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ON FINITELY GENERATED MULTIPLICATION MODULES

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Abstract. We shall prove that if $M$ is a finitely generated multiplication module and $\text{Ann}(M)$ is a finitely generated ideal of $R$, then there exists a distributive lattice $\overline{M}$ such that $\text{Spec}(\overline{M})$ with Zariski topology is homeomorphic to $\text{Spec}(\overline{M})$ to Stone topology. Finally we shall give a characterization of finitely generated multiplication $R$-modules $M$ such that $\text{Ann}(M)$ is a finitely generated ideal of $R$.

Keywords: prime submodules, multiplication modules, distributive lattices, spectral spaces

MSC 2000: 13C13, 13C99

1. Introduction

Throughout this note all rings are commutative with identity and all modules are unital.

For any submodule $N$ of an $R$-module $M$, we define $(N : M) = \{ r \in R : rM \subseteq N \}$ and denote $(0 : M)$ by $\text{Ann}(M)$. A submodule $P$ of $M$ is called prime if $P \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $r \in (P : M)$. It is easy to show that if $P$ is a prime submodule of an $R$-module $M$, then $(P : M)$ is a prime ideal of $R$. The set of all prime submodules of $M$ is denoted by $\text{Spec}(M)$. As defined in [4] the radical of a submodule $N$ of an $R$-module $M$ is given by $\text{rad}(N) = \bigcap P$, where the intersection is over all prime submodules of $M$ containing $N$. If there is no prime submodule containing $N$, then we define $\text{rad}(N) = M$. The radical of an ideal $I$ of $R$ is denoted by $\sqrt{I}$.

An $R$-module $M$ is called a multiplication module provided for any submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to check that $M$ is a multiplication module if and only if $N = (N : M)M$ for every submodule $N$ of $M$ (see [8]).
In this paper at first we shall construct a distributive lattice \( \overline{M} \) and discuss some properties of \( \text{Spec}(\overline{M}) \), where \( \text{Spec}(\overline{M}) \) is the set of all prime ideals in the lattice \( \overline{M} \). We shall then prove that if \( M \) is a finitely generated multiplication module and \( \text{Ann}(M) \) is a finitely generated ideal of \( R \), then \( \text{Spec}(M) \) and \( \text{Spec}(\overline{M}) \) are homeomorphic. Finally we shall generalize the notion of reticulated and semi-reticulated rings for modules and characterize some classes of semi-reticulated modules.

2. On the lattice \( \overline{M} \) and its prime spectrum

Let \( R \) be a ring and let \( \text{FI}(R) \) be the set of all finitely generated ideals of \( R \). Now let \( M \) be an \( R \)-module and \( \text{su}(M) \) the \( \text{FI}(R) \)-semimodule generated by the principal \( R \)-submodules of \( M \) and \( M \) under the operations \( N + K, IN \), where \( N, K \in \text{su}(M) \) and \( I \in \text{FI}(R) \). Hence

\[
\text{su}(M) = \left\{ \sum_{i=1}^{k} I_i R m_i + J_i M : I_i, J_i \in \text{FI}(R), \ m_i \in M, \ k \in \mathbb{N} \right\}.
\]

It is clear that if \( M \) is a finitely generated \( R \)-module then \( \text{su}(M) \) is the set of all finitely generated submodules of \( M \).

Define the equivalence relation on \( \text{su}(M) \), \( \sim \) by \( N \sim L \) if and only if \( \text{rad}(N) = \text{rad}(L) \) [6, p. 1470], and denote the resulting set of equivalence classes by \( \overline{M} \); i.e.,

\[
\overline{M} = \{ [K] : K \in \text{su}(M) \}.
\]

**Lemma 2.1.** Let \( N, N', K, K' \in \text{su}(M) \) and \( I, I' \in \text{FI}(R) \). If \( N \sim N' \) and \( K \sim K' \), then we have

(i) \( (N + K) \sim (N' + K') \);

(ii) if \( \sqrt{I} = \sqrt{I'} \) then \( IN \sim I'N' \).

**Proof.** (i) By [6, Lemma 1.5].

(ii) Let \( P \in \text{Spec}(M) \) and \( IN \subseteq P \). Hence \( N \subseteq P \) or \( I \subseteq (P : M) \). If \( N \subseteq P \) then \( I'N' \subseteq N' \subseteq \text{rad}(N') = \text{rad}(N) \subseteq P \). Suppose that \( I \subseteq (P : M) \in \text{Spec}(R) \). Hence \( I' \subseteq \sqrt{I'} = \sqrt{I} \subseteq (P : M) \). Thus \( I'N' \subseteq I'M \subseteq P \). Therefore \( \text{rad}(I'N') \subseteq \text{rad}(IN) \). Similarly \( \text{rad}(IN) \subseteq \text{rad}(I'N') \) and hence \( IN \sim I'N' \). \( \square \)

Let \( [N], [K] \) belong to \( \overline{M} \) and \( I \in \text{FI}(R) \).

We define \( [N] + [K] := [N + K] \) and \( I[N] := [IN] \). Then by Lemma 2.1, \( \overline{M} \) becomes an \( \text{FI}(R) \)-semimodule. Furthermore we define \( [N] \leq [K] \) if for each \( P \in \text{Spec}(M) \), \( K \subseteq P \) implies that \( N \subseteq P \). Therefore \( (\overline{M}, \leq) \) is a partially ordered set.

Let \( N \) be a subset of \( M \).
We define $\overline{M}(N) = \{[L] \in \overline{M} : L \sim K, \text{ for some } K \subseteq N\}$. If $0 \in N$ then $[0] \in \overline{M}(N)$ and hence $\overline{M}(N) \neq \emptyset$.

Now let $N$ be a subset of $\overline{M}$. We define $M[N] = \{x \in M : [Rx] \in N\}$. If $[0] \in N$ then $0 \in M[N]$ and hence $M[N] \neq \emptyset$.

**Lemma 2.2.** Let $P \in \text{Spec}(M)$. Then $M[\overline{M}(P)] = P$.

**Proof.** Let $x \in P$. Then $Rx \subseteq P$ and hence $[Rx] \in \overline{M}(P)$. Therefore $x \in M[\overline{M}(P)]$. Now let $x \in M[\overline{M}(P)]$. Then $[Rx] \in \overline{M}(P)$ and so $Rx \sim L$ for some $L \subseteq P$. Thus $Rx \subseteq \text{rad}(Rx) = \text{rad}(L) \subseteq P$. Hence $x \in P$.

For the remainder of this section we let $M$ be a finitely generated multiplication $R$-module and $\text{Ann}(M)$ a finitely generated ideal of $R$.

**Proposition 2.3.** Suppose that $M$ is an $R$-module. Then $(\overline{M}, \leq) \text{ is a distributive lattice.}$

**Proof.** Put $0 := [0_M]$ and $1 := [M]$.

Define for any $N, K \in \text{su}(M)$: $[N] \lor [K] := [N + K]$ and $[N] \land [K] = [(N : M)K]$. Since $M, N$ and $\text{Ann}(M)$ are finitely generated, by [8, Proposition 13], $(N : M)$ is finitely generated. Therefore $(N : M) \in \text{FI}(R)$ and so $(N : M)K \in \text{su}(M)$. Since $M$ is a multiplication module, the infimum of $[N]$ and $[K]$ is well-defined.

We now show that $\overline{M}$ is a distributive lattice. It is enough to show that


Let $P \in \text{Spec}(M)$ be such that $(N : M)K + L \subseteq P$. Then $(N : M)K \subseteq P$ and $L \subseteq P$. Hence $K \subseteq P$ or $(N : M) \subseteq (P : M)$. If $K \subseteq P$ then $(N + L : M)K \subseteq P$ and since $(N + L : M)L \subseteq P$, we get $(N + L : M)(K + L) \subseteq P$. If $(N : M) \subseteq (P : M)$, then since $M$ is a multiplication module, $N = (N : M)M \subseteq P$. Hence $(N + L) \subseteq P$ and so $(N + L : M)K \subseteq P$. Therefore

$$[(N : M)K + L] \subseteq [(N + L : M)(K + L)].$$


Let $N$ be an ideal of $\overline{M}$. Since $R(x + y) \subseteq Rx + Ry$ and $R(rx) \subseteq Rx$, where $x, y \in M$ and $r \in R$, we see that $M(N)$ is an $R$-submodule of $M$. 505
Lemma 2.4. If $M$ is a finitely generated $R$-module, then $\overline{M}(M[N]) = N$, for all ideals $N$ of $\overline{M}$.

Proof. It is clear that $N \subseteq \overline{M}(M[N])$. Let $[L] \in \overline{M}(M[N])$. Then for some finitely generated $K \in \text{su}(M)$, $K \subseteq L$ and $K \subseteq M[N]$. Suppose that $K = \sum_{i=1}^{n} m_i R$. Therefore we have $[L] = [K] = \sum_{i=1}^{n} [m_i R] \in N$. We conclude that $\overline{M}(M[N]) = N$ and the proof is complete. □

Lemma 2.5. Let $M$ be an $R$-module and $N \in \text{Spec}(\overline{M})$. Then $M[N] \in \text{Spec}(M)$.

Proof. If $M[N] = M$ then by Lemma 2.4, $N = \overline{M}(M[N]) = \overline{M}(M) = \overline{M}$, which is a contradiction. Suppose that $N \in \text{Spec}(\overline{M})$ and $rm \in M[N]$, $r \in R$, $m \in M$. Then $[Rrm] \in \overline{M}(M[N]) = N$. Since $M$ is a multiplication module, so $(Rm : M)M = Rm$. Hence $[(Rm : M)rM] = [Rm] \cap [rM] = [Rrm] \in N$, and so $[Rm] \in N$ or $[rM] \in N$. If $[Rm] \in N$ then $m \in M[N]$. Now if $[rM] \in N$, then $rM \subseteq M[N]$. □

Proposition 2.6.
(i) If $N \in \text{Spec}(\overline{M})$ then $\overline{M}(M[N]) = N$.
(ii) For every ideal $N$ of $\overline{M}$,

$$N \subseteq \overline{M}(M[N]) \subseteq \text{rad}(N) = \bigcap \{P \in \text{Spec}(\overline{M}) : N \subseteq P\}.$$ 

Proof. (i) Clearly $N \subseteq \overline{M}(M[N])$. Let $[K] \in \overline{M}(M[N])$. Hence there exists $L \subseteq M[N]$ such that $L \sim K$. By Lemma 2.5, $M[N] \in \text{Spec}(M)$ and so $K \subseteq M[N]$. Since $K \in \text{su}(M)$, we have $K = \sum_{i=1}^{t} I_i m_i R + J_i M$, where $I_i, J_i \in \text{FI}(R)$ and $m_i \in M$. Therefore $I_i m_i R \subseteq M[N]$ and $J_i M \subseteq M[N]$, for all $i$. Thus $m_i \in M[N]$ or $I_i \subseteq (M[N] : M)$. If $m_i \in M[N]$ then $[m_i R] \in N$. Since $[I_i m_i R] \subseteq [m_i R]$, we get $[I_i m_i R] \in N$. Now if $I_i \subseteq (M[N] : M)$ then $[I_i m_i R] \in N$. Therefore $[I_i m_i R] \in N$, for all $i$. By a similar proof $[J_i M] \in N$. We conclude that $[K] \in N$.

(ii) Let $N$ be any ideal of $\overline{M}$. If $N = \overline{M}$ then clearly $N = \overline{M}(M[N])$. Therefore assume that $N \neq \overline{M}$. Let $[K] \in \overline{M}(M[N])$. Hence $K \sim L$, for some $L \subseteq M[N]$. Choose a $P \in \text{Spec}(\overline{M})$, with $N \subseteq P$, then $M[N] \subseteq M[P]$. By Lemma 2.5, $M[P] \in \text{Spec}(M)$ and hence $K \subseteq \text{rad}(L) \subseteq \text{rad}(M[N]) \subseteq M[P]$. Thus $[K] \in \overline{M}(M[P]) = P$ (by (i)). So $[K] \in \text{rad}(N)$. Therefore $\overline{M}(M[N]) \subseteq \text{rad}(N)$. The proof is complete. □
Lemma 2.7. Let $M$ be an $R$-module and $N$ a submodule of $M$. Then $\overline{M}(N)$ is an ideal in the lattice $\overline{M}$.

Proof. Let $[L_1],[L_2] \in \overline{M}(N)$. Then there exist $K_1 \subseteq N$ and $K_2 \subseteq N$ such that $K_1 \sim L_1$ and $K_2 \sim L_2$. By Lemma 2.1, $(K_1 + K_2) \sim (L_1 + L_2)$ and so $[L_1] \lor [L_2] = [L_1 + L_2] \in \overline{M}(N)$. Now assume that $[L] \in \overline{M}(N)$, $[K] \in \overline{M}$ and $[K] \leq [L]$. We must show that $[K] \in \overline{M}(N)$. There exists $L' \subseteq N$, $L' \sim L$. Put $L_1 = (K : M)L'$. It is clear that $L_1 \subseteq N$. Let $Q \in \text{Spec}(M)$ and $L_1 \subseteq Q$. Then $L' \subseteq Q$ or $(K : M) \subseteq (Q : M)$. If $L' \subseteq Q$ then $L \subseteq \text{rad}(L) = \text{rad}(L') \subseteq Q$ and hence $K \subseteq Q$, because $[K] \leq [L]$. Now if $(K : M) \subseteq (Q : M)$ then $K \subseteq Q$. Clearly $(K : M)L' \subseteq K \subseteq Q$. Thus $K \sim L_1$ and so $[K] \in \overline{M}(N)$. □

Let $N$ be a submodule of $\overline{M}$.

Put $(N : \overline{M}) = \{J \in \text{FI}(R) : \text{for all } [K] \in \overline{M}, \text{there exists } [L] \in N; J[K] \leq [L]\}$. It is easy to show that $(N : \overline{M})$ is an ideal of $\text{FI}(R)$, i.e. $J_1 + J_2 \in (N : \overline{M})$, $IJ \in (N : \overline{M})$, where $J_1, J_2, J \in (N : \overline{M})$ and $I \in \text{FI}(R)$.

Proposition 2.8. Let $M$ be an $R$-module. Then $P \in \text{Spec}(\overline{M})$ if and only if $(P : \overline{M}) \in \text{Spec}(\text{FI}(R))$.

Proof. Let $(P : \overline{M}) = \text{FI}(R)$. By assumption $P \neq \overline{M}$, so there exists $[K] \in \overline{M} \setminus P$. Since $R \in (P : \overline{M})$, we have $R[K] = [K] \leq [L]$, for some $[L] \in P$. So $[K] \in P$, which is a contradiction. Therefore $(P : \overline{M}) \neq \text{FI}(R)$. Assume that $I, J \in \text{FI}(R)$ are such that $IJ \in (P : \overline{M})$. Let $[K] \in \overline{M}$. Then there exists $[L] \in P$ such that $IJ[K] \leq [L]$ and so $[IJK] \in P$. Clearly $[(IK : M)JM] \leq [IJK]$. Hence $[IK] \lor [JM] = [(IK : M)JKLM] = [IJK] \in P$, and so $[IK] \in P$ or $[JM] \in P$. We conclude that $I \in (P : \overline{M})$ or $J \in (P : \overline{M})$ and $(P : \overline{M}) \in \text{Spec}(\text{FI}(R))$. Conversely, let $[K] \lor [L] = [(K : M)L] \in P$ and $[T] \in \overline{M}$. Since $[(K : M)L]T \leq [(K : M)L]$, we have $(K : M) \in (P : \overline{M})$ or $(L : M) \in (P : \overline{M})$. If $(K : M) \in (P : \overline{M})$ then $[(K : M)M] \in P$. Since $M$ is a multiplication module, $[K] = [(K : M)M] \in P$. Similarly $[L] \in P$ and hence $P \in \text{Spec}(\overline{M})$. □

Lemma 2.9. Let $M$ be an $R$-module. If $P \in \text{Spec}(M)$ then $\overline{M}(P) \in \text{Spec}(\overline{M})$.

Proof. Assume that $\overline{M}(P) = \overline{M}$. By Lemma 2.2, $P = M[\overline{M}] = M$, which is a contradiction. Now let $[K] \lor [L] = [(K : M)L] \in \overline{M}(P)$. Then there exists $L' \subseteq P$ such that $(K : M)L \sim L'$. Therefore $(K : M)L \subseteq \text{rad}(L') \subseteq P$. So $L \subseteq P$ or $K = (K : M)M \subseteq P$. Thus $[K] \in \overline{M}(P)$ or $[L] \in \overline{M}(P)$. □
3. Topologies on Spec($M$) and Spec($\overline{M}$)

We begin this section by introducing a topology called the Zariski topology on Spec($M$) for any $R$-module $M$, in which closed sets are varieties

$$V(N) = \{ P \in \text{Spec}(M) : (N : M) \subseteq (P : M) \}$$

of all submodules $N$ of $M$ [2, Proposition 1.1]. Similarly, for any ideal $L$ of $\overline{M}$, put

$$\nabla(L) = \{ Q \in \text{Spec}(\overline{M}) : L \subseteq Q \}.$$ 

For the remainder of this section we let $M$ be a finitely generated multiplication $R$-module and $\text{Ann}(M)$ a finitely generated ideal of $R$.

**Lemma 3.1.** Let $M$ be an $R$-module. Put $T = \{ \nabla(L) \mid L \text{ is an ideal of } \overline{M} \}$. Then $T$ is the collection of closed sets of the Stone topology on Spec($\overline{M}$).

**Proof.** It is easy to show that $\nabla([0]) = \text{Spec}(\overline{M})$ and $\nabla(\overline{M}) = \emptyset$. Let $L$ and $N$ be ideals of $\overline{M}$. We show that $\nabla(L) \cup \nabla(N) = \nabla(L \cap N)$. Suppose that $Q \in \text{Spec}(\overline{M})$ is such that $L \cap N \subseteq Q$ and $L \not\subseteq Q$. Then there exists $[K] \in L \setminus Q$. Let $[K_1] \in N$. Clearly $[K] \wedge [K_1] \in L \cap N$. Therefore $[K_1] \in Q$. Hence $\nabla(L \cap N) \subseteq \nabla(L) \cup \nabla(N)$. It is clear that $\nabla(L) \cup \nabla(N) \subseteq \nabla(L \cap N)$. Let $\{ N_i \mid i \in I \}$ be a family of ideals of $\overline{M}$. Then $\bigcap_{i \in I} \nabla(N_i) = \nabla(\sum_{i \in I} N_i)$. \qed

For any subset $X \subseteq \text{Spec}(M)$, let $X = \{ \overline{M}(P) : P \in X \}$. Since $M$ is a finitely generated multiplication $R$-module and $\text{Ann}(M)$ is a finitely generated ideal of $R$, by Lemma 2.9 $X \subseteq \text{Spec}(\overline{M})$.

**Lemma 3.2.** Let $M$ be an $R$-module. Then for each submodule $N$ of $M$, $\overline{V(N)} = \nabla(\overline{M}(N))$.

**Proof.** Let $\overline{M}(P) \in \overline{V(N)}$, so $P \in V(N)$. Thus $(N : M) \subseteq (P : M)$. Let $[L] \in \overline{M}(N)$. Then $L \sim L'$, for some $L' \subseteq N$. But $(N : M)M = N$ and hence $L' \subseteq P$. Therefore $[L] \in \overline{M}(P)$. We conclude that $\overline{M}(N) \subseteq \overline{M}(P)$ and so $\overline{V(N)} \subseteq \nabla(\overline{M}(N))$. Now let $Q \in \nabla(\overline{M}(N))$, then $\overline{M}(N) \subseteq Q$. By Lemma 2.5, $M[Q] = P \in \text{Spec}(M)$. Hence by Lemma 2.4, $\overline{M}(P) = \overline{M}(M[Q]) = Q$. We claim that $(N : M) \subseteq (P : M)$. If $rM \subseteq N$ then $[rM] \in \overline{M}(N) \subseteq Q$ and so $rR[M] \in Q$. Hence $rM \subseteq M[Q] = P$. We conclude that $Q \in \overline{V(N)}$. \qed

Put $T = \{ V(N) \mid N \text{ is a submodule of } M \}$. 508
Theorem 3.3. Let $M$ be a finitely generated multiplication $R$-module and $\text{Ann}(M)$ a finitely generated ideal of $R$. Then the topological spaces $(\text{Spec}(M), T)$ and $(\text{Spec}(\overline{M}), \overline{T})$ are homeomorphic.

Proof. Define

$$\varphi: \text{Spec}(M) \longrightarrow \text{Spec}(\overline{M}); \quad \varphi(P) = \overline{M}(P)$$

and

$$\psi: \text{Spec}(\overline{M}) \longrightarrow \text{Spec}(M); \quad \psi(L) = M[L].$$

By Lemmas 2.9 and 2.5, $\varphi$ and $\psi$ are well-defined. By Lemmas 2.2 and 2.4, we have

$$\psi \circ \varphi(P) = \psi(\overline{M}(P)) = M[\overline{M}(P)] = P$$

and

$$\varphi \circ \psi(L) = \varphi(M[L]) = \overline{M}(M[L]) = L.$$

Hence the two mappings $\varphi$ and $\psi$ are inverses of each other. The bijection $\varphi$ induces a map $\overline{\varphi}: T \longrightarrow \overline{T}$ by $\overline{\varphi}(V(N)) = \overline{V(N)}$. By Lemma 3.2, $\overline{V(N)} = \overline{V(M(N))}$ and so $\overline{\varphi}$ is well-defined. We claim that this induced map is also a bijection. Suppose $\overline{V(N)} = \overline{V(L)}$. By Lemma 3.2, we have $\overline{V(M(N))} = \overline{V(M(L))}$. We must show that $V(N) = V(L)$. Let $P \in \text{Spec}(M)$ and $(N : M) \subseteq (P : M)$. Suppose that $rM \subseteq L$. Hence $[rM] \in \overline{M}(L)$. Since $\overline{M}(P) \in \overline{V(M(N))} = \overline{V(M(L))}$, we get $[rM] \in \overline{M}(P)$. Therefore $rM \subseteq P$. We conclude that $P \in V(L)$ and so $V(N) \subseteq V(L)$. By symmetry we infer that $V(L) = V(N)$. Hence $\overline{\varphi}$ is one-to-one. Now let $\overline{V(L)} \in \overline{T}$. Since $L$ is an ideal of $\overline{M}$, we have $\overline{\varphi}(V(M[L])) = \overline{V(M(M[L]))} = \overline{V(L)}$ and so $\overline{\varphi}$ is onto. □

Following M. Hochster [3], we say that a topological space $W$ is a spectral space if $W$ is homeomorphic to $\text{Spec}(S)$ with the Zariski topology, for some ring $S$.

Definition. A semi-reticulation for an $R$-module $M$ is a pair $(M, \lambda)$ where $M$ is a distributive lattice with 0, 1 and $\lambda: M \longrightarrow M$ is a mapping such that

(I) $\lambda(x + y) \leq \lambda(x) \lor \lambda(y)$;

(II) $\lambda(rx) \leq \lambda(x) \land \lambda(y)$, for some $y \in rM$;

(III) $\lambda(0) = 0$;

(IV) the inverse image map induced by $\lambda$ is a homeomorphism between $\text{Spec}(M)$ and $\text{Spec}(\overline{M})$ (with the Stone and the Zariski topologies respectively).

Moreover, if $\lambda(m) = 1$, for some $m \in M$, then we say that $M$ has a reticulation (this generalizes [7]).
**Theorem 3.5.** Let $M$ be a finitely generated $R$-module and $\text{Ann}(M)$ be a finitely generated ideal of $R$. Then the following are equivalent.

(i) $M$ is a multiplication module;

(ii) there exists a semi-reticulation for $M$;

(iii) $\text{Spec}(M)$ is spectral.

**Proof.** (i) $\Rightarrow$ (ii) Define $\lambda: M \rightarrow \mathbb{M}$ by $\lambda(x) = [Rx]$, where $x \in M$. Clearly (I), (II) and (III) are satisfied. By Theorem 3.3, we have $\lambda^{-1}(Q) = \psi(Q)$. Hence the inverse image map induced by $\lambda$ is a homeomorphism between $\text{Spec}(\mathbb{M})$ and $\text{Spec}(M)$.

(ii) $\Rightarrow$ (iii) It is well known that the prime ideal space of a distributive lattice with $0, 1$, is spectral under the Stone topology (see [1]). By Proposition 2.3 and Theorem 3.3, $\text{Spec}(M)$ is spectral.

(iii) $\Rightarrow$ (i) By [5, Corollary 6.6].

**Corollary 3.6.** Let $M$ be a finitely generated $R$-module. Suppose that $R$ is a Noetherian ring or $M$ is a faithful module (i.e. $\text{Ann}(M) = 0$). Then $M$ is multiplication if and only if $M$ has a semi-reticulation.

**Corollary 3.7.** Let $M$ be a cyclic $R$-module. Suppose that $R$ is a Noetherian ring or $M$ is a faithful module, then $M$ has a reticulation.

**Proof.** By Corollary 3.6, $M$ has a semi-reticulation. Since $M$ is a cyclic $R$-module, there exists $m \in M$ such that $Rm = M$. Therefore $\lambda(m) = [Rm] = [M] = 1$. We conclude that $M$ has a reticulation.

**References**


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