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Implicative hyper $K$-algebras


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IMPLICATIVE HYPER $K$-ALGEBRAS

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Abstract. In this note we first define the notions of (weak, strong) implicative hyper $K$-algebras. Then we show by examples that these notions are different. After that we state and prove some theorems which determine the relationship between these notions and (weak) hyper $K$-ideals. Also we obtain some relations between these notions and (weak) implicative hyper $K$-ideals. Finally, we study the implicative hyper $K$-algebras of order 3, in particular we obtain a relationship between the positive implicative hyper $K$-algebras and (weak, strong) implicative hyper $K$-algebras under a simple condition.

Keywords: hyper $K$-algebra, hyper $K$-ideal, (weak, strong) implicative hyper $K$-algebras, (weak) implicative hyper $K$-ideal

MSC 2000: 06F35, 03G25

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [5] in 1966 introduced the notion of a $BCK$-algebra. Recently [3], [6], [11] Borzooei, Jun and Zahedi et al. applied the hyperstructure to $BCK$-algebras and introduced the concept of the hyper $K$-algebra which is a generalization of the $BCK$-algebra. It is well-known [9] that the category of bounded commutative $BCK$-algebras is equivalent to the category of $MV$-algebras. In particular, any bounded commutative $BCK$-algebra is an $MV$-algebra and vice-versa. On the other hand, an $MV$-algebra is an algebraic structure of the Lukasiewicz many-valued logic. Hence any bounded commutative $BCK$-algebra is somehow related to a many-valued logic. Since the concept of the hyper $K$-algebra is a generalization of the notion of the $BCK$-algebra, it is natural to search for a logic whose algebraic structure is a hyper $K$-algebra. To this end, we first need a deeper understanding of hyper $K$-algebras. Now, in this note we define the notions of (weak, strong) implicative
hyper $K$-algebras, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

**Definition 2.1** ([3]). Let $H$ be a nonempty set and “$\circ$” a hyperoperation on $H$, that is, “$\circ$” is a function from $H \times H$ to $\mathcal{P}(H) = \mathcal{P}(H) \setminus \{\emptyset\}$. Then $H$ is called a hyper $K$-algebra if it contains a constant “0” and satisfies the following axioms:

1. **(HK1)** $(x \circ z) \circ (y \circ z) < x \circ y$,
2. **(HK2)** $(x \circ y) \circ z = (x \circ z) \circ y$,
3. **(HK3)** $x < x$,
4. **(HK4)** $x < y, y < x \Rightarrow x = y$,
5. **(HK5)** $0 < x$ for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A, \exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A, b \in B} a \circ b$ of $H$.

**Example 2.2** ([3]). Define the hyperoperation “$\circ$” on $H = [0, +\infty)$ as follows:

$$x \circ y = \begin{cases} [0, x] & \text{if } x \leq y, \\ (0, y] & \text{if } x > y \neq 0, \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper $K$-algebra.

**Theorem 2.3** ([3]). Let $(H, \circ, 0)$ be a hyper $K$-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets $A, B$ and $C$ of $H$ the following relations hold:

(i) $x \circ y < z \iff x \circ z < y$,
(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \circ y < x$,
(v) $A \subseteq B \Rightarrow A < B$,
(vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $(A \circ C) \circ (B \circ C) < A \circ B$,
(ix) $A \circ B < C \iff A \circ C < B$.

**Definition 2.4** ([3]). Let $I$ be a nonempty subset of a hyper $K$-algebra $(H, \circ, 0)$ and $0 \in I$. Then
(i) \( I \) is called a weak hyper \( K \)-ideal of \( H \) if \( x \circ y \subseteq I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in H \);

(ii) \( I \) is called a hyper \( K \)-ideal of \( H \) if \( x \circ y < I \) and \( y \in I \) imply that \( x \in I \) for all \( x, y \in H \).

**Theorem 2.5** ([3]). Any hyper \( K \)-ideal of a hyper \( K \)-algebra \( H \) is a weak hyper \( K \)-ideal.

**Definition 2.6** ([4]). Let \( I \) be a nonempty subset of \( H \). Then we say that \( I \) satisfies the additive condition, if for all \( x, y \in H \), \( x < y \) and \( y \in I \) imply that \( x \in I \).

**Definition 2.7** ([2]). Let \( H \) be a hyper \( K \)-algebra. An element \( a \in H \) is called a left (right) scalar if \( |a \circ x| = 1 \) (\( |x \circ a| = 1 \)) for all \( x \in H \). If \( a \in H \) is both a left and a right scalar, we say that \( a \) is a scalar element.

**Definition 2.8** ([2]). We say that a hyper \( K \)-algebra \( H \) satisfies the transitive condition if for all \( x, y, z \in H \), \( x < y \) and \( y < z \) imply that \( x < z \).

**Definition 2.9** ([2]). A hyper \( K \)-algebra \( H \) is called a positive implicative hyper \( K \)-algebra, if it satisfies \( (x \circ z) \circ (y \circ x) = (x \circ y) \circ z \) for all \( x, y, z \in H \).

**Definition 2.10** ([1]). We say that a hyper \( K \)-algebra \( H \) satisfies the strong transitive condition if for all \( A, B, C \subseteq H \), \( A < B \) and \( B < C \) imply that \( A < C \).

**Definition 2.11** ([1]). Let \( H \) be a hyper \( K \)-algebra, then a nonempty subset \( I \) of \( H \) is called

(a) a weak implicative hyper \( K \)-ideal if it satisfies

(i) \( 0 \in I \),

(ii) \( (x \circ z) \circ (y \circ x) \subseteq I \) and \( z \in I \) imply \( x \in I \) for all \( x, y, z \in H \),

(b) an implicative hyper \( K \)-ideal if it satisfies

(i) \( 0 \in I \),

(ii) \( (x \circ z) \circ (y \circ x) < I \) and \( z \in I \) imply \( x \in I \) for all \( x, y, z \in H \).

**Theorem 2.12** ([1]). Let \( I \) be a weak hyper \( K \)-ideal of \( H \). Then the following statements hold:

(i) If for all \( x, y, z \in H \), \( x \circ (y \circ x) \subseteq I \) implies \( x \in I \), then \( I \) is a weak implicative hyper \( K \)-ideal.

(ii) Let \( 0 \in H \) be a right scalar element and \( I \) a weak implicative hyper \( K \)-ideal. Then for all \( x, y \in H \), \( x \circ (y \circ x) \subseteq I \) implies that \( x \in I \).
Theorem 2.13 ([1]). Let $I$ be a hyper $K$-ideal of $H$. Then $I$ is an implicative hyper $K$-ideal if and only if

$$x \circ (y \circ x) < I \text{ implies that } x \in I \text{ for any } x, y \in H.$$ 

Definition 2.14 ([10]). Let $H = \{0, 1, 2\}$ be a hyper $K$-algebra of order 3. We say that $H$ satisfies the \textit{simple condition} if $1 \not< 2$ and $2 \not< 1$.

Definition 2.15 ([10]). Let $H = \{0, 1, 2\}$ be a hyper $K$-algebra of order 3. We say that $H$ satisfies the \textit{normal condition} if $1 < 2$ or $2 < 1$.

3. Implicative hyper $K$-algebra

From now on $H$ is a hyper $K$-algebra, unless stated otherwise.

Definition 3.1. $H$ is said to be
(i) weak implicative if $x < x \circ (y \circ x)$ for all $x, y \in H$,
(ii) implicative if $x \in x \circ (y \circ x)$ for all $x, y \in H$,
(iii) strong implicative if $x \circ 0 \subseteq x \circ (y \circ x)$ for all $x, y \in H$.

Example 3.2. Let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1, 2}</td>
<td>{0, 1, 2}</td>
<td>{0, 1, 2}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0}</td>
<td>{2}</td>
</tr>
<tr>
<td>3</td>
<td>{2, 3}</td>
<td>{1, 2}</td>
<td>{0, 2, 3}</td>
<td>{0, 1, 2}</td>
</tr>
</tbody>
</table>

It can be checked that $H$ is a weak implicative, implicative and strong implicative hyper $K$-algebra.

Theorem 3.3.
(i) Any implicative hyper $K$-algebra is a weak implicative hyper $K$-algebra.
(ii) Any strong implicative hyper $K$-algebra is an implicative hyper $K$-algebra.

Proof. The proof is trivial. \qed

The following example shows that the notions given in Definition 3.1 are not equivalent.

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Example 3.4. (i) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0, 1}$</td>
<td>${0, 1}$</td>
<td>${0, 1}$</td>
</tr>
<tr>
<td>1</td>
<td>${1, 2}$</td>
<td>${0, 2}$</td>
<td>${0, 2}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

We can see that $H$ is a weak implicative hyper $K$-algebra. However it is not an implicative hyper $K$-algebra, because $1 \not\in 1 \circ (2 \circ 1) = \{0, 2\}$.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0, 1}$</td>
<td>${0}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${1, 2}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

Now, $H$ is an implicative hyper $K$-algebra. However it is not a strong implicative one because $0 \circ 0 = \{0, 1\} \not\subseteq 0 \circ (1 \circ 0) = \{0\}$.

**Proposition 3.5.** Let $0 \in H$ be a right scalar element. Then the notions of implicative and strong implicative hyper $K$-algebras are equivalent.

**Proof.** The proof follows from the fact that $x \circ 0 = x$. □

**Proposition 3.6.** $H$ is a weak implicative hyper $K$-algebra if and only if $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$.

**Proof.** Let $x \circ 0 < x \circ (y \circ x)$ for all $x, y \in H$. Then we have $x \circ (x \circ (y \circ x)) < 0$. Thus there exists $t \in x \circ (x \circ (y \circ x))$ such that $t < 0$. Hence $t = 0$, therefore $x < x \circ (y \circ x)$. The proof of the converse is trivial. □

**Theorem 3.7.** Let $H$ be a hyper $K$-algebra of order 3 that satisfies the simple condition. Then $H$ is implicative if and only if it is weak implicative.

**Proof.** Let $H$ be a weak implicative hyper $K$-algebra. We show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. If $x = 0$, then $0 \in 0 \circ (y \circ 0)$ for all $y \in H$. Let $x \neq 0$ and $x \not\in x \circ (y \circ x)$. Since $x < x \circ (y \circ x)$, there exists $t \in x \circ (y \circ x)$ such that $x < t$. Clearly since $x \neq 0$, we must have $t \neq 0$ and $t \neq x$. Since $H$ satisfies the simple condition, $x < t$ is impossible. Thus $x \in x \circ (y \circ x)$ for all $x, y \in H$. For the converse see Theorem 3.3 (i). □
**Example 3.8.** This example shows that in the above theorem, the simple condition cannot be omitted. Indeed let \( H = \{0, 1, 2\} \). Then the following table shows a hyper \( K \)-algebra structure on \( H \):

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0, 1}</td>
<td>{0, 2}</td>
</tr>
</tbody>
</table>

Then \( H \) is weak implicative while it is not implicative, since \( 2 \not\in 2 \circ (1 \circ 2) \).

**Example 3.9.** (i) It is not necessary that a (weak, strong) implicative hyper \( K \)-algebra be a positive implicative hyper \( K \)-algebra. Example 3.2 shows a hyper \( K \)-algebra which is strong implicative while it is not a positive implicative hyper \( K \)-algebra. Indeed \( (3 \circ 2) \circ (1 \circ 2) = \{0, 1, 2, 3\} \neq (3 \circ 1) \circ 2 = \{0, 1, 2\} \).

(ii) In general it is not needed that a positive implicative hyper \( K \)-algebra be a (weak, strong) implicative hyper \( K \)-algebra. Because let \( H = \{0, 1, 2\} \). Then the following table shows a positive implicative hyper \( K \)-algebra structure on \( H \):

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{0}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0, 2}</td>
</tr>
</tbody>
</table>

but \( H \) is not a (weak, strong) implicative hyper \( K \)-algebra. Indeed \( 1 \not< 1 \circ (2 \circ 1) \).

**Theorem 3.10.** Let \( H \) be a positive implicative hyper \( K \)-algebra of order 3 that satisfies the simple condition. Then \( H \) is a (weak, strong) implicative hyper \( K \)-algebra.

**Proof.** Since \( H \) satisfies the simple condition, we know that \( 1 \circ 0 = \{1\} \), \( 2 \circ 0 = \{2\} \), \( 1 \circ 2 \neq \{2\} \) and \( 2 \circ 1 \neq \{1\} \) by Theorem 3.17 of [10]. Now we show that \( H \) is a strong implicative hyper \( K \)-algebra, that is \( x \circ y \subseteq x \circ (y \circ x) \) for all \( x, y \in H \).

To do this we consider three different cases:

(i) If \( x = 0 \), then we must show that \( 0 \circ 0 \subseteq 0 \circ (y \circ 0) \) for all \( y \in H \). If \( y = 0 \), then we are done. We know that \( 0 \in 0 \circ 0 \), so \( 0 \circ 0 = \{0\}, \{0, 1\}, \{0, 2\} \) or \( \{0, 1, 2\} \). If \( 0 \circ 0 = \{0\} \), then clearly \( 0 \in 0 \circ 1 \) and \( 0 \in 0 \circ 2 \), and so we are done. Now let \( 0 \circ 0 = \{0, 1\} \). If \( y = 1 \), then we must show that \( 0 \circ 0 \subseteq 0 \circ (1 \circ 0) = 0 \circ 1 \). We have \( (0 \circ 0) \circ 0 = \{0, 1\} \circ 0 = (0 \circ 0) \cup (1 \circ 0) \cup \{1\} = \{0, 1\} \cup \{1\} = \{0, 1\} \). On the other hand, since \( H \) is positive implicative then \( \{0, 1\} = (0 \circ 0) \circ 0 = (0 \circ 0) \circ (0 \circ 0) = \{0, 1\} \circ \{0, 1\} = \{0, 1\} \).
Thus we conclude that \((0 \circ 1)\) and \((1 \circ 1) \subseteq \{0, 1\}\). If \(1 \not\in (0 \circ 1)\), we get that \(0 \circ 1 = \emptyset\). So \((0 \circ 1) = 0 \circ 1 = \emptyset\) and on the other hand, since \(H\) is positive implicative we have \(\{0\} = (0 \circ 1) \circ 1 = (0 \circ 1) \circ (0 \circ 1) \supseteq 0 \circ 0 = \{0, 1\}\), which is a contradiction. Thus \(0 \circ 1 = \{0, 1\}\), and hence \(0 \circ 0 = 0 \circ 1\). Now let \(y = 2\). Since \(0 \in 0 \circ 2\) then \(0 \circ 2 = \emptyset\), \(\{0, 1\}\), \(\{0, 2\}\) or \(\{0, 1, 2\}\). If \(0 \circ 2 = \{0\}\), then \((0 \circ 2) \circ 2 = 0 \circ 2 = \emptyset\) and on the other hand, since \(H\) is positive implicative we have \(\{0\} = (0 \circ 2) \circ 2 = (0 \circ 2) \circ (2 \circ 2) \supseteq 0 \circ 0 = \{0, 1\}\), which is a contradiction. Hence \(0 \circ 2 \neq \{0\}\). Let \(0 \circ 2 = \{0, 2\}\). Since \(1 \not\in 0 \circ 2\), then \(0 \not\in 1 \circ 2\). So \(1 \circ 2 = \{1\}\) or \(\{1, 2\}\).

If \(1 \circ 2 = \{1\}\), then \(0 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 1 \circ 1 \subseteq \{0, 1\}\), which is a contradiction. Hence \(1 \circ 2 = \{1, 2\}\). Now we have \(0 \in 2 \circ 2 \subseteq (1 \circ 2) \circ (0 \circ 2) = (2 \circ 0) = 1 \circ 2 = \{1, 2\}\), which is a contradiction. Hence \(0 \circ 2 = \{0, 1\}\) or \(\{0, 1, 2\}\). Thus in the case \(0 \circ 0 = \{0, 1\}\), we conclude that \(0 \circ 0 \subseteq 0 \circ 2\). The proof for the case \(0 \circ 0 = \{0, 2\}\) is similar as above. If \(0 \circ 0 = \{0, 1, 2\}\), then since \(H\) is a positive implicative we have \(\{1\} = (1 \circ 0) = (1 \circ 0) \circ 0 = (1 \circ 0) \circ (0 \circ 0) = 1 \circ 0 \{0, 1, 2\} = (1 \circ 0) \cup (1 \circ 1) \cup (1 \circ 2)\), thus we must have \(1 \circ 1 = \{1\}\) and this is a contradiction with (HK3). Hence \(0 \circ 0 \neq \{0, 1, 2\}\). Thus if \(x = 0\), then \(0 \circ 0 = 0 \circ (y \circ 0)\) for all \(y \in H\).

(ii) If \(x = 1\), then we must show that \(1 \in 1 \circ (y \circ 1)\) for all \(y \in H\). If \(y = 0\) or \(1\) it is trivial, so let \(y = 2\). Since \(2 \not\in 1\), then \(0 \not\in 2 \circ 1\) and \(2 \circ 1 \neq \{1\}\). Thus we conclude that \(2 \circ 1 \neq \{2\}\) or \(\{1, 2\}\). Since \(1 \not\in 2\), then \(0 \not\in 1 \circ 2\) and \(1 \circ 2 \neq \{2\}\). Therefore \(1 \circ 2 \neq \{1\}\) or \(\{1, 2\}\). Hence in all cases by some manipulations we can get that \(1 \in 1 \circ (2 \circ 1)\).

(iii) If \(x = 2\), then by the same argument as in (ii) we can show that \(2 \in 2 \circ (y \circ 2)\) for all \(y \in H\).

Remark 3.11. The following example shows that in the above theorem the simple condition can not be omitted. Let \(H = \{0, 1, 2\}\). Then the following table shows a positive implicative hyper \(K\)-algebra structure on \(H\) where \(H\) does not satisfy the simple condition:

<table>
<thead>
<tr>
<th>(\circ)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0}</td>
<td>{0, 2}</td>
</tr>
</tbody>
</table>

and \(H\) is not an implicative hyper \(K\)-algebra, either, because \(2 \not\in 2 \circ (1 \circ 2) = \{0\}\).

Theorem 3.12. Let \(H\) be a weak implicative hyper \(K\)-algebra. Then each hyper \(K\)-ideal of \(H\) is a weak implicative hyper \(K\)-ideal.

Proof. Let \(I\) be a hyper \(K\)-ideal and \((x \circ z) \circ (y \circ x) \subseteq I\), \(z \in I\). Then for all \(t \in x \circ (y \circ x)\) we have \(t \circ z \subseteq (x \circ (y \circ x)) \circ z = (x \circ z) \circ (y \circ x) \subseteq I\) and \(z \in I\). Thus
$t \in I$ and hence $x \circ (y \circ x) \subseteq I$. Since $H$ is weak implicative, then $x < x \circ (y \circ x) \subseteq I$. So there exists $r \in I$ such that $x < r$. Thus $0 \in x \circ r$, hence $x \circ r \subseteq I$ and $r \in I$ which implies that $x \in I$. \hfill $\square$

**Remark 3.13.** (i) The following example shows that in the above theorem we can not use “weak hyper $K$-ideal” instead of “hyper $K$-ideal”. Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1, 2}$</td>
<td>${2}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1}$</td>
</tr>
</tbody>
</table>

Now $I = \{0, 1\}$ is a weak hyper $K$-ideal and $(2 \circ 0) \circ (1 \circ 2) = \{0, 1\} \subseteq I$ and $0 \in I$, but $2 \not\in I$. Hence $I$ is not a weak implicative hyper $K$-ideal.

(ii) The following example shows that in the above theorem, if we use “weak hyper $K$-ideal” instead of “hyper $K$-ideal”, we can not conclude that “any weak hyper $K$-ideal is implicative”. Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0, 1}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${1, 2}$</td>
</tr>
<tr>
<td>2</td>
<td>${1, 2}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

Then $I = \{0\}$ is a weak hyper $K$-ideal and $1 \circ (0 \circ 1) = \{0, 1, 2\} < I$, but $1 \not\in I$. Hence $I$ is not an implicative hyper $K$-ideal.

(iii) The following example shows that the conditions of the above theorem do not imply that any hyper $K$-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows a weak implicative hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
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</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0}$</td>
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</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${0, 1}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

We see that $I = \{0\}$ is a hyper $K$-ideal and $2 \circ (2 \circ 2) = \{0, 1, 2\} < I$, but $2 \not\in I$. Hence $I$ is not an implicative hyper $K$-ideal.
Theorem 3.14. Let $H$ be an implicative hyper $K$-algebra. Then each weak hyper $K$-ideal of $H$ is a weak implicative hyper $K$-ideal.

Proof. Let $I$ be a weak hyper $K$ ideal and $(x \circ z) \circ (y \circ x) \subseteq I$, $z \in I$. Then $(x \circ (y \circ x)) \circ z \subseteq I$. Since $H$ is implicative, we have $x \in (x \circ (y \circ x))$. Therefore $x \circ z \subseteq (x \circ (y \circ x)) \circ z \subseteq I$ and since $z \in I$, we conclude that $x \in I$. □

Corollary 3.15. Let $H$ be an implicative hyper $K$-algebra. Then each hyper $K$ ideal of $H$ is a weak implicative hyper $K$-ideal.

Remark 3.16. The following example shows that the conditions of the above corollary do not imply that any hyper $K$-ideal is implicative. Let $H = \{0, 1, 2\}$. Then the following table shows an implicative hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
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<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0, 2}</td>
</tr>
</tbody>
</table>

Now $I = \{0\}$ is a hyper $K$-ideal while it is not an implicative hyper $K$-ideal, since $(2 \circ 0) \circ (2 \circ 2) = \{0, 2\} < I$ and $0 \in I$, but $2 \notin I$.

Note that the following theorem says that if we restrict ourselves to the hyper $K$-algebras of order 3, then the above corollary holds even if $H$ is not implicative.

Theorem 3.17. If $H$ is a hyper $K$-algebra of order 3, then each nonzero hyper $K$-ideal is a weak implicative hyper $K$-ideal.

Proof. Let $H = \{0, 1, 2\}$. Without loss of generality let $I = \{0, 1\}$ be a hyper $K$-ideal of $H$. By Theorem 2.11 it is enough to show that $x \circ (y \circ x) \subseteq I$ implies that $x \in I$. If $x = 0, 1$ then we are done. Now let $x = 2$, then $2 \circ (y \circ 2) \subseteq I$ for all $y \in H$ and we will get a contradiction. To obtain it, consider three different cases:

(i) Let $y = 0$. Then $2 \in 2 \circ (0 \circ 2) \subseteq I$, and this is a contradiction.

(ii) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$. Therefore $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, and this is a contradiction. If $1 \notin 2$, then $0 \notin 1 \circ 2$, so we must have $1 \circ 2 = \{1\}, \{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then $0 \in 0 \circ 2 \subseteq (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1$. Hence $2 \in I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{1, 2\}$, consider $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$, therefore $2 \circ 1 \subseteq I$ and $1 \in I$ imply that $2 \in I$, which is a contradiction.

(iii) If $y = 2$ then $2 \in 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. □
Remark 3.18. (i) The converse of the above theorem is not correct. Indeed let $H = \{0, 1, 2, 3\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${2}$</td>
<td>${0, 2}$</td>
<td>${2}$</td>
</tr>
<tr>
<td>3</td>
<td>${2, 3}$</td>
<td>${1, 2, 3}$</td>
<td>${0, 1, 3}$</td>
<td>${0, 1, 2, 3}$</td>
</tr>
</tbody>
</table>

Then $I = \{0, 1\}$ is a weak implicative hyper $K$-ideal, which is not a hyper $K$-ideal, since $3 \circ 1 = \{1, 2, 3\} < I$ and $1 \in I$, but $3 \not\in I$.

(ii) The following example shows that the condition “nonzero hyper $K$-ideal” in the above theorem can not be omitted. Let $H = \{0, 1\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${1}$</td>
<td>${0, 1}$</td>
</tr>
</tbody>
</table>

Now it is easy to see that $I = \{0\}$ is a hyper $K$-ideal while it is not a weak implicative one since $(1 \circ 0) \circ (2 \circ 1) \subseteq I$ and $0 \in I$, but $1 \not\in I$.

Lemma 3.19. Let $H$ be a positive implicative hyper $K$-algebra of order 3 that satisfies the normal condition. Then the following statements hold:

(i) $1 \circ 0 = \{1\}$,
(ii) $2 \circ 0 = \{2\}$.

Proof. (i) We know that $1 \in 1 \circ 0$ and $0 \not\in 1 \circ 0$, thus $1 \circ 0 = \{1\}$ or $\{1, 2\}$. Let $1 \circ 0 = \{1, 2\}$. Since $H$ satisfies the normal condition, then $1 < 2$ or $2 < 1$. Now we consider the following two cases.

Case 1: Let $1 < 2$. Then $0 \not\in 2 \circ 1$. Since $0 \in 2 \circ 2 \subseteq (2 \circ 0) \circ \{1, 2\} = (2 \circ 0) \circ (1 \circ 0) = (2 \circ 1) \circ 0$, thus $2 \circ 1 < 0$. So there is $x \in 2 \circ 1$ such that $x < 0$, therefore $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction.

Case 2: Let $2 < 1$. Then $0 \not\in 1 \circ 2$. Since $0 \in 2 \circ 2 \subseteq \{1, 2\} \circ (2 \circ 0) = (1 \circ 0) \circ (2 \circ 0) = (1 \circ 2) \circ 0$, thus there is $x \in 1 \circ 2$ such that $x < 0$, so $x = 0$. Hence $0 \in 2 \circ 1$, which is a contradiction. Thus we must have $1 \circ 0 = \{1\}$.

(ii) The proof is similar to the proof of (i).
**Theorem 3.20.** Let $H$ be a hyper $K$-algebra of order 3 and $I \subset H$. Then

(i) If $H$ satisfies the simple condition, then $I$ is a weak implicative hyper $K$-ideal if and only if $I$ is a weak hyper $K$-ideal;

(ii) if $H$ is positive implicative and satisfies the normal condition then $I \neq \{0\}$ is a weak implicative hyper $K$-ideal if and only if $I$ is a weak hyper $K$-ideal.

**Proof.** (i) Let $I = \{0\}$ be a weak hyper $K$-ideal and $(x \circ z) \circ (y \circ x) \subseteq I$ and $z \in I$. Then $x \circ (y \circ x) \subseteq (x \circ 0) \circ (y \circ x) = \{0\}$. We must show that $x = 0$. On the contrary, let $x = 1$. Then $1 \circ (y \circ 1) = \{0\}$. If $y = 0$ or 1, we get the contradiction $1 \in \{0\}$. If $y = 2$, since $H$ satisfies the simple condition, then $1 \circ (2 \circ 1) \neq \{0\}$, which is a contradiction, hence $x = 1$ is impossible. By a similar argument we show that $x = 2$ is also impossible. Thereby $x = 0 \in I$. Note that since $I = \{0\}$ is always a weak hyper $K$-ideal the converse is trivial. For the case $I \neq \{0\}$ see Theorem 4.11 of [1].

(ii) Without loss of generality let $I = \{0, 1\}$ be a weak hyper $K$-ideal. By Theorem 2.11 (i), it is enough to show that if $x \circ (y \circ x) \subseteq I$ for $x, y \in H$, then $x \in I$. If $x = 0$ or 1 we are done. Now let $x = 2$. So $2 \circ (y \circ 2) \subseteq I$ for arbitrary $y \in H$. We consider three cases for $y$ and show that none of these cases holds.

(a) Let $y = 0$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (0 \circ 2) \subseteq I$, which is a contradiction.

(b) Let $y = 1$. If $1 < 2$, then $0 \in 1 \circ 2$, hence $2 \in 2 \circ 0 \subseteq 2 \circ (1 \circ 2) \subseteq I$, which is a contradiction. If $1 \neq 2$, then $1 \circ 2 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $1 \circ 2 = \{1\}$, then $2 \circ 1 = 2 \circ (1 \circ 2) \subseteq I$. Since $1 \in I$, we get that $2 \in I$, which is a contradiction. If $1 \circ 2 = \{2\}$, then we have $2 \circ 2 = 2 \circ (1 \circ 2) \subseteq I = \{0, 1\}$. Since $H$ is positive implicative we have $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) = (1 \circ 2) \circ (2 \circ 2) = (1 \circ 2) \circ 2 = 2 \circ 2 \subseteq I = \{0, 1\}$ which is a contradiction. If $1 \circ 2 = \{1, 2\}$, then $(2 \circ 1) \cup (2 \circ 2) = 2 \circ \{1, 2\} = 2 \circ (1 \circ 2) \subseteq I$. Hence $2 \circ 1 \subseteq I$. Since $I$ is a weak hyper $K$-ideal and $1 \in I$, so $2 \in I$, which is a contradiction.

(c) Let $y = 2$. Then $2 \in 2 \circ 0 \subseteq 2 \circ (2 \circ 2) \subseteq I$, which is a contradiction. Hence $x = 2$ is impossible and therefore $I$ is a weak implicative hyper $K$-ideal.

Conversely, let $I = \{0, 1\}$ be a weak implicative hyper $K$-ideal and $x \circ y \subseteq I$ and $y \in I$. We must show that $x \in I$. If $x = 0$ or 1 then $x \in I$ and we are done. Now we show that $x = 2$ is impossible. If $x = 2$ then we have $2 \circ y \subseteq I$ and $y \in I$. We consider three different cases for $y$:

$(a')$ If $y = 0$, then $2 \in 2 \circ 0 \subseteq I$, which is a contradiction.

$(b')$ The case $y = 2$ never occurs since we must have $y \in I$.

$(c')$ If $y = 1$, since $2 \circ 1 \subseteq I$ we conclude that $2 \circ 1 = \{0\}$, $\{1\}$ or $\{0, 1\}$. Now consider the following cases:
(c'1) If $2 \circ 1 = \{0\}$, then $0 \not\in 1 \circ 2$, therefore $1 \circ 2 = \{1\}, \{2\}$ or $\{1, 2\}$. Thus we have to consider the following three subcases:

(c'1.1) If $1 \circ 2 = \{1\}$, since by Lemma 3.19, $2 \circ 0 = \{2\}$, hence $(2 \circ 0) \circ (1 \circ 2) = 2 \circ 1 = \{0\} \subseteq I$. Since $I$ is a weak implicative hyper $K$-ideal and $0 \in I$, we have $2 \in I$, which is a contradiction.

(c'1.2) If $1 \circ 2 = \{2\}$, since $H$ is positive implicative we have $\{0\} = 2 \circ 1 = (1 \circ 2) \circ 1 = (1 \circ 1) \circ 2 = (1 \circ 2) \circ (1 \circ 2) = 2 \circ 2$. Since by Lemma 3.19, $2 \circ 0 = \{2\}$, $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ 2 = \{0\} \subseteq I$ and $0 \in I$, we conclude that $2 \in I$, which is a contradiction.

(c'1.3) If $1 \circ 2 = \{1, 2\}$, since $1 \circ 0 = \{1\}$ and $2 \circ 0 = \{2\}$ hence $(1 \circ 0) \circ 2 = 1 \circ 2 = \{1, 2\}$ and $(1 \circ 0) \circ 2 = (1 \circ 2) \circ (0 \circ 2) = \{1, 2\} \circ (0 \circ 2)$. If $1$ or $2$ belongs to $0 \circ 2$, then $0 \in (1 \circ 0) \circ 2 = \{1, 2\}$ which is a contradiction, thus $0 \circ 2 = \{0\}$. We know that $0 \in 2 \circ 2$; if $2 \circ 2 = \{0\}$ then $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ \{1, 2\} = (2 \circ 1) \cup (2 \circ 2) = \{0\} \subseteq I$. Since $0 \in I$ and $I$ is weak implicative, we get that $2 \in I$, which is a contradiction. Thus $2 \circ 2 \neq \{0\}$. Since $H$ is positive implicative we have $\{0\} = 0 \circ 2 = (2 \circ 1) \circ 2 = (2 \circ 2) \circ (1 \circ 2) = (2 \circ 2) \circ \{1, 2\} \neq \{0\}$, hence this case is impossible.

(c'2) If $2 \circ 1 = \{1\}$, we know that $1 \circ 0 = \{1\}$. Then $\{1\} = 1 \circ 0 = (2 \circ 1) \circ 0 = (2 \circ 0) \circ 1 = (2 \circ 1) \circ (0 \circ 1) = 1 \circ (0 \circ 1)$, therefore $0 \circ 1 = \{0\}$. We have $0 \in 0 \circ 2$. If $0 \circ 2 = \{0\}$, then by considering $(2 \circ 1) \circ (0 \circ 2) = 1 \circ 0 = \{1\} \subseteq I$, $1 \in I$ and $I$ is weak implicative, we conclude that $2 \in I$, which is a contradiction. Hence $0 \circ 2 \neq \{0\}$. Consider $(0 \circ 2) \circ 1 = (0 \circ 1) \circ (2 \circ 1) = 0 \circ \{1\} = \{0\}$. But if $0 \circ 2 \neq \{0\}$ then $(0 \circ 2) \circ 1 \neq \{0\}$, hence this case is impossible.

(c'3) If $2 \circ 1 = \{0, 1\}$, since $0 \in 2 \circ 1$, hence $0 \not\in 1 \circ 2$. Therefore $1 \circ 2 = \{1\}, \{2\}$ or $\{1, 2\}$. Now we discuss the following three different subcases.

(c'3.1) If $1 \circ 2 = \{1\}$, since $(2 \circ 0) \circ (1 \circ 2) = 2 \circ 1 \subseteq I$, $0 \in I$ and $I$ is weak implicative, we get that $2 \in I$, which is impossible.

(c'3.2) Suppose $1 \circ 2 = \{2\}$. Consider $(2 \circ 0) \circ (1 \circ 2) = 2 \circ (1 \circ 2) = 2 \circ 2 = (1 \circ 2) \circ (1 \circ 2) = (1 \circ 1) \circ 2 = (1 \circ 2) \circ 1 = 2 \circ 1 = \{0, 1\} \subseteq I$. Since $0 \in I$ and $I$ is weak implicative, we get that $2 \in I$, which is a contradiction.

(c'3.3) If $1 \circ 2 = \{1, 2\}$, then since $H$ is positive implicative we have $\{1, 2\} = 1 \circ 2 = (1 \circ 0) \circ 2 = (1 \circ 2) \circ (0 \circ 2) = \{1, 2\} \circ (0 \circ 2)$. If $1$ or $2 \in 0 \circ 2$, then $0 \in (1 \circ 0) \circ 2 = \{1, 2\}$ which is a contradiction, hence $0 \circ 2 = \{0\}$. Consider $\{1\} = 1 \circ 0 = (1 \circ 0) \circ 0 = (1 \circ 0) \circ (0 \circ 0) = 1 \circ (0 \circ 0)$. Then we conclude that $0 \circ 0 = \{0\}$. Then $(2 \circ 1) \circ (0 \circ 2) = \{0, 1\} \circ 0 = (0 \circ 0) \cup (1 \circ 0) = \{0, 1\} \subseteq I$. Since $1 \in I$ we get that $2 \in I$, which is a contradiction.

Thus the above arguments show that $x = 2$ is impossible, hence $I$ is a weak hyper $K$-ideal of $H$. 

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Remark 3.21. (i) In part (ii) of the above theorem the condition “positive implicative” cannot be omitted. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$ which satisfies the normal condition:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1, 2}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>2</td>
<td>${1, 2}$</td>
<td>${0, 1}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

We can check that $I = \{0, 1\}$ is a weak implicative hyper $K$-ideal, but it is not a weak hyper $K$-ideal, since $2 \circ 1 \subseteq I$ and $1 \in I$ but $2 \not\in I$. Note that $H$ is not a positive implicative hyper $K$-algebra, since $\{1, 2\} = (1 \circ 2) \circ 0 \neq (1 \circ 0) \circ (2 \circ 0) = \{0, 1, 2\}$.

(ii) In part (ii) of above theorem the condition “$I \neq \{0\}$” cannot be omitted, since hyper $K$-algebra $H$ of Example 3.9(ii) is positive implicative and normal and $I = \{0\}$ is weak hyper $K$-ideal but is not weak implicative hyper $K$-ideal, since $2 \circ (1 \circ 2) = \{0\} \subseteq I$ but $2 \not\in I$.

Theorem 3.22. Let $H$ be an implicative hyper $K$-algebra that satisfies the strong transitive condition and let $I$ be a hyper $K$-ideal of $H$. Then $I$ is an implicative hyper $K$-ideal.

Proof. Let $x \circ (y \circ x) < I$. Since $H$ is implicative, then $x \in x \circ (y \circ x)$. Hence $x < I$ and $I$ is a hyper $K$-ideal, so $x \in I$. Thus by Theorem 2.13, $I$ is an implicative hyper $K$-ideal. \qed

Note that the example given in Remark 3.16 shows that the “strong transitive condition” is necessary in the above proposition.

Theorem 3.23. Let $H$ be a hyper $K$-algebra of order 3 and $0 \in H$ a right scalar element. If $I = \{0\}$ is an implicative hyper $K$-ideal, then $H$ is a strong implicative hyper $K$-algebra.

Proof. Since $0$ is a right scalar element, it is enough to show that $x \in x \circ (y \circ x)$ for all $x, y \in H$. To do this consider the following cases:

(i) If $x = 0$, then it is clear that $0 \in 0 \circ (y \circ 0)$ for all $y \in H$.

(ii) If $x = 1$, we consider three cases: (a) if $y = 0$, then $1 \in 1 \circ 0 \subseteq 1 \circ (0 \circ 1)$. (b) if $y = 1$, then $1 \in 1 \circ 0 \subseteq 1 \circ (1 \circ 1)$. (c) Let $y = 2$, consider two cases $2 < 1$ and $2 \not\in 1$. If $2 < 1$, then $0 \in 2 \circ 1$. Therefore $1 \in 1 \circ 0 \subseteq 1 \circ (2 \circ 1)$. If $2 \not\in 1$, then $0 \not\in 2 \circ 1$. Thus $2 \circ 1 = \{1\}$, $\{2\}$ or $\{1, 2\}$. If $2 \circ 1 = \{1\}$, then $0 \in 1 \circ (2 \circ 1)$ and therefore $1 \circ (2 \circ 1) < I$. Since $I$ is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \circ 1 = \{2\}$, we show that $1 \circ 2 = \{1\}$.
To do this, we show that $0 \not\in 1 \circ 2$ and $2 \not\in 1 \circ 2$. If $0 \in 1 \circ 2$, then we have $0 \in 1 \circ 2 = 1 \circ (2 \circ 1)$, so $1 \circ (2 \circ 1) < I$. Since $I$ is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction. If $2 \in 1 \circ 2$, then $0 \in 2 \circ (1 \circ 2)$, therefore $2 \circ (1 \circ 2) < I$. Since $I$ is implicative, by Theorem 2.12 we conclude that $2 \in I$, which is a contradiction. Therefore $1 \circ 2 = \{1\}$, so $1 \in 1 \circ (2 \circ 1)$. Now, let $2 \circ 1 = \{1, 2\}$. Hence $0 \in (1 \circ 1) \cup (1 \circ 2) = 1 \circ \{1, 2\} = 1 \circ (2 \circ 1)$, thus $1 \circ (2 \circ 1) < I$. Since $I$ is implicative, by Theorem 2.13 we conclude that $1 \in I$, which is a contradiction.

(iii) If $x = 2$ then by the same argument as in the case (ii) we can obtain that $2 \in 2 \circ (y \circ 2)$ for all $y \in H$.  

\begin{proof}

Remark 3.24. If in the above theorem we replace $I = \{0\}$ by $I = \{0, 1\}$, then the theorem does not hold. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper $K$-algebra structure on $H$:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${2}$</td>
<td>${0, 1}$</td>
</tr>
</tbody>
</table>

Here $0 \in H$ is a scalar element and $I = \{0, 1\}$ is an implicative hyper $K$-ideal, but $H$ is not an implicative hyper $K$-algebra since $1 \not\in 1 \circ (2 \circ 1)$.

References


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