant sign on some interval $[t_0, \infty)$, $t_0 \geq a$. We define

$$I_0 = 1$$

and

$$I_k(t, s; p_k, \dots, p_1) = \int_s^t p_k(x) I_{k-1}(x, s, p_{k-1}, \dots, p_1) dx, \quad k = 1, \dots, n - 2.$$

Lemma 1. Suppose that

$$(1) \quad y = (y_1, \dots, y_n) \in W$$

is a nonoscillatory solution of (S) and

$$(2) \quad (-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty), \quad t_0 \geq a, \quad i = 1, \dots, n.$$

Then

$$(3) \quad y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] \geq |y_n(h_n(t))|^\alpha \int_{h_1(t)}^t p_{n-1}(x) I_{n-2}(x, h_1(t); p_{n-2}, \dots, p_1) dx$$

for all large t .

Proof. Let $t_0 \leq s \leq t$. It is evident that

$$y_1(s) = y_1(t) - \int_s^t y_1'(x) dx = y_1(t) - \int_s^t p_1(x)y_2(x) dx.$$

We calculate the second integral by parts. Denote

$$v(x) = \int_s^x p_1(\tau) d\tau = I_1(x, s; p_1), \quad u(x) = y_2(x).$$

Then we get

$$\begin{aligned} y_1(s) &= y_1(t) - y_2(t)I_1(t, s; p_1) + \int_s^t y_2'(x)I_1(x, s; p_1) dx \\ &= y_1(t) - y_2(t)I_1(t, s; p_1) + \int_s^t p_2(x)y_3(x)I_1(x, s; p_1) dx. \end{aligned}$$

Applying further $(n-3)$ -times the method of integration by parts to the last integral we obtain the identity

$$\begin{aligned} y_1(s) &= \sum_{j=0}^{n-2} (-1)^j y_{j+1}(t) I_j(t, s; p_j, \dots, p_1) \\ &\quad + \int_s^t p_{n-1}(x) |y_n(h_n(x))|^\alpha \operatorname{sgn}[y_n(h_n(x))] I_{n-2}(x, s; p_{n-2}, \dots, p_1) dx, \\ &\hspace{25em} t_0 \leq s \leq t. \end{aligned}$$

In view of (2) and the monotonicity of $y_n(t)$, we obtain for $T \geq t_0$ sufficiently large,

$$\begin{aligned} y_1(s) \operatorname{sgn}[y_1(s)] &= \sum_{j=0}^{n-2} (-1)^j y_{j+1}(t) \operatorname{sgn}[y_1(t)] I_j(t, s; p_j, \dots, p_1) \\ &\quad + \int_s^t p_{n-1}(x) |y_n(h_n(x))|^\alpha I_{n-2}(x, s; p_{n-2}, \dots, p_1) dx, \\ &\hspace{25em} T \leq s \leq t, \\ y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] &\geq |y_n(h_n(t))|^\alpha \int_{h_1(t)}^t p_{n-1}(x) I_{n-2}(x, s; p_{n-2}, \dots, p_1) dx, \quad t > T. \end{aligned}$$

The proof is complete. □

The following notation will be used:

$$\begin{aligned} \bar{p}_i(t) &= \min\{p_i(s) : h_1(t) \leq s \leq t\}, \quad t \geq a, \quad i = 1, \dots, n-1, \\ P_{n-1}(t) &= \bar{p}_{n-1}(t) \bar{p}_{n-2}(t) \dots \bar{p}_1(t). \end{aligned}$$

Lemma 2. *Suppose that assumptions (1) and (2) are fulfilled. Then*

$$(4) \quad y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] \geq \frac{(t - h_1(t))^{n-1}}{(n-1)!} P_{n-1}(t) |y_n(h_n(t))|^\alpha$$

for all large t .

Proof. In view of (3) we get

$$y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] \geq |y_n(h_n(t))|^\alpha \bar{p}_{n-1}(t) \int_{h_1(t)}^t I_{n-2}(x, h_1(t); p_{n-2}, \dots, p_1) dx.$$

Integrating by parts we obtain

$$\begin{aligned} & y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] \\ & \geq |y_n(h_n(t))|^\alpha \bar{p}_{n-1}(t) \int_{h_1(t)}^t (t-x) p_{n-2}(x) I_{n-3}(x, h_1(t); p_{n-3}, \dots, p_1) dx \\ & \geq \dots \geq |y_n(h_n(t))|^\alpha \bar{p}_{n-1}(t) \dots \bar{p}_1(t) \int_{h_1(t)}^t \frac{(t-x)^{n-2}}{(n-2)!} dx. \end{aligned}$$

Calculating the last integral we have

$$y_1(h_1(t)) \operatorname{sgn}[y_1(h_1(t))] \geq \frac{(t - h_1(t))^{n-1}}{(n-1)!} P_{n-1}(t) |y_n(h_n(t))|^\alpha, \quad t \geq T,$$

where T is sufficiently large. □

The next lemma follows from Theorem 3 in [4].

Lemma 3. *Suppose that $0 < \alpha\beta < 1$ and*

$$(5) \quad \int_T^\infty (h_1(t))^{(n-1)\beta} p_n(t) (P_{n-1}((h_1(t))))^\beta dt = \infty, \quad T \geq a.$$

Then every nonoscillatory solution of system (S) has the property $\lim_{t \rightarrow \infty} y_k(t) = 0$, $k = 1, 2, \dots, n$, and (2) holds.

The next lemma is derived from Theorem 2 in [1].

Lemma 4. Assume that $g \in C([a, \infty), [0, \infty))$, $\delta \in C([a, \infty), \mathbb{R})$, $\lim_{t \rightarrow \infty} \delta(t) = \infty$, $\delta(t) < t$ for $t \leq a$ and

$$\liminf_{t \rightarrow \infty} \int_{\delta(t)}^t g(s) \, ds > \frac{1}{e}.$$

Then the functional inequality

$$y'(t) + g(t)y(\delta(t)) \leq 0, \quad t \geq a,$$

cannot have an eventually positive solution and

$$y'(t) + g(t)y(\delta(t)) \geq 0, \quad t \geq a,$$

cannot have an eventually negative solution.

The next lemma is presented in [4] as Lemma 1.

Lemma 5. Let $y = (y_1, \dots, y_n) \in W$ be a weakly nonoscillatory solution of (S), then y is nonoscillatory.

Theorem 1. Suppose that $0 < \alpha\beta < 1$, (5) holds and

$$(6) \quad \liminf_{t \rightarrow \infty} \int_{h_n(t)}^t p_n(s) \left[\int_{h_1(s)}^s p_{n-1}(x) I_{n-2}(x, h_1(s); p_{n-2}, \dots, p_1) \, dx \right]^\beta \, ds > \frac{1}{e}.$$

Then all solutions of system (S) are oscillatory.

Proof. Assume that the system (S) has a solution $y = (y_1, \dots, y_n) \in W$ at least one component of which is eventually of constant sign. Then by Lemma 5 y is nonoscillatory. We may suppose that $y_1(t) > 0$ for $t \geq t_0 \geq a$. By Lemma 3 the solution y has the property

$$\lim_{t \rightarrow \infty} y_k(t) = 0, \quad k = 1, 2, \dots, n$$

and(2) holds. Applying Lemma 1 to the n th inequality of the system (S) we obtain

$$y'_n(t) + y_n^{\alpha\beta}(h_n(t))p_n(t) \left[\int_{h_1(t)}^t p_{n-1}(x) I_{n-2}(x, h_1(t); p_{n-2}, \dots, p_1) \, dx \right]^\beta \leq 0, \\ t \geq T \geq t_0.$$

With regard to the facts that $0 < \alpha\beta < 1$ and $\lim_{t \rightarrow \infty} y_n(t) = 0$, we get

$$(7) \quad y'_n(t) + p_n(t) \left[\int_{h_1(t)}^t p_{n-1}(x) I_{n-2}(x, h_1(t); p_{n-2}, \dots, p_1) \, dx \right]^\beta y_n(h_n(t)) \leq 0, \\ t \geq T,$$

where T is sufficiently large. By Lemma 4 the inequality (7) cannot have a positive solution. This contradicts the fact that $y_n(t) > 0$ for $t \geq T$. The proof is complete. \square

Theorem 2. *Suppose that $0 < \alpha\beta < 1$, (5) holds and*

$$(8) \quad \liminf_{t \rightarrow \infty} \int_{h_n(t)}^t (s - h_1(s))^{(n-1)\beta} P_{n-1}^\beta(s) p_n(s) ds > \frac{[(n-1)!]^\beta}{e}.$$

Then all solutions of system (S) are oscillatory.

Proof. Assume that the system (S) has a solution $y = (y_1, \dots, y_n) \in W$ at least one component of which is nonoscillatory. Then by Lemma 5 y is nonoscillatory. We may suppose that $y_1(t) > 0$ for $t \geq t_0 \geq a$. Due to Lemma 3 the solution y has the property $\lim_{t \rightarrow \infty} y_k(t) = 0$, $k = 1, 2, \dots, n$, and (2) holds. Applying (4) to the n th inequality of (S) we get

$$y'_n(t) + \frac{(t - h_1(t))^{(n-1)\beta}}{[(n-1)!]^\beta} P_{n-1}^\beta(t) p_n(t) y_n^{\alpha\beta}(h_n(t)) \leq 0, \quad t \geq T \geq t_0.$$

By virtue of the conditions $0 < \alpha\beta < 1$ and $\lim_{t \rightarrow \infty} y_n(t) = 0$, we obtain

$$(9) \quad y'_n(t) + \frac{(t - h_1(t))^{(n-1)\beta}}{[(n-1)!]^\beta} P_{n-1}^\beta(t) p_n(t) y_n(h_n(t)) \leq 0, \quad t \geq T$$

where T is sufficiently large.

By Lemma 4 the inequality (9) cannot have a positive solution. This is a contradiction with property (2). \square

The next lemma follows from Lemma 2 and Lemma 5 in [4].

Lemma 6. *Suppose that the assumption (1) of Lemma 1 is fulfilled. Then there exists $l \in \{1, 2, \dots, n\}$, l is odd and $t_0 \geq a$ such that*

$$(10) \quad y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = 1, 2, \dots, l,$$

$$(11) \quad (-1)^{n+i} y_i(t) y_1(t) > 0 \quad \text{on } [t_0, \infty) \quad \text{for } i = l+1, \dots, n,$$

and

$$(12) \quad |y_i(t/2)| \geq c_i t^{n-i} P_{n-1}^i(t) |y_n(t)|^\alpha \quad \text{for } t \geq t_0, \quad i = 1, 2, \dots, l-1,$$

where

$$c_i = \frac{2^{-2(n-i)}}{(n-1)!(n-i)!}, \quad i = 1, 2, \dots, n-1,$$

$$P_{n-1}^i(t) = \bar{p}_{n-1}(t)\bar{p}_{n-2}(t)\dots\bar{p}_i(t) \quad \text{for } i = 1, 2, \dots, n-1,$$

$$P_{n-1}^1(t) = P_{n-1}(t).$$

Remark. The inequality (10) implies

$$|y_i(t)| \geq |y_i(t/2)| \quad \text{for } i = 1, 2, \dots, l-1.$$

Hence(12) can be written in the form

$$(13) \quad |y_i(t)| \geq c_i t^{n-i} P_{n-1}^i(t) |y_n(t)|^\alpha \quad \text{for } t \geq t_0, \quad i = 1, \dots, l-1.$$

Theorem 3. Suppose that $\alpha\beta = 1$, (6) holds and

$$(14) \quad \liminf_{t \rightarrow \infty} \int_{h_1(t)}^t [h_1(s)]^{(n-1)\beta} [P_{n-1}(h_1(s))]^\beta p_n(s) \, ds > \frac{1}{ec_1^\beta}.$$

Then all solutions of the system (S) are oscillatory.

Proof. Assume that the system (S) has a solution $y = (y_1, \dots, y_n) \in W$ at least one component of which is nonoscillatory. Then by Lemma 5 the solution y is nonoscillatory. We may assume that $y_1(t) > 0$ for $t \geq t_0 \geq a$ and $y_1(h_1(t)) > 0$ for $t \geq t_1 \geq t_0$. Then the n th inequality of (S) implies that $y_n'(t) \leq 0$ for $t \geq t_1$ and it is not identically zero on any subinterval of $[t_1, \infty]$. As $y_1(t) > 0$ and $y_n'(t) \leq 0$ for $t \geq t_1$, then by Lemma 6 we get (10), (11), and (12) or (13).

Let $l \geq 2$. From (13) we have for $i = 1$,

$$y_1(t) \geq c_1 t^{n-1} P_{n-1}(t) y_n^\alpha(t), \quad t \geq t_2 \geq t_1.$$

Then the n th inequality of system (S) implies

$$y_n'(t) + c_1^\beta [h_1(t)]^{(n-1)\beta} [P_{n-1}(h_1(t))]^\beta p_n(t) [y_n(h_1(t))] \leq 0, \quad t \geq t_3 \geq t_2.$$

This inequality by Lemma 4 cannot have an eventually positive solution $y_n(t)$, which is a contradiction. The case when $l = 1$ is also impossible. This case can be treated as in the proof of Theorem 1. So the proof is complete. \square

Theorem 4. *Suppose that $\alpha\beta = 1$ and (5), (8), (14) hold. Then all solutions of system (S) are oscillatory.*

The result of the theorem follows from Theorems 3 and 2.

References

- [1] *R. G. Koplataдзе and T. A. Chanturia:* On the oscillatory and monotone solutions of the first order differential equations with deviating arguments. *J. Diff. Equations* 8 (1982), 1463–1465. (In Russian.)
- [2] *I. Foltynska and J. Werbowski:* On the oscillatory behaviour of solution of system of differential equation with deviating arguments. *Colloquia Math. Soc. J. B., Qualitative theory of Diff. Eq. Szeged* 30 (1979), 243–256.
- [3] *Y. Kitamura and T. Kusano:* On the oscillation of a class of nonlinear differential systems with deviating argument. *J. Math. Annal Appl.* 66 (1978), 20–36.
- [4] *P. Marušiak:* On the oscillation of nonlinear differential systems with retarded arguments. *Math. Slovaca* 34 (1984), 73–88.
- [5] *P. Marušiak and R. Olach:* *Functional Differential Equations.* University of Žilina, EDIS, Žilina, 2000. (In Slovak.)

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