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# ON SIGNPOST SYSTEMS AND CONNECTED GRAPHS

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*Abstract.* By a *signpost system* we mean an ordered pair $(W, P)$, where $W$ is a finite nonempty set, $P \subseteq W \times W \times W$ and the following statements hold:

$$\text{if } (u,v,w) \in P, \text{ then } (v,u,u) \in P \text{ and } (v,u,w) \notin P, \text{ for all } u,v,w \in W;$$
$$\text{if } u \neq v, \text{ then there exists } r \in W \text{ such that } (u,r,v) \in P, \text{ for all } u,v \in W.$$

We say that a signpost system $(W, P)$ is *smooth* if the folowing statement holds for all $u, v, x, y, z \in W$: if $(u,v,x), (u,v,z), (x,y,z) \in P$, then $(u,v,y) \in P$. We say thay a signpost system $(W, P)$ is *simple* if the following statement holds for all $u, v, x, y \in W$: if $(u,v,x), (x,y,v) \in P$, then $(u,v,y), (x,y,u) \in P$.

By the underlying graph of a signpost system $(W, P)$ we mean the graph $G$ with $V(G) = W$ and such that the following statement holds for all distinct $u, v \in W$: $u$ and $v$ are adjacent in $G$ if and only if $(u,v,v) \in P$. The main result of this paper is as follows: If $G$ is a graph, then the following three statements are equivalent:

$G$ is connected;

$G$ is the underlying graph of a simple smooth signpost system;

$G$ is the underlying graph of a smooth signpost system.

*Keywords*: connected graph, signpost system

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By a graph we mean here a *finite* undirected graph with no multiple edges or loops. The letters $f - n$ will serve for denoting non-negative integers.

## 1. Ternary systems

By a *ternary system* we mean an ordered pair $(W, P)$ such that $W$ is a finite nonempty set and $P \subseteq W \times W \times W$. Obviously, if $(W, P)$ is a ternary system, then $P$ is a ternary relation in $W$.

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Let $S = (W, P)$ be a ternary system. Consider $u_0, \dots, u_i, v \in W$, where $i \geqslant 1$. If

$$(u_j, u_{j+1}, v) \in P \quad \text{for each } j,\ 0 \leqslant j \leqslant i-1,$$

we will write

$$u_0 \dots u_i S v.$$

Thus, instead of $(u, v, w) \in P$, where $u, v, w \in W$, we write $uvSw$. Moreover, instead of $(x, y, z) \notin P$, where $x, y, z \in W$, we will write $\neg(xySz)$. Finally, we will write $V(S) = W$.

We say that a ternary system $S$ is *smooth* if it satisfies the following axiom (SMO):

(SMO) if $uvSx,\ uvSz$ and $xySz,$ then $uvSy$ for all $u, v, x, y, z \in V(S)$.

**Lemma 1.** *Let $S$ be a smooth ternary system and let $u_0, \dots, u_i, x, y, z \in V(S)$, where $i \geqslant 1$. Assume that $u_0 \dots u_i Sx$, $u_0 \dots u_i Sz$ and $xySz$. Then $u_0 \dots u_i Sy$.*

Proof. We proceeed by induction on $i$. The case when $i = 1$ follows immediately from (SMO). Let $i \geqslant 2$. Obviously, $u_0 \dots u_{i-1} Sx$ and $u_0 \dots u_{i-1} Sz$. By the induction hypothesis, $u_0 \dots u_{i-1} Sy$. Recall that $u_{i-1} u_i Sx$, $u_{i-1} u_i Sz$ and $xySz$. As follows from (SMO), $u_{i-1} u_i Sy$. Thus $u_0 \dots u_i Sy$, which completes the proof. □

**Proposition 1.** *Let $S$ be a smooth ternary system, and let $u_0, \dots, u_i, x_0, \dots, x_j \in V(s)$, where $i, j \geqslant 1$. If*

(1) $$u_0 \dots u_i S x_0, \quad u_0 \dots u_i S z \quad \text{and} \quad x_0 \dots x_j S z,$$

*then*

(2) $$u_0 \dots u_i S x_j.$$

Proof. Assume that (1) holds. We will prove that (2) holds. We proceed by induction on $j$. The case when $j = 1$ follows immediately from Lemma 1. Let $j \geqslant 2$. Obviously, $x_0 \dots x_{j-1} Sz$. By the induction hypothesis, $u_0 \dots u_i S x_{j-1}$. Since $u_0 \dots u_i Sz$ and $x_{j-1} x_j Sz$, Lemma 1 implies that $u_0 \dots u_i S x_j$, which completes the proof. □

We say that a ternary system $S$ is *simple* if it satisfies the following axioms (SIM1) and (SIM2):

(SIM1) if $uvSx,\ xySv,$ then $uvSy$ for all $u, v, x, y \in V(S)$;

(SIM2) if $uvSx,\ xySv,$ then $xySu$ for all $u, v, x, y \in V(S)$.

The next lemma depends on a result proved in [4].

**Lemma 2.** *Let $S$ be a simple ternary system, let $w_0, \ldots, w_k$, $y, z \in V(S)$, where $k \geqslant 1$. Assume that $w_0 \ldots w_k S y$. If $zySw_0$, then $w_0 \ldots w_k S z$ and $zySw_k$. If $yzSw_k$, then $w_0 \ldots w_k S z$ and $yzSw_0$.*

Proof. The lemma immediately follows from Lemma 1 in [4]. □

**Proposition 2.** *Let $S$ be a simple ternary system, and let $u_0, \ldots u_i, x_0, \ldots, x_j \in V(S)$, where $i, j \geqslant 1$. If*

$$u_i \ldots u_0 S x_0 \quad \text{and} \quad x_0 \ldots x_j S u_0, \tag{3}$$

*then*

$$u_i \ldots u_0 S x_j \quad \text{and} \quad x_0 \ldots x_j S u_i. \tag{4}$$

Proof. Put $h = i + j$. Obviously, $h \geqslant 2$. Let (3) hold. We will prove that (4) holds. We proceeed by induction on $h$. First, let $h = 2$. Then $i = 1 = j$. Clearly, (SIM1) and (SIM2) imply (4).

Let $h \geqslant 3$. If $i = 1$ or $j = 1$, then (4) immediatelly follows from Lemma 2. Assume that $i, j \geqslant 2$. Then $h \geqslant 4$. Since $u_i \ldots u_0 S x_0$ and $x_0 \ldots x_{j-1} S u_0$, the induction hypothesis implies that $u_i \ldots u_0 S x_{j-1}$. Since $u_{i-1} \ldots u_0 S x_0$ and $x_0 \ldots x_j S u_0$, the induction hypothesis implies that $x_0 \ldots x_j S u_{i-1}$. It follows from (3) that $u_i u_{i-1} S x_0$ and $x_{j-1} x_j S u_0$. Since $x_0 \ldots x_j S u_{i-1}$ and $u_i u_{i-1} S x_0$, Lemma 2 implies that $x_0 \ldots x_j S u_i$. Since $u_i \ldots u_0 S x_{j-1}$ and $x_{j-1} x_j S u_0$, Lemma 2 implies that $u_i \ldots u_0 S x_j$. Hence (4) holds. □

## 2. Signpost systems and their underlying graphs

By a *signpost system* we mean a ternary system $S$ satisfying the following axioms (SIG1), (SIG2) and (SIG3):

(SIG1) if $uvSw$, then $vuSu$ for all $u, v, w \in V(S)$,

(SIG2) if $uvSw$, then $\neg(vuSw)$ for all $u, v, w \in V(S)$,

(SIG3) if $u \neq v$, then there exists $r \in V(S)$ such that $urSv$ for all $u, v \in V(S)$.

**Remark.** If $S$ is a signpost system, $u, v, w \in V(S)$ and $uvSw$, then the ordered triple $(u, v, w)$ can be considered as a signpost showing a "direction" from $u$ to $w$; this direction is determined by $v$.

The notion of a signpost system appeared in [2] but an implicit form of the idea of a signpost system can be found in [4] already. Our definition follows the definition in [6], which is slightly different from that in [2]. Note that $(W, P)$ is a signpost system (in the sense of our definition) if and only if $P$ is a signpost system on $W$ in the sense of the definition in [2].

**Proposition 3.** *Let $S$ be a signpost system. Then*

$$\text{(5)} \qquad \text{if } uvSw, \text{ then } uvSv \text{ for all } u, v, w \in V(S),$$

$$\text{(6)} \qquad uvSv \text{ if and only if } vuSu \text{ for all } u, v \in V(S)$$

*and*

$$\text{(7)} \qquad \text{if } uvSw, \text{ then } v \neq u \neq w \text{ for all } u, v, w \in V(S).$$

Proof. Obviously, (1) and (2) are implied by (SIG1). Combining (SIG1) and (SIG2) we get (3). □

**Lemma 3.** *Let $S$ be a smooth signpost system. Then $S$ satisfies the axiom* (SIM1).

Proof. Consider arbitrary $u, v, x, y \in V(S)$. Let $uvSx$, $xySv$. By (5), $uvSv$. If we put $z = v$ in (SMO), we get $uvSy$. □

Let $S$ be a signpost system. By the *underlying* graph of $S$ we mean the graph $G$ defined as follows: $V(G) = V(S)$ and

$$u \text{ and } v \text{ are adjacent in } G \text{ if and only if } uvSv$$

for all $u, v \in V(S)$.

Let $S$ be a signpost system. By a *confusing circuit* in $S$ we mean an ordered pair

$$\text{(8)} \qquad (u_0 u_1 \dots u_k, v)$$

such that $u_0, u_1, \dots u_k, v \in V(S)$, $k \geqslant 1$, $u_0 \dots u_k S v$ and $u_k = u_0$.

**Lemma 4.** *Let $S$ be a signpost system, and let* (8) *be a confusing circuit in $S$. Then $u_0 \neq v$ and $k \geqslant 3$.*

P r o o f. As follows from (7), $u_0 \neq v$ and $k \geqslant 2$. Let $k = 2$. Since $u_2 = u_0$, we get $u_1u_0Sv$ and $u_0u_1Sv$, which contradicts (SIG2). Thus $k \geqslant 3$. □

**Example.** Let $X$, $Y$ and $Z$ be pairwise disjoint finite sets such that $|X| \geqslant 3$, $|Y| \geqslant 1$ and $|Z| \geqslant 3$. Put $j = |X|$ and $m = |Y|$. Assume that

$$X = \{x_0, \ldots, x_{j-1}\} \quad \text{and} \quad Z = \{z_0, \ldots, z_{m-1}\}.$$

Moreover, put $x_j = x_0$ and $z_m = z_0$. Let $S$ be the ternary system defined as follows: $V(S) = X \cup Y \cup Z$ and the following statement holds for all $u, v, w \in V(S)$: $uvSw$ if and only if one of the conditions (9)–(12) holds:

(9) $$u, v \in X \cup Y \quad \text{and} \quad u \neq v = w;$$

(10) $$u, v \in Y \cup Z \quad \text{and} \quad u \neq v = w;$$

(11) there exists $f$, $0 \leqslant f \leqslant j-1$, such that $u = x_f$, $v = x_{f+1}$ and $w \in Z$;

(12) there exists $h$, $0 \leqslant h \leqslant m-1$, such that $u = z_h$, $v = z_{h+1}$ and $w \in X$.

Obviously, $S$ is a simple signpost system and its underlying graph is connected. It is clear that $(x_0 \ldots x_{j-1}x_j, z)$ and $(z_0 \ldots z_{m-1}z_m, x)$ are confusing circuits in $S$ for every $z \in Z$ and every $x \in X$.

Let $S$ be a simple signpost system. Assume that the underlying graph of $S$ is connected. It was proved in [4] that there exists no confusing circuit in $S$ if $S$ satisfies the following axiom (ALT):

(ALT) if $uvSx$ and $xySy$, then $uvSy$ or $xySu$ or $yxSv$ for all $u, v, x, y \in V(S)$.

The assumption that the underlying graph of a considered signpost system is connected is not needed in the next proposition:

**Proposition 4.** *Let $S$ be a smooth signpost system. Then there exists no confusing circuit in $S$.*

P r o o f. Suppose, to the contrary, that there exist $u_0, u_1, \ldots, u_k, v \in V(S)$ such that $k \geqslant 1$ and (8) is a confusing circuit in $S$. Then $u_0u_1 \ldots u_kSv$ and $u_0 = u_k$. By Lemma 4, $k \geqslant 3$. Since $u_0u_1Sv$, Proposition 3 implies that $u_0u_1Su_1$. Since $u_0 = u_k$, Proposition 3 implies that $\neg(u_0u_1Su_k)$. Then there exists $j$, $1 \leqslant j \leqslant k-1$, such that $u_0u_1Su_j$ and $\neg(u_0u_1Su_{j+1})$. Recall that $u_0u_1Sv$. Since $S$ is smooth, we get $\neg(u_ju_{j+1}Sv)$, which contradicts the fact that $u_0u_1 \ldots u_kSv$. Thus the proposition is proved. □

**Corollary 1.** *Let $S$ be a smooth signpost system, let $u_0, \dots, u_i, v \in V(S)$, where $i \geqslant 1$, and let $u_0 \dots u_i S v$. Then $(u_0, \dots, u_i)$ is a path in the underlying graph of $S$.*

P r o o f. Let $G$ denote the underlying graph of $S$. Clearly, $(u_0, \dots, u_i)$ is a walk in $G$. Suppose, to the contrary, that $(u_0, \dots, u_i)$ is not a path. Then there exist $j$ and $k$ such that $0 \leqslant j < k \leqslant i$ and $u_j = u_k$. Then $(u_j \dots u_k, v)$ is a confusing circuit in $S$, which is a contradiction. □

Let $S$ be a smooth signpost system. By a *path* in $S$ we mean a sequence $(u_0, \dots, u_k)$, where $k \geqslant 1$, $u_0, \dots, u_k \in V(S)$ and $u_0 \dots u_k S u_k$. As follows from Corollary 1, every path in $S$ is a nontrivial path in the underlying graph of $S$. Let $v_0, \dots, v_m, u, v, w \in V(S)$, where $m \geqslant 1$; if $(v_0, \dots, v_m)$ is a path in $S$, $u = v_0$, $v = v_1$ and $w = v_m$, then we say that $(v_0, \dots, v_m)$ is an $uv - w$ path in $S$.

**Theorem 1.** *Let $S$ be a smooth signpost system, and let $u, v, w \in V(S)$. If $uvSw$, then there exists an $uv - w$ path in $S$.*

P r o o f. Suppose, to the contrary, that there exist no $uv - w$ path in $S$. Then $v \neq w$. As follows from (SIG3), there exists an infinite sequence

$$v_0, v_1, v_2, \dots$$

of elements of $V(S)$ such that $v_0 = u$, $v_1 = v$ and

$$v_0 \dots v_i S w \text{ and } v_i \neq w \text{ for each } i = 1, 2, 3, \dots.$$

Since $V(S)$ is finite, there exist distinct $j$ and $k$ such that $0 \leqslant j < k$ and $v_j = v_k$. Since $v_j \dots v_k S w$, we see that $(v_j \dots v_k, w)$ is a confusing circuit in $S$, which contradicts Proposition 4. Thus the theorem is proved. □

**Remark.** Theorem 1 can be interpreted as follows: Let $S$ be a signpost system, let $G$ denote the underlying graph of $S$, let $u, v, w \in V(S)$ and $uvSw$. If we know that $S$ is smooth, then we are certain that the signpost $(u, v, w)$ shows a path going from $u$ to $w$ in $G$; the direction (in $u$) of this path is determined by $v$.

The next lemma was proved in [4]:

**Lemma 5** (see Lemma 2 in [4]). *Let $S$ be a simple signpost system, and let $u_0, \dots, u_{i-1}, u_i \in V(S)$, where $i \geqslant 1$. If $u_0 \dots u_{i-1} u_i S u_i$, then $u_i u_{i-1} \dots u_0 S u_0$.*

Hint for the proof. We proceed by induction on $i$. If $i = 1$, then we use (6). If $i \geqslant 2$, then we combine the induction hypothesis with Lemma 2 and (6). □

**Corollary 2.** *Let $S$ be a simple smooth signpost system, and let $u_0, \ldots, u_{i-1}, u_i \in V(S)$, where $i \geqslant 1$. If $(u_0, \ldots, u_{i-1}, u_i)$ is a path in $S$, then $(u_i, u_{i-1}, \ldots, u_0)$ is a path in $S$.*

P r o o f. Proof is obvious. □

## 3. Extensions of simple smooth signpost systems

The next lemma will be used in the proof of the main result of this section.

**Lemma 6.** *Let $S$ be a simple smooth signpost system, and let $u_0, \ldots, u_i, v_0, \ldots, v_j, x, y \in V(S)$, where $i, j \geqslant 1$. Assume that $u_0 = v_0$ and $u_i = v_j$. If*

$$u_0 \ldots u_i S x \text{ and } v_0 \ldots v_j S y, \tag{13}$$

*then*

$$u_0 \ldots u_i S y \text{ and } v_0 \ldots v_j S x. \tag{14}$$

P r o o f. Let (13) hold. Since $S$ is a smooth signpost system, Theorem 1 implies that there exist an $u_{i-1}u_i - x$ path in $S$ and a $v_{j-1}v_j - y$ path in $S$.

If $u_i = x$, we put $m = i$. Let $u_i \neq x$. Then there exist $u_{i+1}, \ldots u_m \in V(S)$, where $m \geqslant i+1$, such that $u_m = x$ and $(u_{i-1}, u_i, \ldots, u_m)$ is a path in $S$.

If $v_j = y$, we put $n = j$. Let $v_j \neq y$. Then there exist $v_{j+1}, \ldots, v_n \in V(S)$, where $n \geqslant j+1$, such that $v_n = y$ and $(v_{j-1}, v_j, \ldots, v_n)$ is a path in $S$.

Clearly,

$$(u_0, \ldots, u_{i-1}, u_i, \ldots, u_m) \quad \text{and} \quad (v_0, \ldots, v_{j-1}, v_j, \ldots, v_n)$$

are paths in $S$. Corollary 2 implies that

$$(u_m, \ldots, u_i, u_{i-1}, \ldots, u_0) \quad \text{and} \quad (v_n, \ldots, v_j, v_{j-1}, \ldots, v_0)$$

are paths in $S$. Since $u_0 = v_0$ and $u_i = v_j$, we see that

$$(v_n, \ldots, v_j = u_i, u_{i-1}, \ldots, u_0) \quad \text{and} \quad (u_m, \ldots, u_i = v_j, v_{j-1}, \ldots v_0)$$

are paths in $S$ and therefore, by Corollary 2,

$$(u_0, \ldots, u_{i-1}, u_i = v_j, \ldots, v_n) \quad \text{and} \quad (v_0, \ldots, v_{j-1}, v_j = u_i, \ldots, u_m)$$

are paths in $S$ as well. Since $v_n = y$ and $u_m = x$, (14) follows, which completes the proof. □

**Corollary 3.** *Let $S$ be a simple smooth signpost system, and let $u_0, \dots, u_i, x \in V(S)$, where $i \geqslant 1$. If $u_0 \dots u_i S x$ and $u_0 u_i S u_i$, then $u_0 u_i S x$.*

P r o o f. Proof is obvious. □

A graph $G$ is called a *factor* of a graph $H$ if $V(H) = V(G)$ and $E(G) \subseteq E(H)$.

Let $S$ be a simple smooth signpost system, and let $G$ denote the underlying graph of $S$. Consider a graph $H$ such that $G$ is a factor of $H$. By the *extension* of $S$ to $H$ we mean a ternary system $S'$ with the following properties:

(a) $V(S') = V(S)$;

(b) if $u, v, w \in V(S')$, then $uvS'w$ if and only if $u$ and $v$ are adjacent vertices of $H$ and there exist $u_0, \dots, u_i \in V(S')$, where $i \geqslant 1$, such that $u_0 = u$, $u_i = v$ and $u_0 \dots u_i S w$.

Clearly, the extension of $S$ to $H$ is defined uniquely.

As follows from Corollary 3, the extension of a simple smooth signpost system $S$ to the underlying graph of $S$ is identical to $S$.

**Theorem 2.** *Let $S$ be a simple smooth signpost system, and let $G$ denote the underlying graph of $S$. Consider a graph $H$ such that $G$ is a factor of $H$. Then the extension of $S$ to $H$ is a simple smooth signpost system and $H$ is its underlying graph.*

P r o o f. Put $W = V(S)$. Let $S'$ denote the extension of $S$ to $H$. Obviously, $S'$ is a ternary system with $V(S') = W$. The proof of the theorem will be divided into four parts. We will prove that

(1) $S'$ is a signpost system,

(2) $H$ is the underlying graph of $S'$,

(3) $S'$ is smooth, and

(4) $S'$ is simple.

*Part 1.* Consider arbitrary $u, v, w \in W$ such that $uvS'w$. Then there exist $u_0, \dots u_i \in W$, where $i \geqslant 1$, such that $u_0 = u$, $u_i = v$ and $u_0 \dots u_i S w$. Hence $u_{i-1} u_i S w$. Recall that $S$ is a smooth signpost system. If $u_i = w$, we put $m = i$. If $u_i \neq w$, then, by Theorem 1, there exist $u_{i+1}, \dots u_m \in W$, where $m > i$, such that $u_m = w$ and $(u_{i-1}, u_i, \dots, u_m)$ is a path in $S$. Hence $(u_0, \dots, u_i, \dots, u_m)$ is a path in $S$.

By Corollary 2, $(u_m, \dots, u_i, \dots, u_0)$ is a path in $S$ as well. Hence $u_i \dots u_0 S u_0$. This means that $vuS'u$. We see that $S'$ satisfies (SIG1).

Assume that $vuS'w$. Then there exist $v_0, \dots, v_j \in W$, where $j \geqslant 1$, such that $v_0 = v$, $v_j = u$ and $v_0 \dots v_j S w$. Since $u_i = v_0$ and $v_j = u_0$, we see that $(u_0 \dots u_i v_1 \dots v_j, w)$ is a confusing circuit in $S$, which contradicts Proposition 4. Hence $\neg(vuS'w)$. We see that $S'$ satisfies (SIG2).

Consider arbitrary $u^*, v^* \in W$ such that $u^* \neq v^*$. Then there exists $r \in W$ such that $u^* r S v^*$. Since $u*$ and $r$ are adjacent vertices of $H$, we get $u^* r S' v^*$. We see that $S'$ satisfies (SIG3). Therefore, $S'$ is a signpost system.

*Part 2.* Obviously, $V(H) = W$. Consider arbitrary $u, v \in W$, $u \neq v$. As follows from the definition of $S'$, if $uvS'v$, then $uv \in E(H)$. Conversely, let $uv \in E(H)$. As follows from (SIG3), there exists $r \in W$ such that $urS'v$. By Theorem 1, there exists an $ur - v$ path in $S$. This implies that there exist $u_0, \ldots, u_k$, where $k \geqslant 1$, such that $u_0 = u$, $u_1 = r$, $u_k = v$ and $u_0 \ldots u_k S u_k$. Thus $u_0 u_k S' u_k$; we have $uvS'v$. Therefore, $H$ is the underlying graph of $S'$.

*Part 3.* Consider arbitrary $u, v, x, y \in W$ such that $uvS'x$, $uvS'z$ and $xyS'z$. Then $uv, xy \in E(H)$ and there exist $u_0, \ldots, u_i, v_0, \ldots, v_j, x_0, \ldots, x_k \in W$, where $i, j, k \geqslant 1$, such that $u_0 = u$, $u_i = v$, $v_0 = u, \ldots v_j = v$, $x_0 = x$, $x_k = y$,

$$u_0 \ldots u_i S x, v_0 \ldots v_j S z \quad \text{and} \quad x_0 \ldots x_k S z.$$

Since $S$ is a simple smooth signpost system, Lemma 6 implies that $u_0 \ldots u_i S z$. By Proposition 1, $u_0 \ldots u_i S x_k$. Recall that $uv \in E(H)$. Hence $uvS'y$. We see that $S'$ satisfies (SMO). Therefore, $S'$ is smooth.

*Part 4.* Consider arbitrary $u, v, x, y \in W$ such that $uvS'x$ and $xyS'v$. Then $uv, xy \in E(H)$ and there exist $u_0, \ldots u_i, x_0, \ldots, x_j \in W$, where $i, j \geqslant 1$, such that $u_0 = v$, $u_i = u$, $x_0 = x$, $x_j = y$, $u_i \ldots u_0 S x_j$ and $x_0 \ldots x_j S u_0$. Since $S$ is simple, Proposition 2 implies that $u_i \ldots u_0 S x_j$ and $x_0 \ldots x_j S u_i$. Recall that $uv, xy \in E(H)$. Hence $uvS'y$ and $xyS'u$. We see that $S'$ satisfies (SIM1) and (SIM2). Therefore, $S'$ is simple. □

## 4. Connected graphs

Let $G$ be a connected graph. By the *step system* of $G$ we mean the ternary system $S$ such that $V(S) = V(G)$ and

$$uvSw \text{ if and only if } d(u,v) = 1 \text{ and } d(v,w) = d(u,w) - 1 \text{ for all } u, v, w \in V(G),$$

where $d(x, y)$ denotes the distance between vertices $x$ and $y$ in $G$. It is easy to show that the step system of a connected graph is a simple signpost system.

**Lemma 7.** *If $G$ is a tree, then the step system of $G$ is smooth.*

Proof. Proof is simple. □

As was proved in [3] (and also in [5]), a simple signpost system is the step system of a connected graph if and only if

(15) the underlying graph of $S$ is connected

and $S$ satisfies the axiom (ALT) and a certain pair of additional axioms. Almost the same result was proved in [6]; but the proof in [6] is very different both from the proof in [3] and the proof in [5]; the pair of additional axioms was replaced by the following axiom (COM) in [6]:

(COM) if $uvSx$, $vuSy$ and $xySu$, then $yxSv$ for all $u, v, x, y \in V(S)$.

The axioms (SIG1), (SIG2), (SIG3), (SIM1), (SIM2), (ALT) and (COM) (and, of course, the axiom (SWO)) could be formulated in a language of the first-order logic. It is unknown whether the condition (15) can be replaced by a finite set of axioms of the same kind.

The step systems of connected graphs of two special classes were characterized without help of the condition (15) in [2]; one of these classes is the class of median graphs (for the notion of a median graph the reader is referred to [1]); note that every tree is a median graph.

As was shown in [6], the step system of every median graph is smooth. On the other hand, if $G$ is a connected graph that contains an induced K(2, 3), then the step system of $G$ is not smooth.

The following theorem is the main result of the present paper.

**Theorem 3.** *Let $G$ be a graph. Then the following statements are equivalent:*
(a) *$G$ is connected;*
(b) *$G$ is the underlying graph of a simple smooth signpost system;*
(c) *$G$ is the underlying graph of a smooth signpost system.*

Proof. (a) → (b): Let $G$ be connected. Consider an arbitrary spanning tree $T$ of $G$. Let $S_T$ denote the step system of $T$. Then $S_T$ is a simple signpost system. By Lemma 7, $S_T$ is smooth. Let $S$ denote the extension of $S_T$ to $G$. By Theorem 2, $S$ is a simple smooth signpost system and the underlying graph of $S$ is $G$.

(b) → (c): Obvious.

(c) → (a): Let $G$ be the underlying graph of a smooth signpost system $S$. Consider arbitrary $u, v \in V(S)$ such that $u \neq v$. By (SIG3), there exists $t \in V(S)$ such that $utSv$. Theorem 1 implies that there exists an $ut - v$ path in $S$. Clearly, there exists a path connecting $u$ and $v$ in $G$. This implies that $G$ is connected. □

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