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# UNIQUE $a$-CLOSURE FOR SOME $\ell$-GROUPS OF RATIONAL VALUED FUNCTIONS

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*Abstract.* Usually, an abelian $\ell$-group, even an archimedean $\ell$-group, has a relatively large infinity of distinct $a$-closures. Here, we find a reasonably large class with unique and perfectly describable $a$-closure, the class of archimedean $\ell$-groups with weak unit which are "$\mathbb{Q}$-convex". ($\mathbb{Q}$ is the group of rationals.) Any $C(X,\mathbb{Q})$ is $\mathbb{Q}$-convex and its unique $a$-closure is the Alexandroff algebra of functions on $X$ defined from the clopen sets; this is sometimes $C(X)$.

*Keywords*: archimedean lattice-ordered group, $a$-closure, rational-valued functions, zero-dimensional space

*MSC 2000*: 06F20, 06F25, 20F60, 54C30, 54F65

## Introduction

A *lattice-ordered group* (or *$\ell$-group* for short) is a group $(G,+)$ with a partial order that is a lattice (infimum and supremum are denote by $\wedge$ and $\vee$, respectively) such that the ordering is compatible with the group operation. That is, for all $g,h,k\in G$ with $g\leqslant h$ we have $g+k\leqslant h+k$. The set of positive elements of $G$ is written as $G^+$; note that the additive identity is an element of this set.

Elements $g,h\in G^+$ are *archimedean equivalent* (or *$a$-equivalent*), denoted $g\sim_a h$, if there exist natural numbers $n$, $m$ for which $g\leqslant nh$ and $h\leqslant mg$. If $G$ is an $\ell$-subgroup of $H$ then $H$ is an *$a$-extension of* $G$ if every positive element of $H$ is $a$-equivalent to a positive element of $G$. We write $G\leqslant_a H$ in this case. The divisible hull of an abelian $\ell$-group is an $a$-extension, for example. If $G$ has no proper

$a$-extension, then $G$ is *$a$-closed*. By Holland's Embedding Theorem, $a$-closures exist (see [7]); however, $a$-closures are not necessarily unique (see [4]).

Throughout, we use $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ to represent the naturals, rationals and reals, respectively.

Over the past 30 years, several researchers have sought $a$-closures in various classes of $\ell$-groups. Recently, the authors of [6] sought $a$-closures via valuation mappings of an $\ell$-group onto a distributive lattice. Also, in [14] the authors considered a class of $\ell$-groups that generalizes the class of hyperarchimedean $\ell$-groups (see also [5]) and determined the $a$-closures of these groups. In particular, they explicitly describe the $a$-closures of $C(X, \mathbb{Z})$, the ring of continuous integer-valued functions on $X$. In the present article we are interested in determining $a$-extensions and $a$-closures of certain more general objects in the category, $\mathbf{W}$, of archimedean $\ell$-groups with weak unit.

In this section we introduce standard concepts needed throughout the paper.

The $\ell$-group $G$ is *archimedean* if whenever $0 \leqslant g \leqslant nh$ for all $n \in \mathbb{N}$, then $g = 0$. All archimedean $\ell$-groups are necessarily abelian. This is explained in [7].

An element $u \in G^+$ is a *weak order unit* if $u \wedge g = 0$ implies $g = 0$. $\mathbf{W}$ denotes the category whose objects are the archimedean $\ell$-groups with designated weak order unit and whose morphisms are the lattice-preserving group homomorphisms that also preserve the unit. $(G, u)$ denotes an object in $\mathbf{W}$.

Recall that an $\ell$-subgroup $K \leqslant G$ is *convex* if $0 \leqslant g \leqslant k \in K$ implies that $g \in K$. Let $(G, u)$ be a $\mathbf{W}$-object. By Zorn's Lemma, there exist convex $\ell$-subgroups of $G$ that are maximal with respect to not containing $u$. We let $YG$ denote the set of these. In the hull-kernel topology, $YG$ is a compact Hausdorff space. Define

$$D(YG) = \{f\colon\, YG \to \mathbb{R} \cup \{\pm\infty\}\colon\, f \text{ is continuous and } f^{-1}\mathbb{R} \subseteq YG \text{ is dense}\}.$$

Though $D(YG)$ is rarely a group under pointwise addition, it is known that $G$ may be mapped bijectively, via an $\ell$-group isomorphism, onto an $\ell$-group $\hat{G}$ of $D(YG)$, which maps $u$ to the constant function 1 and so that the elements of $\hat{G}$ separate the points of $YG$. This representation is unique: If $G \cong \tilde{G} \leqslant D(X)$ is an $\ell$-isomorphism with $X$ compact Hausdorff and $\tilde{u} = 1$, then there is a continuous surjection $\tau\colon\, X \to YG$ such that $\tilde{g} = \hat{g} \circ \tau$ for each $g \in G$; moreover, $\tilde{G}$ separates the points of $X$ if and only if $\tau$ is a homeomorphism. We identify $G$ with its image $\hat{G}$. This representation is the "Yosida Embedding" (see [21] and [16]) and $YG$ is called the *Yosida space* of $G$.

We now turn to topological considerations and to $C(X)$, the $\ell$-group of real-valued continuous functions on the space $X$ with the pointwise ordering. See [9] for details.

We assume that all spaces are Tychonoff, that is, completely regular and Hausdorff. $\beta X$ denotes the Stone-Čech compactification of $X$, and we note that the Yosida space of $C(X)$ is homeomorphic to $\beta X$. $C^*(X)$ is the $\ell$-subgroup containing the bounded elements of $C(X)$. There is a natural isomorphism between $C^*(X)$ and $C(\beta X)$, given by extension (and inversely, restriction) of functions to $\beta X$ (inversely, to $X$). Whenever $C(X) = C^*(X)$, we call $X$ *pseudocompact*.

Recall that a space is called *zero-dimensional* if it has a base of clopen sets and that every zero-dimensional space has a maximal zero-dimensional compactification called the *Banaschewski compactification* (see [20]) denoted by $\beta_0 X$. The space $\beta_0 X$ is homeomorphic to the Yosida spaces of $C(X,\mathbb{Z})$ and $C(X,\mathbb{Q})$ and the map $\beta_0$ is the compact zero-dimensional reflection. When $\beta X = \beta_0 X$, the space $\beta X$ is zero-dimensional and we call $X$ *strongly zero-dimensional*.

## 2. Unique $a$-closure and convex $\ell$-groups

Let $(G,u)$ be in $\mathbf{W}$ and $g \in G$. The *zeroset* of $g$ is $Z(g) = \{p \in YG\colon\ g(p) = 0\}$ and the *cozeroset* of $g$ is $YG \setminus Z(g)$. We use $\mathscr{Z}G$ to denote the set of all zerosets of $G$.

**Theorem 2.1.** *Let $(G,u)$ be in $\mathbf{W}$. If $G \leqslant_a H$ then $G$ majorizes $H$ (that is, for every $h \in H^+$ there exists $g \in G^+$ such that $h \leqslant g$); $u$ is a weak unit in $H$, $Y(G,u) = Y(H,u)$ and in the Yosida representation $G \leqslant H \leqslant D(Y(G,u))$ and $\mathscr{Z}H = \mathscr{Z}G$.*

Proof. Let $(G,u)$ be in $\mathbf{W}$ and assume that $G \leqslant_a H$. That $G$ majorizes $H$ follows directly from the definition of $a$-extension. If there is $h \in H^+$ such that $u \wedge h = 0$, then for any $g \in G$ such that $g \sim_a h$, we have that $u \wedge g = 0$. Hence $g = 0$ and $0 \leqslant h \leqslant mg = 0$ for some $m$ and therefore, $h = 0$. It follows from Theorem 2.1 of [4] that $Y(G,u) = Y(H,u)$; hence, $G \leqslant H \leqslant D(Y(G,u))$ and $\mathscr{Z}H = \mathscr{Z}G$. □

For $g \in G$, let $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$. Then $g = g^+ - g^-$ and we define $|g| = g^+ + g^-$.

**Definition 2.2.** Let $(G,u)$ be in $\mathbf{W}$.
(a) $G^c = \{f \in D(YG)\colon\ |f| \leqslant g$ for some $g \in G\}$.
(b) From [2]: $G$ is *convex* if $G = G^c$.

$G^c$ is usually not an $\ell$-group, as we discuss shortly.

**Corollary 2.3.** *In $\mathbf{W}$:*
(a) *If $G \leqslant_a H$ then $H \subseteq G^c$.*
(b) *If $G$ is convex, then $G$ is $a$-closed.*
(c) *If $G^c$ is an $\ell$-group and if $G \leqslant_a G^c$ then $G^c$ is the unique $a$-closure of $G$.*
(d) *If $H$ is convex and $G \leqslant_a H$, then $H$ is the unique $a$-closure of $G$.*

P r o o f. It is clear that Theorem 2.1 implies statements (a) and (b) which together imply (c). To verify (d), note that $G \leqslant_a H$ implies $H \subseteq G^c$ by (a). But also, $G^c \subseteq H^c = H$. Thus, $G^c = H$, and (c) applies. □

The statement of Corollary 2.3 (c) and (d) present us with the following two versions of the same questions, which the sequel examines.

**Question 2.4.** Let $G$ be an archimedean $\ell$-group.

1. (a) For which $G$ is $G^c$ an $\ell$-group?
   (b) For which $G$ is $G^c$ an $\ell$-group and $G \leqslant_a G^c$?
2. For convex $H$, what **W**-subobjects $G$ have $G \leqslant_a H$?

The following compendium from the literature illustrates what the class of convex $\ell$-groups encompasses. Recall that an $f$*-ring* is a subdirect product of totally ordered rings, [3].

**Theorem 2.5.** *For the following classes of* **W***-objects, for each* $n$*, the class* $(n)$ *is contained the class* $(n+1)$*.*

(1) *Rings of continuous functions,* $C(X)$*.*
(2) *Alexandroff algebras:* $\ell$*-subalgebras of* $\mathbb{R}^X$ *containing* 1 *that are closed under uniform convergence and inversion (see* § 5 *below).*
(3) **W***-objects closed under countable composition.*
(4) *Archimedean* $f$*-rings with identity, that are divisible and uniformly complete.*
(5) *Convex* **W***-objects.*

P r o o f. That (1) ⊆ (2) is clear; (2) ⊆ (3) ⊆ (4) can be found in [18]; and (4) ⊆ (5) is in [17]. (One has to recognize that the representation in [17] and [18] of an $f$-algebra is the Yosida representation of the underlying **W**-object). □

As a class of study, "convex" was introduced in [2], and there shown to be monoreflective in **W**: for each $(G, u)$ there is a group $cG$ such that $G \leqslant cG$ with $cG$ convex such that each $\varphi\colon G \to H$ in **W** with $H$ convex has a unique extension $c\varphi\colon cG \to H$ in **W**. Usually, $YcG$ is much larger than $YG$, but it is easy to see that if $G^c$ is an $\ell$-group then $G^c = cG$.

**Remark 2.6.** (a) Recall that $V \in YG$ is *real* if $G/V \hookrightarrow \mathbb{R}$ and $\mathscr{R}G \subseteq YG$ denotes the set of all such points. Let $G|_{\mathscr{R}G} = \{g|_{\mathscr{R}G}\colon g \in G\}$. In Theorem 2.1 and Corollary 2.3 (a), suppose that $\bigcap \mathscr{R}G = (0)$, so that $G|_{\mathscr{R}G} \subseteq C(\mathscr{R}G)$ is a representation of $G$; then $G^c|_{\mathscr{R}G} \subseteq C(\mathscr{R}G)$ also and $G \leqslant_a H$ implies that $H \subseteq C(\mathscr{R}G)$. Within the category **W**, this sharpens an observation in Example 6.2 of [4].

(b) By Theorem 2.5, $C(X)$ is convex for any $X$. Here's another proof: The Yosida embedding of $C(X)$ is given by $\{\beta f \in D(\beta X)\colon f \in C(X)\}$, therefore, $C(X)^c =$

$C(X)$. Thus, by Corollary 2.3 (b), $C(X)$ is convex. This improves Example 6.2 of [4] in which Conrad shows that $C(X)$ is $a$-closed.

(c) If $G$ is hyperarchimedean, then the converse of Theorem 2.1 holds (see [13]), but the converse fails in general. Let $\alpha\mathbb{N}$ be the one-point compactification of $\mathbb{N}$ and let $G \leqslant C(\alpha\mathbb{N})$ be given by $g \in G$ if and only if there exist $r, s \in \mathbb{R}$ such that eventually $g(n) = r + s/n$. Then $\mathscr{X}G = \mathscr{X}C(\alpha\mathbb{N})$, though $G$ is not $a$-extended by $C(\alpha\mathbb{N})$ since $f(n) = \mathrm{e}^{-1/n} \in C(\alpha\mathbb{N})$ has no $a$-equivalent element in $G$.

## 3. Relatively convex $\ell$-groups

**Definition 3.1.** Let $A$ be a subgroup of $\mathbb{R}$ containing 1 and $(G, u)$ in $\mathbf{W}$.

(a) For a compact Hausdorff space $X$, let

$$D_A(X) = \{f \in D(X)\colon\ f(p) \in \mathbb{R} \Rightarrow f(p) \in A\}.$$

(b) $G$ is *$A$-convex* if for $f \in D_A(YG)$, $|f| \leqslant g \in G$ implies $f \in G$. When $A \neq \mathbb{R}$, we assume that $YG$ is zero-dimensional.

(c) $W_A G = G \cap D_A(YG)$.

Note that an $A$-convex group is $Z$-convex. In fact, we are really only interested in $Z$- and $\mathbb{Q}$-convex objects.

In this section, we show that $G$ is $A$-convex if and only if $G^c$ is a convex $\ell$-group for which $YG^c = YG$ is zero-dimensional and

$$W_A G = W_A G^c \leqslant G \leqslant G^c.$$

This relates the two queries in Question 2.4 and addresses Question 2.4.1 (a). We also note the rarity of $W_Z G \leqslant_a G$.

In the next section we show that $W_{\mathbb{Q}} G \leqslant_a G$ for convex groups $G$. Thus, $\mathbb{Q}$-convex is the answer to Question 2.4.

**Remark 3.2.** The operator $W_Z$ is studied in [15], there denoted $\mathbf{W}_s$. It is a coreflection of $\mathbf{W}$ onto the full subcategory whose objects satisfy $G = W_Z G$ (called *singular*). The situation with $W_A$ is analogous, but we won't pursue that here. Note that $Z$-convexity is an extension of the *singularly convex* condition in [14].

**Proposition 3.3.** *$G^c$ is an $\ell$-group (hence it is convex and $YG^c = YG$) if and only if $\beta g^{-1}\mathbb{R} = YG$ for each $g \in G$.*

P r o o f. $\Rightarrow$: Suppose that $g^{-1}\mathbb{R}$ is not $C^*$-embedded (without loss of generality, we may take $g \in G^+$), say $f \in C^*(g^{-1}\mathbb{R})$ fails to extend over $YG$. Choose $m \geqslant |h|$ and define $h(x) = f(x) + g(x)$ if $x \in g^{-1}\mathbb{R}$ and $f(x) = +\infty$ if $x \notin g^{-1}\mathbb{R}$. Then $|h| \leqslant g + m \in G$, so that $h \in G^c$. But $h - g \notin D(YG)$, so $G^c$ is not closed under addition.

$\Leftarrow$: The lattice operations are inherited from $D(YG)$. Suppose that $f_i \in D(YG)$ with $|f_i| \leqslant g_i \in G^+$ for $i = 1, 2$. Then $f_i^{-1}\mathbb{R} \supseteq g_i^{-1}\mathbb{R}$ so that

$$f_1 + f_2 \in C(g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R}).$$

Since $g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R} = (|g_1| + |g_2|)^{-1}\mathbb{R}$ and we assume that this set is $C^*$-embedded, we have the extension to $h \in D(YG)$ and $|h| \leqslant g_1 + g_2$ (since that holds on the dense set $g_1^{-1}\mathbb{R} \cap g_2^{-1}\mathbb{R}$). Thus $h \in G^c$ and $h = f_1 + f_2$ in $G^c$. $\square$

**Proposition 3.4.** *Suppose that $G$ is $Z$-convex.*

(a) *If $g \in G^+$ and there is $0 < r \in \mathbb{R}$ such that $g(x) > 0$ implies $g(x) \geqslant r$, then there is $f \in W_Z G$ such that $f \sim_a g$.*
(b) *For all $g \in G$, there exists $f \in W_Z G$ such that $f^{-1}\mathbb{R} = g^{-1}\mathbb{R}$.*
(c) *For all $g \in G$, $\beta g^{-1}\mathbb{R} = YG$.*
(d) *$G^c$ is a convex $\ell$-group with $YG^c = YG$.*

P r o o f. The definition of $Z$-convex includes the assumption that $YG$ is zero-dimensional, so any $g^{-1}\mathbb{R}$ is zero-dimensional and Lindelöf, thus strongly zero-dimensional. See [9] and [20].

(a) Without loss of generality, $r \geqslant 3$. For every $n \geqslant 3$, choose a clopen set $U_n$ with $g^{-1}[n-1, n+1] \subseteq U_n \subseteq g^{-1}(n-2, n+2)$ so that

$$n - 2 \leqslant \bigwedge_n g|_{U_n} \leqslant \bigvee_n g|_{U_n} \leqslant n + 2.$$

Let $V_n = U_n \setminus \bigcup\limits_{j<n} U_j$. Then the functions $g|_{V_n}$ retain the preceding inequalities and $g^{-1}\mathbb{R} = Z(g) \bigsqcup_n V_n$. Clearly, this set is open.

Now define $f \in D_Z(YG)$ by $f|_{V_n} = n - 2$, $f|_{YG - g^{-1}\mathbb{R}} = +\infty$ and $f|_{Z(g)} = 0$. So then $f \leqslant g$ on $g^{-1}\mathbb{R}$ and hence, $f \leqslant g$. Also

$$g|_{V_n} \leqslant n + 2 = (n-2) + 4 = f|_{V_n} + 4.$$

Then $g \leqslant f + 4 \leqslant 5f$, since $1 \leqslant f$.

(b) Apply (a) to $|g| \vee 3$ to get $f$.

(c) Since $g^{-1}\mathbb{R}$ is strongly zero-dimensional, it suffices to demonstrate that any $h \in C(g^{-1}\mathbb{R}, \{0,1\})^+$ extends over $YG$. See [9] and [20]. By (b), we can assume that $g \in W_Z G$. Define $f \in D_Z(YG)$ by $f(x) = g(x) + h(x)$ if $x \in g^{-1}\mathbb{R}$ and $f(x) = +\infty$, otherwise. Then $f \leqslant g + 2 \in G$. Since $G$ is $Z$-convex, $f \in G$. Thus, $g - f \in G$ and this is the desired extension of $h$.

(d) By (c) and Proposition 3.3. □

**Theorem 3.5.** *Let $A$ be a proper subgroup of $\mathbb{R}$ containing* 1.

(a) *If $H$ is convex with $YH$ zero-dimensional, then $W_A H$ is $A$-convex and $H = (W_A H)^c$.*

(b) *If $G$ is $A$-convex, then $G^c$ is a convex $\ell$-group with $YG^c$ zero-dimensional and $W_A(G^c) \leqslant G$.*

Proof. (a) If $YH$ is zero-dimensional, then $C(YH, Z)$ separates points of $YH$. Since $C(YH, Z) \leqslant W_Z H \leqslant W_A H$, the group $W_Z H$ also separates points of $YH$ and thus $YW_A H = YH$. Now suppose that $H$ is convex and $f \in D_A(YW_A H)$, such that $|f| \leqslant g \in W_A H$ for some $g$. Then $f \in D(YH)$ and $|f| \leqslant g \in H$. Since $H$ is convex, $f \in H$. Since also $f \in D_A(YH)$, we have that $f \in W_A H$ and, hence, $W_A H$ is convex.

We know that $W_A H \subseteq H$ and so $(W_A H)^c \subseteq H$ since $H$ is convex. For the reverse, $H^+ \subseteq (W_Z H)^c$ by Proposition 3.4 (a); so $H \subseteq (W_Z H)^c$ since the larger set is an $\ell$-group by the above and by Proposition 3.4 (d). Since we have the containment $(W_Z H)^c \subseteq (W_A H)^c$, the proof is complete.

(b) Assume that $G$ is $A$-convex. Since $Z \leqslant A$, $G$ is $Z$-convex, so Proposition 3.4 (d) applies. Let $f \in W_A G^c$, that is, $f \in D_A(YG^c)$ and $|f| \leqslant g \in G^c$. Thus, $|f| \leqslant g \leqslant g' \in G$. Since $G$ is $A$-convex, $f \in G$. □

**Corollary 3.6.** *Let $A$ be a proper subgroup of $\mathbb{R}$ containing* 1.

(a) *The following are equivalent:*
  - ($a_1$) *$H$ is convex with $YH$ zero-dimensional.*
  - ($a_2$) *$H = G^c$ for some $A$-convex $G$.*
  - ($a_3$) *$H = G^c$ for a unique $A$-convex $G$ with $G = W_A G$, namely $G = W_A H$.*

(b) *The following are equivalent:*
  - ($b_1$) *$G$ is $A$-convex.*
  - ($b_2$) *$W_A H \leqslant G \leqslant H$ for some convex $H$ with $YH$ zero-dimensional; such an $H$ is unique, namely $H = G^c$.*

Proof. ($a_3$) $\Rightarrow$ ($a_2$) is clear and ($a_2$) $\Rightarrow$ ($a_1$) by Theorem 3.5 (b).

($a_1$) ⇒ ($a_3$): We know that $H = (W_A H)^c$ by Theorem 3.5 (a). If also $H = G^c$ for some $A$-convex $G = W_A G$, then

$$W_A H = W_A(G^c) \leqslant G \leqslant W_A G \leqslant W_A(G^c),$$

using Theorem 3.5 (b) and the fact that $G \leqslant G^c$ implies that $W_A G \leqslant W_A G^c$.

($b_1$) ⇒ ($b_2$): Assume that $G$ is $A$-convex. By Theorem 3.5 (a), if $H$ satisfies ($b_2$) then $H = G^c$ and by Theorem 3.5 (b) $G^c$ does satisfy ($b_2$).

($b_2$) ⇒ ($b_1$): Suppose that $G$ and $H$ satisfy ($b_2$). Then $YG = YH$ and if $f \in D_A(YG)$ with $|f| \leqslant g \in G$ then $f \in W_A H$ so $f \in G$. Thus, $G$ is $A$-convex. □

**Remark 3.7.** (a) Proposition 3.3 is the content of Remark 2.6 (e) in [2], where no proof was given.

(b) Proposition 3.4 is related to a lemma in [2].

(c) A **W**-object $(G, u)$ for which every $g \in G^+$ satisfies the hypothesis of Proposition 3.4 (a) is called *bounded away*. So we have shown that when $G$ is $Z$-convex and bounded away, $W_Z G \leqslant_a G$. This is closely related to Corollary 4.5 of [14].

In Proposition 3.4 (a), the bounded away condition can not be dropped: Let $X$ be a compact and zero-dimensional space, then $C(X)$ is $Z$-convex. However, $W_Z C(X) = C(X, Z)$ and $C(X, Z) \leqslant_a C(X)$ if and only if $X$ is finite. (See [13].)

(d) In fact, for $H$ convex, $W_Z H \leqslant_a H$ if and only if $YH$ is finite (whence $H \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$): sufficiency is easy to show, so let's show necessity. If $H$ is convex, then $H^* = C(YH)$ and if $W_Z H \leqslant_a H$, then

$$C(YH, Z) = W_Z H^* = (W_Z H)^* \leqslant_a H^* = C(YH)$$

and we have the situation of the above. So $YH$ is finite.

(e) Proposition 3.4 shows that $Z$-convex answers Question 2.4.1 (a), while Corollary 3.6 and Remark (d) above show that $Z$-convex fails to answer Question 2.4.1 (b), equivalently, the condition $W_Z H = G$ fails to answer Question 2.4.2.

## 4. The main theorem

We now replace $Z$ by $\mathbb{Q}$.

**Theorem 4.1.** *In* **W***:*

(a) *If $H$ is convex with $YH$ zero-dimensional, then $W_{\mathbb{Q}} H \leqslant_a H$ and $H$ is the unique $a$-closure of $W_{\mathbb{Q}} H$.*

(b) *If $G$ is $\mathbb{Q}$-convex, then $G \leqslant_a G^c$, so $G^c$ is the unique $a$-closure of $G$.*

(c) *If $H$ is $\mathbb{Q}$-convex, then $W_{\mathbb{Q}} H \leqslant_a H$.*

Proof. By § 3, (a) and (b) two are the same statement, so we prove (a). Statement (c) is a direct consequence of (a) and (b).

Let $H$ be convex with $YH$ zero-dimensional and $h \in H^+$. Choose a clopen set $U \subseteq YH$ with $h^{-1}[0, \frac{1}{2}] \subseteq U \subseteq h^{-1}[0, 1)$. Let $h_1(p) = h(p)$ if $p \in U$, $h_1(p) = 0$ otherwise and let $h_2(p) = h(p)$ if $p \notin U$ and $h_2(p) = 0$ if $p \in U$. Since $U$ is clopen, $h_1, h_2 \in D(YH)$ and since $0 \leqslant h_1, h_2 \leqslant h$, and $H$ is convex, $h_1, h_2 \in H$. It suffices to find $g_1, g_2 \in W_{\mathbb{Q}} H^+$ with $g_i \sim_a h_i$ when $i = 1, 2$ and then $g_1 + g_2 \sim_a h_1 + h_2 = h$.

Now $h_2(p) > 0$ implies that $h_2(p) \geqslant \frac{1}{2}$. So by Proposition 3.4 (a), there is $g_2 \in W_Z H$ with $g_2 \sim_a h_2$.

For $i = 1$: since $H$ is convex, $H^* = C(YH)$. We finish by using the following Lemma (with $f = h_1$). □

**Lemma 4.2.** *If $X$ is compact and zero-dimensional and $f \in C(X)$ such that $0 \leqslant f \leqslant 1$, then there is $g \in C(X, \mathbb{Q})$ with $g \sim_a f$.*

Proof. By induction, choose clopen sets $K_0 \supseteq K_2 \supseteq \ldots$ as follows: $K_0 = X$ and for each $n$,

$$f^{-1}[0, 1/2^{n+1}] \subseteq K_{n+1} \subseteq K_n \cap f^{-1}[0, 1/2^n).$$

Then we see that $Z(f) = \bigcap\limits_n K_n$,

$$1/2^{n+1} \leqslant f|_{K_n \setminus K_{n+1}} \leqslant 1/2^{n-1}$$

and $\operatorname{coz}(f) = \bigcup\limits_n (K_n \setminus K_{n+1})$. Define $g \in C(X, \mathbb{Q})$ by $g(x) = 0$ when $x \in Z(f)$ and $g(x) = 1/2^{n+1}$ when $x \in K_n \setminus K_{n+1}$. Then $g \leqslant f$ and $f \leqslant 4g$. Thus, $g \sim_a f$. □

## 5. Alexandroff algebras and $C(X, \mathbb{Q})$

Throughout, we assume that $X$ is zero-dimensional; otherwise, $C(X, \mathbb{Q})$ may be too small.

**Theorem 5.1.** *Suppose $X$ is zero-dimensional.*

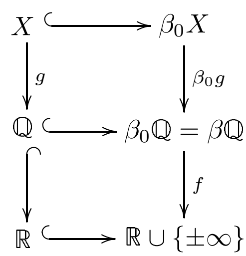
(a) *Each $g \in C(X, \mathbb{Q})$ has an extension $\hat{g} \in D(\beta_0 X)$, and $\{\hat{g}\colon\ g \in C(X, \mathbb{Q})\}$ is the Yosida representation. In particular, $YC(X, \mathbb{Q}) = \beta_0 X$.*

(b) *$C(X, \mathbb{Q})$ is $\mathbb{Q}$-convex and so has a unique $a$-closure $C(X, \mathbb{Q})^c$.*

(c) *$W_{\mathbb{Q}} C(X) = W_{\mathbb{Q}} C(X, \mathbb{Q})$ and $W_{\mathbb{Q}} C(X) \leqslant_a C(X, \mathbb{Q})$.*

(d) *$W_{\mathbb{Q}} C(X) = C(X, \mathbb{Q})$ if and only if $X$ is pseudocompact.*

Proof. (a) Consider the commutative diagram of continuous functions:




in which $\beta_0 g$ exists with $\beta_0 g|_X = g$, because $\beta_0$ is the reflection functor to compact zero-dimensional spaces. Since $\mathbb{Q}$ is strongly zero-dimensional, we have that $\beta_0 \mathbb{Q} = \beta \mathbb{Q}$. Then $f$ is the extension of the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R} \subseteq \mathbb{R} \cup \{\pm\infty\}$, and $\hat{g} = f \circ \beta_0 g \in D(\beta_0 X)$.

We have $C(X, \mathbb{Q}) \supseteq C^*(X, Z) \cong C(\beta_0 X, Z)$, and the last separates the points of $\beta_0 X$, thus so does $\{\hat{g}\colon\ g \in C(X, \mathbb{Q})\}$ hence this is the Yosida representation.

(b) Let $f \in D_{\mathbb{Q}}(\beta_0 X)$ and $|f| \leqslant \hat{g}$, where $g \in C(X, \mathbb{Q})$. Then $f|_X \in C(X, \mathbb{Q})$ and $\widehat{f|_X} = f$. Thus, $C(X, \mathbb{Q})$ is $\mathbb{Q}$-convex. Then $C(X, \mathbb{Q})^c$ is the unique $a$-closure by Theorem 4.1.

(c) Since $C(X, \mathbb{Q}) \leqslant C(X)$ we have $W_{\mathbb{Q}} C(X, \mathbb{Q}) \leqslant W_{\mathbb{Q}} C(X)$. For the reverse, let $f \in W_{\mathbb{Q}} C(X)$. This means that $f = \beta g$ for $f|_X = g \in C(X)$ and for $p \in \beta X$, whenever $f(p) \in \mathbb{R}$ necessarily means that $f(p) \in \mathbb{Q}$. Thus $g \in C(X, \mathbb{Q})$. We have $f = \beta g = \hat{g} \circ \varphi$, where $\varphi\colon\ \beta X \to \beta_0 X$ is the canonical map. Then whenever $\hat{g}(q) \in \mathbb{R}$, we necessarily have that $\hat{g}(q) \in \mathbb{Q}$ for all $q \in \beta_0 X$.

That $W_{\mathbb{Q}} C(X) \leqslant_a C(X, \mathbb{Q})$ follows from (b) and Theorem 4.1.

(d) By (*c*), having $W_{\mathbb{Q}} C(X) = C(X, \mathbb{Q})$ is equivalent to having the inclusion $W_{\mathbb{Q}} C(X) \supseteq C(X, \mathbb{Q})$, which means that for $f \in D(\beta_0 X)$, $f(X) \subseteq \mathbb{Q}$ implies $f|_{f^{-1}\mathbb{R}} \subseteq \mathbb{Q}$.

Suppose that $X$ is pseudocompact, $f \in D(\beta_0 X)$ and $f(X) \subseteq \mathbb{Q}$. Then $f(X)$ is a pseudocompact subset of $\mathbb{Q}$, hence compact, so $f^{-1}\mathbb{R} = \beta_0 X$ and $f(\beta_0 X) = f(X) \subseteq \mathbb{Q}$.

Suppose that $X$ is not pseudocompact. Then, since $X$ is zero-dimensional, $X = \bigcup\limits_n U_n$ for nonempty pairwise disjoint clopen sets $U_n$. Let $x_n \to r$ in $\mathbb{R}$ with $x_n \in \mathbb{Q}$

and $r \notin \mathbb{Q}$, and define $g \in C(X,\mathbb{Q})$ by $g|_{U_n} = x_n$. Then the extension $\hat{g} \in D(\beta_0 X)$ must have $\hat{g}(p) = r$ for some $p$ therefore $g \notin W_{\mathbb{Q}} C(X)$. □

**Corollary 5.2.** *$C(X,\mathbb{Q})^c = C(X)$ (equivalently, $C(X,\mathbb{Q}) \leqslant_a C(X)$) if and only if $X$ is strongly zero-dimensional.*

We now describe $C(X,\mathbb{Q})^c$, in general.

If $X$ is zero-dimensional, let $\operatorname{clop}(X)$ be the Boolean algebra of clopen sets of $X$ Then for $U \in \operatorname{clop}(X)$, the map $U \mapsto \operatorname{cl} U \in \operatorname{clop}(\beta_0 X)$ is its Stone representation.

Define $(\operatorname{clop}(X))_\sigma = \bigl\{\bigcup_n U_n \colon U_n \in \operatorname{clop}(X)\bigr\}$. Clearly, $(\operatorname{clop}(X))_\sigma \subseteq \operatorname{coz}(X)$, with equality if and only if $X$ is strongly zero-dimensional. In fact,

$$(\operatorname{clop}(X))_\sigma = \{K \cap X \colon K \in \operatorname{coz}(\beta_0 X)\}.$$

Define $A(X) = \{f \in \mathbb{R}^X \colon f^{-1}K \in (\operatorname{clop}(X))_\sigma \text{ for } K \subseteq \mathbb{R} \text{ open}\}$. Then $A(X)$ is a **W**-object and $A(X) \leqslant C(X)$ with equality if and only if $X$ is strongly zero-dimensional. See § 7 of [10] for a discussion.

**Theorem 5.3.**

(a) *$A(X)$ is of the type in Theorem 2.5. (2), thus is convex.*

(b) *$C(X,\mathbb{Q}) \leqslant A(X)$ and for each $f \in A(X)$ there is a sequence of functions $\{g_n\}_{n=1}^{\infty} \in C(X,\mathbb{Q})$ such that $g_n \to f$ uniformly on $X$.*

(c) *Each $f \in A(X)$ has an extension $\hat{f} \in D(\beta_0 X)$ and $\{\hat{f} \colon f \in A(X)\}$ is the Yosida representation. In particular, $Y A(X) = \beta_0 X$.*

(d) *$W_{\mathbb{Q}} A(X) = W_{\mathbb{Q}} C(X) \leqslant C(X,\mathbb{Q}) \leqslant A(X)$.*

(e) *$A(X) = C(X,\mathbb{Q})^c$, that is, $A(X)$ is the unique a-closure of $C(X,\mathbb{Q})$.*

P r o o f. (a) This is easily verified, or one may see § 7 of [10].

(b) Let $g \in C(X,\mathbb{Q})$ and let $A$ be an open set in $\mathbb{R}$. Since $\mathbb{Q}$ is strongly zero-dimensional, $A \cap \mathbb{Q} = \bigcup_n U_n$ for clopen sets $U_n \in \mathbb{Q}$. Thus, we can write $g^{-1}A = \bigcup_n g^{-1}U_n \in (\operatorname{clop}(X))_\sigma$.

Let $f \in A(X)$ and $\varepsilon > 0$. Let $\mathscr{A}$ be a countable cover of $\mathbb{R}$ by open intervals of length less than $\varepsilon$. So, for $A \in \mathscr{A}$, $f^{-1}A = \bigcup_n U(n,A)$ for clopen $U(n,A)$ and $\mathscr{U} = \{U(n,A) \colon A \in \mathscr{A}, n \in \mathbb{N}, U(n,A) \neq \emptyset\}$ is a countable cover of $X$ by clopen sets. We re-index the sets as $\mathscr{U} = \{U_n\}$ and disjointify: $V_n = U_n \setminus \bigcup_{i<n} U_i$. Let $\mathscr{V} = \{V_n\}_n$.

For each $A \in \mathscr{A}$, choose $r_A \in A \cap \mathbb{Q}$. Let $g = \sum_n \{r_A \chi_{V_n} \colon V_n \in \mathscr{V}\}$, where $\chi_{V_n}$ is the characteristic function of $V_n$. Then $g \in C(X,\mathbb{Q})$ and $|g(x) - f(x)| < \varepsilon$ for each $x \in X$.

(c) The extensions $\hat{f}$ exist by Theorem 5.1 (a) and the fact that a uniform limit of extendible functions is extendible. These extensions separate the points, since the extensions $\hat{g}$ for $g \in C(X, \mathbb{Q})$ do. The rest follows from this.

(d) This follows from Theorem 5.1 (c) and from $C(X, \mathbb{Q}) \leqslant A(X) \leqslant C(X)$.

(e) By (a), (d) and § 4. □

**Remark 5.4.** (a) Theorem 5.3 (a), (b), and (c) are implicit in § 7 of [11].

(b) From a more general perspective, $(\text{clop}(X))_\sigma$ is an example of what is called a *cozero field*, $A(X)$ is its associated *Alexandroff algebra*, and Theorem 2.5 (2) is a characterization of such things. One may see [10], [11], [12] and the original references therein to Hausdorff, Lebesgue and A. D. Alexandroff.

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